Journal of Nonlinear and Convex Analysis
Volume 15, Number 2, 2014, 429–436

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

APPROXIMATION OF A COMMON FIXED POINT OF A FINITE FAMILY OF NONEXPANSIVE MAPPINGS WITH NONSUMMABLE ERRORS IN A HILBERT SPACE

YASUNORI KIMURA

Dedicated to Professor Simeon Reich on his 65th birthday

Abstract. We study an iterative scheme for a finite family of nonexpansive mappings generated by the shrinking projection method with errors. We consider an error for obtaining the value of metric projections and show that the sequence still has a nice property for approximating a common fixed point of the mappings. In the proposed iterative scheme, we do not need to suppose any summability condition for the error terms.

1. Introduction

Let C be a subset of a real Hilbert space. The fixed point problem is a problem to find a fixed point of a mapping $T : C \to H$, that is, a point $z \in C$ satisfying that $z = Tz$. It is a very simple problem and it has been applied to various types of nonlinear problems such as convex minimization problems, variational inequality problems, equilibrium problems, and others. Further, as a generalization of this problem, we can consider the common fixed point problem for a family of mappings.

Approximation schemes to a solution of this problem have been investigated by a numerous number of reserachers as well as the existence of its solution. One branch of this study is implicit schemes; see Browder [2], Reich [17], Takahashi and Ueda [23], and others. These schemes also have a strong relation to the proximal point algorithm; see Rockafellar [19], Brézis and Lions [1], Pazy [15], Eckstein and Bertsekas [4], Kamimura and Takahashi [8], and others. For the studies in Banach spaces, see Bruck and Reich [3], Nevanlinna and Reich [14], Jung and Takahashi [7], Reich and Zaslavski [18], Kimura and Takahashi [12], and others.

The explicit iterative schemes which guarantee strong convergence have also investigated in many papers; see Halpern [5], Wittmann [25], Shioji and Takahashi [20], and others.

Another important scheme to approximate the solution of fixed point problems is projection method. It was first proposed by Haugazeau [6] and was developed by Solodov and Svaiter [21] as a modified version of the proximal point algorithm for monotone operators. Nakajo and Takahashi [13] first adopted this scheme to solve the fixed point problem for nonexpansive mapping.

2010 *Mathematics Subject Classification.* 47H09.
Key words and phrases. Common fixed point, nonexpansive, shrinking projection method, iterative scheme, metric projection, error term.
The author is supported by Grant-in-Aid for Scientific Research No. 22540175 from Japan Society for the Promotion of Science.

We will focus on another type of projection methods proved by Takahashi, Takeuchi, and Kubota [22], which is called the shrinking projection method.

Theorem 1.1 (Takahashi-Takeuchi-Kubota [22])**.** *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $F(T) = \{z \in C : z = Tz\}$ is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0, a]$, where $0 < a < 1$. For a point $x \in H$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_1 = C$, and*

$$\begin{aligned} y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_{n+1} &= \{z \in C : \|z - y_n\| \leq \|z - x_n\|\} \cap C_n, \\ x_{n+1} &= P_{C_{n+1}} x \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{F(T)}x \in C$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

We remark that the original result of the theorem above deals with a family of nonexpansive mappings, and it has been generalized to the setting of Banach spaces; see also Kimura, Nakajo, and Takahashi [11], Kimura and Takahashi [12].

In this paper, we study an iterative scheme for a finite family of nonexpansive mappings. The approximating sequence is generated by the shrinking projection method with errors. In the practical calculation, it is a task of difficulty to calculate the exact value of metric projections which is required to obtain the iterative sequence by this method. We consider an error for obtaining the value of metric projections and show that the sequence still has a nice property for approximating a common fixed point of the mappings. The technique we used in the main result has been proposed in [9, 10]. We emphasize that, in the proposed iterative scheme, we do not need to suppose any summability condition for the error terms.

2. Preliminaries

In what follows, we always assume that Hilbert spaces are over the real scalar field. and we denote by $\mathbb{N}$ the set of positive intergers.

Let H be a Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. For $x_1, x_2, \dots, x_m \in H$ and $\alpha_1, \alpha_2, \dots, \alpha_m \in [0, 1]$ with $\sum_{k=1}^m \alpha_k = 1$, it follows from the parallelogram law that

$$\left\| \sum_{k=1}^m \alpha_k x_k \right\|^2 = \sum_{k=1}^m \alpha_k \|x_k\|^2 - \sum_{l=k+1}^m \sum_{k=1}^{m-1} \alpha_k \alpha_l \|x_k - x_l\|^2 .$$

Let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $z \in C$ is called a fixed point of T if it holds that $z = Tz$. The set of all fixed point of T is denoted by $F(T)$. We know that if C is closed and convex, then so is $F(T)$.

Let C be a nonempty closed convex subset of H and $u \in H$. Then, there exists a unique point $z_u \in C$ such that

$$\|u - z_u\| = \inf_{z \in C} \|u - z\| .$$

Using this correspondence, we can define the mapping $P_C : H \to C$ by $u \mapsto z_u$ for each $u \in H$ and call it the metric projection onto C. We know that P_C is nonexpansive.

The following lemma is easily deduced from the theorem proved by Tsukada [24].

Lemma 2.1 (Tsukada [24]). *Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a Hilbert space H such that $C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$. Let u be a point in H. Then, if $C_0 = \bigcap_{n=1}^{\infty} C_n$ is nonempty, then the sequence $\{P_{C_n} u\}$ of metric projections onto the subsets $\{C_n\}$ of u converges strongly to $P_{C_0} u$.*

3. The shrinking projection method with errors

We obtain an iterative scheme approximating a solution to the common fixed point problem for a finite family of nonexpansive mappings. We consider calculation errors for the metric projections used in the scheme. The main result shows that, for an iterative sequence $\{x_n\}$, we are able to estimate an upper bound of the sequence $\{\|x_n - T_k x_n\|\}$ for each mapping T_k without any summability conditions for the error terms.

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a Hilbert space H with $D = \operatorname{diam} C = \sup_{x,y \in C} \|x - y\| < \infty$, and let $\{T_1, T_2, \ldots, T_m\}$ be a finite family of nonexpansive mappings of C to H such that $\bigcap_{k=1}^{m} F(T_k)$ is nonempty. Let $\{\alpha_{n,k} : n \in \mathbb{N}, k \in \{1, 2, \ldots, m\}\}$ be a family of positive real numbers such that $\sum_{k=1}^{m} \alpha_{n,k} = 1$. Let $\alpha_k = \liminf_{n\to\infty} \alpha_{n,k} > 0$ for $k \in \{1, 2, \ldots, m\}$. Let $\{\epsilon_n\}$ be a nonnegative real sequence such that $\epsilon_0 = \limsup_{n\to\infty} \epsilon_n < \infty$. For given $u \in H$, generate an iterative sequece $\{x_n\}$ as follows: $x_1 \in C$ such that $\|x_1 - u\| < \epsilon_1$, $C_1 = C$,*

$$
\begin{aligned}
&y_n = \sum_{k=1}^{m} \alpha_{n,k} T_k x_n, \\
&C_{n+1} = \{z \in C : \|y_n - z\| \le \|x_n - z\|\} \cap C_n, \\
&x_{n+1} \in C_{n+1} \text{ such that } \|x_{n+1} - u\|^2 \le d(u, C_{n+1})^2 + \epsilon_{n+1}^2
\end{aligned}
$$

for $n \in \mathbb{N}$. Then

$$
\limsup_{n\to\infty} \|x_n - T_k x_n\| \le \left(2 + \frac{4D(m-1)}{\alpha_k}\right) \epsilon_0
$$

for each $k \in \{1, 2, \ldots, m\}$. Further, if $\epsilon_0 = 0$, then $\{x_n\}$ converges strongly to $P_{\bigcap_{k=1}^{m} F(T_k)} u \in \bigcap_{k=1}^{m} F(T_k)$.

Proof. Let $F = \bigcap_{k=1}^{m} F(T_k)$. First we show that C_n is a closed convex subset such that $F \subset C_n$ for every $n \in \mathbb{N}$ by induction. It is trivial that $F \subset C_1 = C$ and a given point x_1 is defined. It is also obvious that C_n is closed and convex for any $n \in \mathbb{N}$. Suppose that each of $C_1, C_2, \ldots, C_j$ contains F. Then, since C_j is nonempty, we can choose a point $x_j \in C_j$ satisfying the condition in the theorem. Then y_j and C_{j+1} is also defined. Let $z \in \bigcap_{k=1}^{m} F(T_k)$. Since it follows that

$$
\|y_j - z\| = \left\| \sum_{k=1}^{m} \alpha_{j,k} T_k x_j - z \right\| \le \sum_{k=1}^{m} \alpha_{j,k} \|T_k x_j - z\| \le \|x_j - z\|,
$$

we have that $z \in C_{j+1}$. Thus we have that $F \subset C_{j+1}$. Hence $\{C_n\}$ is a sequence of nonempty closed convex subset of H such that $F \subset \bigcap_{n=1}^{\infty} C_n$.

Next, Let $C_0 = \bigcap_{n=1}^{\infty} C_n$ and let $w_n = P_{C_n} u$ for each $n \in \mathbb{N}$. Then, since $\{C_n\}$ is decreasing with respect to inclusion and $C_0 \supset F \neq \emptyset$, we obtain that $\{w_n\}$ converges strongly to $w_0 = P_{C_0} u$ by Lemma 2.1. From the definition of the metric projection, we have that

$$\|x_n - u\|^2 \leq d(u, C_n)^2 + \epsilon_n^2 = \|u - w_n\|^2 + \epsilon_n^2$$

for $n \in \mathbb{N}$. Since $x_n \in C_n$ and $w_n = P_{C_n} u$, we have that

$$\begin{aligned}
0 &\leq 2 \langle u - w_n, w_n - x_n \rangle \\
&= \|u - x_n\|^2 - \|u - w_n\|^2 - \|w_n - x_n\|^2
\end{aligned}$$

and hence

$$\|w_n - x_n\|^2 \leq \|u - x_n\|^2 - \|u - w_n\|^2 \leq \epsilon_n^2$$

for any $n \in \mathbb{N}$. Letting $\delta_n = \|w_n - w_0\|$ for every $n \in \mathbb{N}$, we obtain that $\lim_{n\to\infty} \delta_n = 0$ and since $w_0 \in C_0$, it follows that

$$\begin{aligned}
\|y_n - w_0\| &\leq \|x_n - w_0\| \\
&\leq \|x_n - w_n\| + \|w_n - w_0\| \\
&\leq \epsilon_n + \delta_n
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, for $z \in F$ and $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\|y_n - z\|^2 &= \left\| \sum_{k=1}^{m} \alpha_{n,k} T_k x_n - z \right\|^2 \\
&= \sum_{k=1}^{m} \alpha_{n,k} \|T_k x_n - z\|^2 - \sum_{l=k+1}^{m} \sum_{k=1}^{m-1} \alpha_{n,k}\alpha_{n,l} \|T_k x_n - T_l x_n\|^2 \\
&\leq \|x_n - z\|^2 - \sum_{l=k+1}^{m} \sum_{k=1}^{m-1} \alpha_{n,k}\alpha_{n,l} \|T_k x_n - T_l x_n\|^2 .
\end{aligned}$$

Thus, for $k, l \in \{1, 2, \ldots, m\}$ with $k \neq l$, we have that

$$\begin{aligned}
\alpha_{n,k}\alpha_{n,l} \|T_k x_n - T_l x_n\|^2 &\leq \|x_n - z\|^2 - \|y_n - z\|^2 \\
&\leq (\|x_n - z\| + \|y_n - z\|)(\|x_n - z\| - \|y_n - z\|) \\
&\leq 2D \|x_n - y_n\| \\
&\leq 2D(\|x_n - w_n\| + \|w_n - w_0\| + \|w_0 - y_n\|) \\
&\leq 2D(\epsilon_n + \delta_n + \epsilon_n + \delta_n) \\
&= 4D(\epsilon_n + \delta_n).
\end{aligned}$$

So we get that

$$\|T_k x_n - T_l x_n\|^2 \leq \frac{4D(\epsilon_n + \delta_n)}{\alpha_{n,k}\alpha_{n,l}}.$$

Therefore, for each $k \in \{1, 2, \dots, m\}$, we have that

$$\begin{aligned}
\|y_n - T_k x_n\|^2 &= \left\| \sum_{l=1}^{m} \alpha_{n,l} T_l x_n - T_k x_n \right\|^2 \\
&\leq \sum_{l \neq k} \alpha_{n,l} \|T_l x_n - T_k x_n\|^2 \\
&\leq \sum_{l \neq k} \alpha_{n,l} \frac{4D(\epsilon_n + \delta_n)}{\alpha_{n,k}\alpha_{n,l}} \\
&= 4D(\epsilon_n + \delta_n) \sum_{l \neq k} \frac{1}{\alpha_{n,k}} \\
&= \frac{4D(m-1)(\epsilon_n + \delta_n)}{\alpha_{n,k}}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_n - T_k x_n\| &\leq \|x_n - w_n\| + \|w_n - w_0\| + \|w_0 - y_n\| + \|y_n - T_k x_n\| \\
&\leq \epsilon_n + \delta_n + (\epsilon_n + \delta_n) + \sqrt{\frac{4D(m-1)(\epsilon_n + \delta_n)}{\alpha_{n,k}}} \\
&\leq 2 \left((\epsilon_n + \delta_n) + \sqrt{\frac{D(m-1)(\epsilon_n + \delta_n)}{\alpha_{n,k}}} \right)
\end{aligned}$$

for every $n \in \mathbb{N}$ and hence we have that

$$\limsup_{n \to \infty} \|x_n - T_k x_n\| \leq 2 \left(\epsilon_0 + \sqrt{\frac{D(m-1)\epsilon_0}{\alpha_k}} \right)$$

for every $k \in \{1, 2, \dots, m\}$.

For the latter part of the theorem, assume $\epsilon_0 = 0$. Then, from the last inequality, we have that

$$\lim_{n \to \infty} \|x_n - T_k x_n\| = 0$$

for $k \in \{1, 2, \dots, m\}$. We also have that

$$\begin{aligned}
0 &\leq \lim_{n \to \infty} \|x_n - w_0\| \\
&\leq \lim_{n \to \infty} (\|x_n - w_n\| + \|w_n - w_0\|) \\
&\leq \lim_{n \to \infty} (\epsilon_n + \delta_n) \\
&= \epsilon_0 = 0,
\end{aligned}$$

and thus $\{x_n\}$ converges strongly to $w_0 = P_{C_0} u$. Therefore, using the continuity of T_k, we obtain that

$$\|w_0 - T_k w_0\| = \lim_{n \to \infty} \|x_n - T_k x_n\| = 0$$

and hence $w_0 \in F(T_k)$ for all $k \in \{1, 2, \ldots, m\}$. Since $w_0 = P_{C_0}u \in F$ and $F \subset C_0$, we get that

$$w_0 = P_{C_0}u = P_F u,$$

which is the desired result. □

As a special case, an approximating sequence of a solution to a fixed point problem for a single mapping can be generated as follows:

Theorem 3.2. *Let C be a nonempty bounded closed convex subset of a Hilbert space H with $D = \operatorname{diam} C = \sup_{x,y\in C} \|x - y\| < \infty$, and let $T : C \to H$ be a nonexpansive mapping having a fixed point. Let $\{\epsilon_n\}$ be a nonnegative real sequence such that $\epsilon_0 = \limsup_{n\to\infty} \epsilon_n < \infty$. For given $u \in H$, generate an iterative sequece $\{x_n\}$ as follows: $x_1 \in C$ such that $\|x_1 - u\| < \epsilon_1$, $C_1 = C$,*

$$C_{n+1} = \{z \in C : \|Tx_n - z\| \leq \|x_n - z\|\} \cap C_n,$$
$$x_{n+1} \in C_{n+1} \text{ such that } \|x_{n+1} - u\|^2 \leq d(u, C_{n+1})^2 + \epsilon_{n+1}^2$$

for $n \in \mathbb{N}$. Then

$$\limsup_{n\to\infty} \|x_n - Tx_n\| \leq 2\epsilon_0.$$

Further, if $\epsilon_0 = 0$, then $\{x_n\}$ converges strongly to $P_{F(T)}u \in F(T)$.

We can also obtain the following strong convergence theorem for finding a common fixed point of a finite family of nonexpansive mappings; see [16].

Theorem 3.3. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $\{T_1, T_2, \ldots, T_m\}$ be a finite family of nonexpansive mappings of C to H such that $\bigcap_{k=1}^m F(T_k)$ is nonempty. Let $\{\alpha_{n,k} : n \in \mathbb{N}, k \in \{1, 2, \ldots, m\}\}$ be a family of positive real numbers such that $\sum_{k=1}^m \alpha_{n,k} = 1$. For given $u \in H$, generate an iterative sequece $\{x_n\}$ as follows: $x_1 \in C$, $C_1 = C$,*

$$y_n = \sum_{k=1}^m \alpha_{n,k} T_k x_n,$$
$$C_{n+1} = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \cap C_n,$$
$$x_{n+1} = P_{C_{n+1}} u$$

for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_{\bigcap_{k=1}^m F(T_k)} u \in \bigcap_{k=1}^m F(T_k)$.

Proof. Apply Theorem 3.1 with $\epsilon_n = 0$ for all $n \in \mathbb{N}$. □

Remark. In Theorem 3.3, it is easy to see that we do not need the assumption of the boundedness of C to prove the convergence of the iterative scheme. We only use this assumption to calculate an upper bound of the sequence $\{\|x_n - T_k x_n\|\}$.

References

[1] H. Brézis and P.-L. Lions, *Produits infinis de résolvantes*, Israel J. Math. **29** (1978), 329–345.

[2] F. E. Browder, *Convergence of pooroximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Rational Mech. Anal. **24** (1967), 82–90.

[3] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470.

[4] J. Eckstein and D. P. Bertsekas, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Math. Programming **55** (1992), 293–318.
[5] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.
[6] Y. Haugazeau, *Sur les inéquations variationnelles et la minimisation de fonctionnelles convexes*, Ph.D. thesis, Université de Paris, Paris, France, 1968.
[7] J. S. Jung and W. Takahashi, *Dual convergence theorems for the infinite products of resolvents in Banach spaces*, Kodai Math. J. **14** (1991), 358–365.
[8] S. Kamimura and W. Takahashi, *Approximating solutions of maximal monotone operators in Hilbert spaces*, J. Approx. Theory **106** (2000), 226–240.
[9] Y. Kimura, *Approximation of a fixed point of nonexpansive mappings with nonsummable errors in a geodiesic space*, Proceedings of the 10th International Conference on Fixed Point Theory and Applications, to appear.
[10] Y. Kimura, *Approximation of a fixed point of nonlinear mappings with nonsummable errors in a Banach space*, Proceedings of the Fourth International Symposium on Banach and Function Spaces, to appear.
[11] Y. Kimura, K. Nakajo, and W. Takahashi, *Strongly convergent iterative schemes for a sequence of nonlinear mappings*, J. Nonlinear Convex Anal. **9** (2008), 407–416.
[12] Y. Kimura and W. Takahashi, *A generalized proximal point algorithm and implicit iterative schemes for a sequence of operators on Banach spaces*, Set-Valued Anal. **16** (2008), 597–619.
[13] K. Nakajo and W. Takahashi, *Approximation of a zero of maximal monotone operators in Hilbert spaces*, Nonlinear analysis and convex analysis, Yokohama Publ., Yokohama, 2003, pp. 303–314.
[14] O. Nevanlinna and S. Reich, *Strong convergence of contraction semigroups and of iterative methods for accretive operators in Banach spaces*, Israel J. Math. **32** (1979), 44–58.
[15] A. Pazy, *Remarks on nonlinear ergodic theory in Hilbert space*, Nonlinear Anal. **3** (1979), 863–871.
[16] S. Plubtieng and K. Ungchittrakool, *Hybrid iterative methods for convex feasibility problems and fixed point problems of relatively nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. (2008), Art. ID 583082, 19.
[17] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.
[18] S. Reich and A. J. Zaslavski, *Infinite products of resolvents of accretive operators*, Topol. Methods Nonlinear Anal. **15** (2000), 153–168, Dedicated to Juliusz Schauder, 1899–1943.
[19] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.
[20] N. Shioji and W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc. **125** (1997), 3641–3645.
[21] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilbert space*, Math. Program. **87** (2000), 189–202.
[22] W. Takahashi, Y. Takeuchi, and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.
[23] W. Takahashi and Y. Ueda, *On Reich's strong convergence theorems for resolvents of accretive operators*, J. Math. Anal. Appl. **104** (1984), 546–553.
[24] M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory **40** (1984), 301–309.
[25] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. (Basel) **58** (1992), 486–491.

Manuscript received June 11, 2013
revised July 27, 2013

Yasunori Kimura
Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
E-mail address: yasunori@is.sci.toho-u.ac.jp