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NET CONSTRUCTION OF AN ERGODIC NONEXPANSIVE RETRACTION ONTO A FIXED POINT SET OF ASYMPTOTICALLY NONEXPANSIVE IN THE INTERMEDIATE SENSE MAPPINGS

WIESŁAWA KACZOR AND JUSTYNA WALCZUK

This paper is dedicated to Professor Simeon Reich on the occasion of his 65th birthday

Abstract. Suppose X is uniformly convex, Y is a Banach space and $\|\cdot\|_Z$ is a norm in $X \oplus Y$ which satisfies the condition (*). Let C be a nonempty, convex and weakly compact subset of $X \oplus Y$ such that the standard projection $P_2(C)$ is compact in Y and let $\mathcal{S} = \{T_\alpha : \alpha \in A\}$ be a family of commuting, uniformly continuous and asymptotically nonexpansive in the intermediate sense self-mappings of C. Then there exists a nonexpansive $\mathcal{S}$-ergodic retraction $R : C \to \operatorname{Fix}\mathcal{S}$ onto the common fixed point set.

1. Introduction

In 1968 F. E. Browder [3] stated the following demiclosedness principle for nonexpansive mappings.

Definition 1.1 ([3]). Let X be a Banach space and $\emptyset \neq C \subset X$ be a bounded and convex set. A mapping $T : C \to X$ is demiclosed at y if a sequence $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to y, then $x \in C$ and $Tx = y$.

If for every nonexpansive mapping $T : C \to C$ the mapping $I - T$, where I is the identity mapping on C, is demiclosed at 0, then we say that C has the demiclosedness principle for nonexpansive mappings.

If each bounded and convex set $\emptyset \neq C \subset X$ has the demiclosedness principle for nonexpansive mappings, then we say that the Banach space X has the demiclosedness principle for nonexpansive mappings.

In his paper F. E. Browder proved the following theorem.

Theorem 1.2 ([3]). *A uniformly convex Banach space X has the demiclosedness principle for nonexpansive mappings.*

In [21] P.-K. Lin proved the following generalization of the above principle.

Theorem 1.3 ([21]). *Suppose X is uniformly convex and Y has the Schur property. If the norm $\|\cdot\|_Z$ satisfies $(*)$, then $(X \oplus Y, \|\cdot\|_Z)$ has the demiclosedness principle for nonexpansive mappings.*

It is easy to observe that Lin's proof can be applied to the following theorem.

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Theorem 1.4. *Suppose X is uniformly convex and Y is an arbitrary Banach space. If in $(X \oplus Y, \|\cdot\|_Z)$ the norm $\|\cdot\|_Z$ satisfies $(*)$ and for a bounded and convex set $C \subset X \oplus Y$ its projection $P_2(C)$ is a compact set in Y, then C has the demiclosedness principle for nonexpansive mappings.*

In Theorems 1.3 and 1.4 the norm $\|\cdot\|_Z$ in $X \oplus Y$ is defined as follows. Let $\|\cdot\|_Z$ be any norm in $\mathbb{R}^2$ that satisfies

(1) $\|(s,0)\|_Z = \|(0,s)\|_Z = |s|$ for all $s \in \mathbb{R}$,
(2) $\|(s,t)\|_Z = \|(|s|,|t|)\|_Z$ for all $s,t \in \mathbb{R}$,
(3) $\|(s,t)\|_Z \le \|(s_1,t_1)\|_Z$ for $0 \le s \le s_1$ and $0 \le t \le t_1$.

Moreover, assume that the norm $\|\cdot\|_Z$ satisfies the following condition:

$$(*) \qquad \|(s,t)\|_Z > |s| \quad \text{for all} \quad t \neq 0.$$

Now, the norm of $(X \oplus Y, \|\cdot\|_Z)$ is given by

$$\|(x,y)\|_Z = \|(\|x\|_X, \|y\|_Y)\|_Z$$

for all $(x,y) \in X \oplus Y$.

Browder's result has been generalized to wider classes of mappings, Banach spaces and even CAT(0) spaces [22]. First in [27] H.-K. Xu extended demiclosedness principle to asymptotically nonexpansive mappings (see also [11]).

Theorem 1.5 ([27]). *Let X be a uniformly convex Banach space, C a bounded, closed and convex subset of X, and $T : C \to C$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0.*

The nonstandard proof of the above result can be found in [17] (compare [13]). Later for asymptotically nonexpansive in the intermediate sense mappings and nets the similar result was obtained [14].

Theorem 1.6. *Let X be a uniformly convex Banach space, C a bounded, closed and convex subset of X, and $T : C \to C$ a uniformly continuous mapping which is asymptotically nonexpansive in the intermediate sense. If $\{x_\xi\}_{\xi\in I}$ is a net in C converging weakly to x and if*

$$\lim_{\xi \in I} \|x_\xi - Tx_\xi\| = 0,$$

then $x = Tx$.

The above theorem is a direct consequence of the following generalized demiclosedness property.

Theorem 1.7 ([14]). *Let X be a uniformly convex Banach space, C a bounded, closed and convex subset of X, and $T : C \to C$ a continuous mapping which is asymptotically nonexpansive in the intermediate sense. If $\{x_\xi\}_{\xi\in I}$ is a net in C converging weakly to x and if*

$$\lim_{k\to\infty} \left(\limsup_{\xi\in I} \|x_\xi - T^k x_\xi\|\right) = 0,$$

then $x = Tx$.

In our paper we extend the above results: theorems due to P.-K. Lin and H.-K. Xu and Theorem 1.7.

Next, we apply our demiclosedness principle for nets to get a construction of a nonexpansive $\mathcal{S}$-ergodic retraction $R : C \to \operatorname{Fix} \mathcal{S}$ onto the common fixed point set of a family $\mathcal{S}$ of commuting, uniformly continuous and asymptotically nonexpansive in the intermediate sense self-mappings of C. Let us observe that in [16] under the assumption of separability of a uniformly convex Banach space X and using demiclosedness principle for sequences the authors give a simpler, than we show here, construction of the $\mathcal{S}$-ergodic nonexpansive retraction onto the fixed point set of a finite family $\mathcal{S}$ of commuting asymptotically nonexpansive in the intermediate sense mappings.

Finally, it is worth mentioning here that T. Domínguez Benavides and P. Lorenzo Ramírez [10] using different methods proved the following general and deep result about the common fixed point set for commuting families of mappings.

Theorem 1.8 ([10]). *Let X be a Banach space and C a nonempty weakly compact convex subset of X. Assume that every asymptotically nonexpansive self-mapping of C satisfies the (ω)-fpp. Then for any commuting family $\mathcal{G}$ of asymptotically nonexpansive self-mappings of C, the common fixed point set of $\mathcal{G}$ is a nonempty nonexpansive retract of C.*

2. Preliminaries

First we recall some necessary definitions, notions and notations. The modulus of convexity [9] of a Banach space X is the function $\delta : [0, 2] \to [0, 1]$ defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon \right\}.$$

A Banach space X is said to be uniformly convex if $\delta(\epsilon) > 0$ for each $\epsilon \in (0, 2]$.

By $\operatorname{Fix} T$ we denote the set of fixed points of a mapping T.

Definition 2.1. Let $C \subset X$. A mapping $T : C \to C$ is nonexpansive if for any $x, y \in C$,

$$\|Tx - Ty\| \le \|x - y\|.$$

Some results concerning nonexpansive mappings have been extended to wider classes of mappings and Banach spaces. Now we recall definitions of these classes of mappings. The first one was introduced in 1972 by K. Goebel and W. A. Kirk [12].

Definition 2.2. Let $C \subset X$ and $T : C \to C$. If there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ as $n \to \infty$ for which

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in C$, then T is said to be asymptotically nonexpansive.

Later in 1974 W. A. Kirk [19] defined mappings of asymptotically nonexpansive type.

Definition 2.3. Let $C \subset X$ be bounded and $T : C \to C$. If T satisfies

$$\limsup_{n\to\infty} \sup_{y\in C}(\|T^n x - T^n y\| - \|x - y\|) \le 0$$

for each $x \in C$, and T^N is continuous for some $N \ge 1$, then T is a mapping of asymptotically nonexpansive type.

The third class of mappings was introduced in 1993 by R. E. Bruck, T. Kuczumow and S. Reich [8].

Definition 2.4. Let $C \subset X$ be bounded. A mapping $T : C \to C$ is called asymptotically nonexpansive in the intermediate sense if T is continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0.$$

W. A. Kirk [19] proved that if X is a uniformly convex Banach space, $C \subset X$ is nonempty, bounded, closed and convex and $T : C \to C$ is a mapping of asymptotically nonexpansive type, then T has a fixed point.

Now we give necessary information about ultrapowers. Basic facts concerning ultrapowers come from the book [17] and we adopt notations introduced there. The set $\mathbb{N}$ can be treated as a sequence $\{n\}_{n\in\mathbb{N}}$. Hence it has a subnet $\{n_\xi\}$ which is an ultranet (see, e.g.,[1]). Throughout this paper the ultranet $\{n_\xi\}$ will remain fixed.

Given a Banach space X, we define

$$l_\infty(X) = \{x = \{x_n\} : \sup_{n\in\mathbb{N}} \|x_n\| < \infty\}$$

and for $\{x_n\} \in l_\infty(X)$ we put

$$\|\{x_n\}\|_\infty = \sup_{n\in\mathbb{N}} \|x_n\|.$$

It is known that $l_\infty(X)$ is a Banach space equipped with the above norm. Now let

$$\mathcal{N} = \{\{x_n\} \in l_\infty(X) : \lim_\xi \|x_{n_\xi}\| = 0\},$$

and define the Banach space ultrapower $\widetilde{X}$ of X (relative to the fixed ultranet $\{n_\xi\}$) as the quotient space $l_\infty(X)/\mathcal{N}$. The elements of $\widetilde{X}$ consist of equivalence classes $\widetilde{x} = [\{x_n\}]$. It is known that $\widetilde{X}$ with the norm $\|\cdot\|_\xi$ defined by the following formula

$$\|\widetilde{x}\|_\xi = \|[\{x_n\}]\|_\xi = \lim_\xi \|x_{n_\xi}\|,$$

is a Banach space. We also have $\{u_n\} \in [\{x_n\}]$ if and only if $\lim_\xi \|u_{n_\xi} - x_{n_\xi}\| = 0$. Moreover, if X is uniformly convex, then so is $\widetilde{X}$ and $\delta_{\widetilde{X}} = \delta_X$.

Let (x_n) denote the constant sequence whose all terms are equal to x, where $x \in X$. Then $\dot{x} = [(x_n)] \in \widetilde{X}$ and X is linearly isometric to the subspace

$$\dot{X} = \{\dot{x} : x \in X\}$$

of the ultrapower $\widetilde{X}$ via mapping $i(x) = \dot{x}$. Now if $C \subset X$, then we set

$$\dot{C} = \{\dot{x} : x \in C\}$$

and

$$\widetilde{C} = \{\widetilde{x} = [\{x_n\}] : x_n \in C \text{ for each } n\}.$$

To state our next observation we need the following notions and notations. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces and let P_1 and P_2 be the standard projections of $X \oplus Y$ onto X and Y, respectively.

Now we recall the definition of the norm $\|\cdot\|_Z$ in $X \oplus Y$, which was mentioned in the Introduction. Let $\|\cdot\|_Z$ be any norm in $\mathbb{R}^2$ that satisfies

(1) $\|(s,0)\|_Z = \|(0,s)\|_Z = |s|$ for all $s \in \mathbb{R}$,
(2) $\|(s,t)\|_Z = \|(|s|,|t|)\|_Z$ for all $s,t \in \mathbb{R}$,
(3) $\|(s,t)\|_Z \le \|(s_1,t_1)\|_Z$ for $0 \le s \le s_1$ and $0 \le t \le t_1$.

Clearly, there are $m, M > 0$ such that

$$m \cdot \max\{|s|,|t|\} \le \|(s,t)\|_Z \le M \cdot \max\{|s|,|t|\}$$

for all $s,t \in \mathbb{R}$.

The norm $\|\cdot\|_Z$ in $X \oplus Y$ is generated by the above $\mathbb{R}^2$-norm by setting

$$\|(x,y)\|_Z = \|(\|x\|_X, \|y\|_Y)\|_Z$$

for all $(x,y) \in X \oplus Y$.

Suppose also that the norm $\|\cdot\|_Z$ satisfies

$$(*) \qquad \|(s,t)\|_Z > |s| \quad \text{for all} \quad t \ne 0.$$

Then the norm in the ultrapower $\widetilde{X \oplus Y}$ of $X \oplus Y$ has the following form

$$\begin{aligned}\|[\{(x_n,y_n)\}]\| &= \lim_{\xi} \|(\|x_{n_\xi}\|_X, \|y_{n_\xi}\|_Y)\|_Z \\ &= \|(\lim_{\xi} \|x_{n_\xi}\|_X, \lim_{\xi} \|y_{n_\xi}\|_Y)\|_Z = \|(\|[\{x_n\}]\|, \|[\{y_n\}]\|)\|_Z\end{aligned}$$

and therefore $\widetilde{X \oplus Y} = \widetilde{X} \oplus \widetilde{Y}$ and its norm satisfies the condition similar $(*)$. Namely,

$$(**) \qquad \|[\{(x_n,y_n)\}]\| > \|[\{x_n\}]\| \quad \text{for all} \quad [\{y_n\}] \ne 0.$$

Hence we also denote $\|[\{(x_n,y_n)\}]\|$ by $\|\cdot\|_Z$.

If $P_2(C)$, where $C \subset X \oplus Y$, is a compact set in Y, then

$$\widetilde{C} \subset \widetilde{P_1(C)} \oplus P_2\dot{(C)}$$

and therefore the set $\widetilde{C}$ can be treated as a subset of the Banach space $(\widetilde{X} \oplus Y, \|\cdot\|_Z)$ with compact $P_2(\widetilde{C})$. Now if X is additionally uniformly convex and $C \subset X \oplus Y$ is weakly compact and $P_2(C)$ is a compact, then $\widetilde{C} \subset \widetilde{P_1(C)} \oplus P_2\dot{(C)}$ and both sets $\widetilde{C}$ and $\widetilde{P_1(C)}$ are weakly compact and the second one lies in the uniformly convex Banach space $\widetilde{X}$. So, $\widetilde{C}$ can be identified with the weakly compact subset of $(\widetilde{X} \oplus Y, \|\cdot\|_Z)$, where $\widetilde{X}$ is the uniformly convex Banach space and $P_2(\widetilde{C}) = P_2(C)$ is compact. If C is additionally convex, then $\widetilde{C}$ is also convex.

More facts concerning this setting can be found, for example, in [1], [17], [18], [25] and [26].

Now let X be an arbitrary Banach space, $\emptyset \ne C \subset X$ and suppose $T : C \to C$. We can define a canonical extension $\widetilde{T} : \widetilde{C} \to \widetilde{C}$ of T by

$$\widetilde{T}(\widetilde{x}) = [\{T(x_n)\}]$$

for $\widetilde{x} = [\{x_n\}] \in \widetilde{C}$. In the case of a uniformly continuous mapping $T : C \to C$ which is asymptotically nonexpansive in the intermediate sense, we can also define another natural mapping $\hat{T} : \widetilde{C} \to \widetilde{C}$ by setting

$$\hat{T}(\widetilde{x}) = [\{T^n(x_n)\}]$$

for $\widetilde{x} = [\{x_n\}] \in \widetilde{C}$. It is clear that this mapping is nonexpansive.

3. The generalized demiclosedness principle

We begin from the following definition.

Definition 3.1. Let C be a nonempty bounded closed and convex subset of a Banach space $(X, \|\cdot\|)$. C is said to have the generalized demiclosedness property if for any continuous and asymptotically nonexpansive in the intermediate sense mapping $T : C \to C$ and each sequence $\{x_n\}$ in C converging weakly to x with

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|x_n - T^k x_n\| = 0$$

we have $Tx = x$. The space X is said to have the generalized demiclosedness property if every nonempty bounded closed and convex subset of X has the generalized demiclosedness property.

Now we can state the main result of this section.

Theorem 3.2. *Suppose X is uniformly convex, Y is a Banach space and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. If the norm $\|\cdot\|_Z$ satisfies $(*)$, then each nonempty bounded closed and convex subset C of $(X \oplus Y, \|\cdot\|_Z)$ such that $P_2(C)$ is compact in Y has the generalized demiclosedness property for continuous and asymptotically nonexpansive in the intermediate sense mappings.*

Proof. Let C be a nonempty bounded closed and convex subset of a Banach space $(X \oplus Y, \|\cdot\|_Z)$ and let $T : C \to C$ be a continuous and asymptotically nonexpansive in the intermediate sense mapping. Suppose that $\{(x_n, y_n)\}$ is a weakly convergent sequence in C with

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|(x_n, y_n) - T^k(x_n, y_n)\|_Z = 0.$$

For $(x, y) = w\text{-}\lim_{n\to\infty}(x_n, y_n)$, we shall prove that $T(x, y) = (x, y)$. It is obvious that $\{y_n\}$ tends strongly to y. Set

$$\rho = \inf\{\liminf_{n\to\infty} \|x'_n - x\|_X : \{(x'_n, y'_n)\} \in D\},$$

where D is the set of all sequences $\{(x'_n, y'_n)\}$ in C which satisfy the following three conditions:

(i) $\lim_{n\to\infty} y'_n = y$,
(ii) $w\text{-}\lim_{n\to\infty} x'_n = x$,
(iii) $\lim_{k\to\infty} \limsup_{n\to\infty} \|(x'_n, y'_n) - T^k(x'_n, y'_n)\|_Z = 0$.

Assume first that $\rho = 0$. Given $\epsilon > 0$, there is a sequence $\{(x'_n, y'_n)\} \in D$ such that $\lim_{n\to\infty} \|x'_n - x\|$ exists and $\lim_{n\to\infty} \|x'_n - x\| \le \frac{\epsilon}{2M}$. Then we have

$$\|(x, y) - T^k(x, y)\|_Z \le \|(x, y) - (x'_n, y'_n)\|_Z + \|(x'_n, y'_n) - T^k(x'_n, y'_n)\|_Z$$

$$+\|T^k(x'_n,y'_n)-T^k(x,y)\|_Z$$
$$\le 2M\max\{\|x-x'_n\|_X,\|y-y'_n\|_Y\}$$
$$+\|(x'_n,y'_n)-T^k(x'_n,y'_n)\|_Z+\eta_k,$$

where

$$0\le \eta_k=\max\{0,\sup_{(x,y),(x',y')\in C}(\|T^k(x,y)-T^k(x',y')\|-\|(x,y)-(x',y')\|)\}.$$

Therefore

$$\lim_{k\to\infty}\eta_k=0$$

and

$$\limsup_{k\to\infty}\|(x,y)-T^k(x,y)\|_Z\le \limsup_{k\to\infty}\limsup_{n\to\infty}[2M\max\{\|x-x'_n|_X,\|y-y'_n\|_Y\}$$
$$+\|(x'_n,y'_n)-T^k(x'_n,y'_n)\|_Z+\eta_k]\le\epsilon.$$

This means that

$$\lim_{k\to\infty}\|(x,y)-T^k(x,y)\|_Z=0$$

and, by the continuity of T, we get

$$T(x,y)=T(\lim_{k\to\infty}T^k(x,y))=\lim_{k\to\infty}T(T^k(x,y))=\lim_{k\to\infty}T^{k+1}(x,y)=(x,y).$$

Suppose now that $\rho>0$. By the definition of ρ for $0<\epsilon<\rho$ there is a sequence $\{(x'_n,y'_n)\}\in D$ such that $\lim_{n\to\infty}\|x'_n-x\|_X$ exists and

$$\rho\le\lim_{n\to\infty}\|x'_n-x\|_X\le\rho+\epsilon<2\rho.$$

Let

$$(x''_n,y''_n)=\frac{1}{2}[(x'_n,y'_n)+(x'_{n+1},y'_{n+1})].$$

We may assume that $\lim_{n\to\infty}\|x''_n-x\|_X$ exists. Our task is to prove that the sequence $\{(x''_n,y''_n)\}$ is in D, which will lead to a contradiction. To show that $\{(x''_n,y''_n)\}\in D$, we introduce the following notation

$$(\bar{x}_{nk},\bar{y}_{nk})=T^k(x''_n,y''_n)$$

and

$$(\bar{x}'_{nk},\bar{y}'_{nk})=T^k(x'_n,y'_n)$$

for $n.k=1,2,\dots$. By assumptions and by passing to a subsequence, we may assume that $\|x'_n-x'_m\|\ge\frac{\rho}{2}$ for all $n\ne m$. Since

$$m\cdot\max\{\|x'_n-\bar{x}'_{nk}\|_X,\|y'_n-\bar{y}'_{nk}\|_Y\}\le\|(x'_n,y'_n)-(\bar{x}'_{nk},\bar{y}'_{nk})\|_Z$$

and

$$\lim_{k\to\infty}\limsup_{n\to\infty}\|(x'_n,y'_n)-(\bar{x}'_{nk},\bar{y}'_{nk})\|_Z=0,$$

we get

$$\lim_{k\to\infty}\limsup_{n\to\infty}\|x'_n-\bar{x}'_{nk}\|_X=0,$$
$$\lim_{k\to\infty}\limsup_{n\to\infty}\|y'_n-\bar{y}'_{nk}\|_Y=0$$

and finally,

$$\lim_{k\to\infty}\limsup_{n\to\infty}\|y-\bar{y}'_{nk}\|_Y=0.$$

We proceed to show that

$$\lim_{k\to\infty}\limsup_{n\to\infty}\|(x''_n,y''_n)-(\bar{x}_{nk},\bar{y}_{nk})\|_Z=0.$$

First we choose sequences $\{k_i\}$ and $\{n_i\}$ so that

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\|(x''_n,y''_n)-(\bar{x}_{nk},\bar{y}_{nk})\|_Z=\lim_{i\to\infty}\|(x''_{n_i},y''_{n_i})-(\bar{x}_{n_ik_i},\bar{y}_{n_ik_i})\|_Z,$$

$$\lim_{i\to\infty}\|T^{k_i}(x'_{n_i},y'_{n_i})-(x'_{n_i},y'_{n_i})\|_Z=0$$

and the limits

$$\begin{aligned}
&\lim_{i\to\infty}\|\bar{x}_{n_ik_i}-x'_{n_i}\|_X,\\
&\lim_{i\to\infty}\|\bar{x}_{n_ik_i}-x'_{n_i+1}\|_X,\\
&\lim_{i\to\infty}\|\bar{y}_{n_ik_i}-y'_{n_i}\|_Y,\\
&\lim_{i\to\infty}\|\bar{y}_{n_ik_i}-y'_{n_i+1}\|_Y,\\
&\lim_{i\to\infty}\|x'_{n_i}-\bar{x}'_{n_ik_i}\|_X=0,\\
&\lim_{i\to\infty}\|y'_{n_i}-\bar{y}'_{n_ik_i}\|_Y=0
\end{aligned}$$

and

$$\lim_{i\to\infty}\|x'_{n_i}-x'_{n_i+1}\|_X=r\geq\frac{\rho}{2}$$

exist. Then we have

$$\begin{aligned}
\lim_{i\to\infty}\|(\bar{x}_{n_ik_i},\bar{y}_{n_ik_i})-(x'_{n_i},y'_{n_i})\|_Z&=\lim_{i\to\infty}\|T^{k_i}(x''_{n_i},y''_{n_i})-T^{k_i}(x'_{n_i},y'_{n_i})\|_Z\\
&\quad+\lim_{i\to\infty}\|T^{k_i}(x'_{n_i},y'_{n_i})-(x'_{n_i},y'_{n_i})\|_Z\\
&=\lim_{i\to\infty}\|T^{k_i}(x''_{n_i},y''_{n_i})-T^{k_i}(x'_{n_i},y'_{n_i})\|_Z.
\end{aligned}$$

Consequently, applying the property $(*)$ we get

$$\begin{aligned}
\lim_{i\to\infty}\|\bar{x}_{n_ik_i}-x'_{n_i}\|_X&\leq\lim_{i\to\infty}\|(\bar{x}_{n_ik_i},\bar{y}_{n_ik_i})-(x'_{n_i},y'_{n_i})\|_Z\\
&=\lim_{i\to\infty}\|T^{k_i}(x''_{n_i},y''_{n_i})-T^{k_i}(x'_{n_i},y'_{n_i})\|_Z\\
&\leq\lim_{i\to\infty}\|(x''_{n_i},y''_{n_i})-(x'_{n_i},y'_{n_i})\|_Z\\
&=\frac{1}{2}\lim_{i\to\infty}\|x'_{n_i+1}-x'_{n_i}\|_X=\frac{1}{2}r.
\end{aligned}$$

Likewise,

$$\begin{aligned}
\lim_{i\to\infty}\|\bar{x}_{n_ik_i}-x'_{n_i+1}\|_X&\leq\lim_{i\to\infty}\|(\bar{x}_{n_ik_i},\bar{y}_{n_ik_i})-(x'_{n_i+1},y'_{n_i+1})\|_Z\\
&\leq\frac{1}{2}\lim_{i\to\infty}\|x'_{n_i+1}-x'_{n_i}\|_X=\frac{1}{2}r
\end{aligned}$$

Hence

$$\lim_{i\to\infty}\|\bar{x}_{n_ik_i}-x'_{n_i}\|_X=\lim_{i\to\infty}\|\bar{x}_{n_ik_i}-x'_{n_i+1}\|_X=\frac{1}{2}\lim_{i\to\infty}\|x'_{n_i+1}-x'_{n_i}\|_X=\frac{1}{2}r.$$

By the uniform convexity of X we get

$$\lim_{i\to\infty} \|\bar{x}_{n_i k_i} - x''_{n_i}\|_X = 0.$$

By the above we also obtain

$$\frac{1}{2}r = \lim_{i\to\infty} \|\bar{x}_{n_i k_i} - x'_{n_i}\|_X = \lim_{i\to\infty} \|(\bar{x}_{n_i k_i}, \bar{y}_{n_i k_i}) - (x'_{n_i}, y'_{n_i})\|_Z,$$

$$\frac{1}{2}r = \lim_{i\to\infty} \|\bar{x}_{n_i k_i} - x'_{n_i+1}\|_X = \lim_{i\to\infty} \|(\bar{x}_{n_i k_i}, \bar{y}_{n_i k_i}) - (x'_{n_i+1}, y'_{n_i+1})\|_Z$$

and since $\|\cdot\|_Z$ satisfies $(*)$, we have

$$\lim_{i\to\infty} \|\bar{y}_{n_i k_i} - y'_{n_i}\|_Y = 0$$

and

$$\lim_{i\to\infty} \|\bar{y}_{n_i k_i} - y'_{n_i+1}\|_Y = 0.$$

Finally,

$$\lim_{i\to\infty} \|\bar{y}_{n_i k_i} - y''_{n_i}\|_Y = 0$$

and

$$\lim_{i\to\infty} \|(\bar{x}_{n_i k_i}, \bar{y}_{n_i k_i}) - (x''_{n_i}, y''_{n_i})\|_Z = 0.$$

This shows that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|T^k(x''_n, y''_n) - (x''_n, y''_n)\|_Z = 0.$$

So, we have $\{(x''_n, y''_n)\} \in D$. But taking $0 < \epsilon < \delta(\frac{1}{4})\rho$ we get the following contradiction

$$0 < \rho \le \liminf_{n\to\infty} \|x''_n - x\|_X \le \Big(1 - \delta\Big(\frac{1}{4}\Big)\Big)\Big(1 + \delta\Big(\frac{1}{4}\Big)\Big)\rho < \rho.$$

□

Directly from the definition of the Schur space and the above theorem we obtain the following generalized demiclosedness property for Banach spaces.

Corollary 3.3. *Suppose X is uniformly convex, Y has the Schur property and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. If the norm $\|\cdot\|_Z$ satisfies $(*)$, then $(X \oplus Y, \|\cdot\|_Z)$ has the generalized demiclosedness property for continuous and asymptotically nonexpansive in the intermediate sense mappings.*

As a simple consequence of Theorem 3.2 we get.

Theorem 3.4. *Suppose X is uniformly convex, Y is a Banach space and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. If the norm $\|\cdot\|_Z$ satisfies the condition $(*)$, then each nonempty bounded closed and convex subset C of $(X \oplus Y, \|\cdot\|_Z)$ such that $P_2(C)$ is compact in Y has the demiclosedness principle for uniformly continuous and asymptotically nonexpansive in the intermediate sense mappings, that is, for such a C and for a uniformly continuous mapping $T : C \to C$ which is asymptotically nonexpansive in the intermediate sense if $\{(x_n, y_n)\}$ is a sequence in C converging weakly to (x, y) and $\lim_{n\to\infty} \|(x_n, y_n) - T(x_n, y_n)\|_Z = 0$, then $(x, y) = T(x, y)$.*

Corollary 3.5. *Suppose X is uniformly convex, Y has the Schur property and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. If the norm $\|\cdot\|_Z$ satisfies the condition $(*)$, then $(X \oplus Y, \|\cdot\|_Z)$ has the demiclosedness principle for uniformly continuous and asymptotically nonexpansive in the intermediate sense mappings.*

4. Equivalence of the demiclosedness principle for sequences and the demiclosedness principle for nets

In our paper we will need a net version of Theorem 3.4. Therefore we have to introduce the following definition of demiclosedness for nets.

Definition 4.1. Let X be a Banach space and let $\emptyset \neq C \subset X$ be bounded and convex. A mapping $T : C \to X$ is demiclosed for nets at y if a net $\{x_\xi\}$ converges weakly to x and $\{Tx_\xi\}$ converges strongly to y, then $x \in C$ and $Tx = y$.

If for every nonexpansive mapping $T : C \to C$ the mapping $I - T$, where I is the identity mapping on C, is demiclosed for nets at 0, then we say that C has the net-demiclosedness principle for nonexpansive mappings.

If for every uniformly continuous and asymptotically nonexpansive in the intermediate sense mapping $T : C \to C$ the mapping $I - T$ is demiclosed for nets at 0, then we say that C has the net-demiclosedness principle for asymptotically nonexpansive in the intermediate sense mappings.

If each bounded closed convex subset C of X has the net-demiclosedness principle for nonexpansive (asymptotically nonexpansive in the intermediate sense) mappings, then we say that Banach space X has the net-demiclosedness principle for nonexpansive (asymptotically nonexpansive in the intermediate sense) mappings.

Let us observe that in the case of weakly compact and convex sets both definitions of the demiclosedness principle for sequences and for nets are equivalent. Indeed, we have the following general theorem. The idea of its proof is similar to that given in the proof of Lemma 2.5 in [15].

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a Banach space and C be a convex and weakly compact subset of X with the demiclosedness principle for sequences and a certain class of continuous mappings. Then C has the demiclosedness principle for nets and this class of mappings.*

Proof. Clearly, it is sufficient to prove that the demiclosedness principle for sequences implies the demiclosedness principle for nets. Let $T : C \to X$ be a mapping in the class of mappings under cosideration. Suppose that a net $\{x_\xi\}_{\xi \in I}$ in C is weakly convergent to x and $\{Tx_\xi\}_{\xi \in I}$ is strongly convergent to y. If $\lim_{\xi \in I} \|x_\xi - x\| = 0$, then we find an increasing sequence $\{\xi_n\}_{n \in \mathbb{N}}$ so that we have $\lim_{n\to\infty} \|x_{\xi_n} - x\| = 0$ and $\lim_{n\to\infty} \|Tx_{\xi_n} - y\| = 0$. By continuity of T we then get $Tx = y$. In the other case the net $\{x_\xi - x\}_{\xi \in I}$ is not norm-convergent to zero. Let $x_\xi = x + \tilde{x}_\xi$ for $\xi \in I$. Then $\{\tilde{x}_\xi\}_{\xi \in I}$ is a weakly null net. Now we can apply the Mazur method of constructing basic sequences [2] (see also [15]). Passing eventually to a subnet we can assume that $\inf\{\|\tilde{x}_\xi\| : \xi \in I\} > 0$. Applying the Mazur technique and the induction assumption $\|Tx_{\xi_n} - y\| = \|T(x + \tilde{x}_{\xi_n}) - y\| < 1/n$ we can find an increasing sequence $\{\xi_n\}_{n \in \mathbb{N}}$ such that $\{\tilde{x}_{\xi_n}\}$ is a basic sequence and

$$\lim_{n\to\infty} \|Tx_{\xi_n} - y\| = \lim_{n\to\infty} \|T(x + \tilde{x}_{\xi_n}) - y\| = 0.$$

Now we choose a weakly convergent subsequence $\{\tilde{x}_{\xi_{n_k}}\}$. This subsequence is also a basic sequence and therefore its weak limit is zero. Hence, the sequence $\{x_{\xi_n}\} = \{x+\tilde{x}_{\xi_n}\}$ is weakly convergent to x and by the demiclosedness principle for sequences of the set C and of our class of mappings we obtain $Tx = y$. □

Directly from the above theorem get two corollaries.

Corollary 4.3. *Suppose X is uniformly convex, Y is a Banach space and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. If the norm $\|\cdot\|_Z$ satisfies the condition $(*)$ then a nonempty, bounded, closed and convex subset C of $(X \oplus Y, \|\cdot\|_Z)$ such that $P_2(C)$ is compact in Y has the demiclosedness principle for nets in the class of uniformly continuous and asymptotically nonexpansive in the intermediate sense mappings, that is, if $T : C \to C$ a uniformly continuous mapping which is asymptotically nonexpansive in the intermediate sense and if $\{(x_\xi, y_\xi)\}_{\xi \in I}$ is a net in C converging weakly to (x, y) and $\lim_{\xi \in I} \|(x_\xi, y_\xi) - T(x_\xi, y_\xi)\|_Z = 0$, then $(x, y) = T(x, y)$.*

Corollary 4.4. *Suppose X is uniformly convex, Y has the Schur property and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. If the norm $\|\cdot\|_Z$ satisfies the condition $(*)$, then $(X \oplus Y, \|\cdot\|_Z)$ has the demiclosedness principle for nets in the class of uniformly continuous and asymptotically nonexpansive in the intermediate sense mappings.*

5. A family of approximate fixed point nets and a construction of a nonexpansive retraction

In this section we introduce a family of approximate fixed point nets for a family of uniformly continuous and asymptotically nonexpansive in the intermediate sense mappings which will be one of the basic tools in our construction of a nonexpansive retraction. First we recall the following definition.

Definition 5.1. Let C be a nonempty subset of a Banach space X. We say that a nonempty subset D of C is T-invariant for $T : C \to C$ if $T(D) \subset D$. If $\mathcal{S}$ is a family of self-mappings of C and for $\emptyset \neq D \subset C$ we have $T(D) \subset D$ for each $T \in \mathcal{S}$, then D is called $\mathcal{S}$-invariant.

To get the above mentioned family of approximate fixed point nets we will use two famous Bruck's theorems ([4], [5], [6], [7]) in a form suitable for our considerations.

Theorem 5.2. *Let C be a nonempty, convex and weakly compact subset of a Banach space X. Suppose also that C has the following property: If $T : C \to C$ is nonexpansive, then T has a fixed point in every nonempty, closed and convex and T-invariant subset of C. Then for any commuting family $\mathcal{S}$ of nonexpansive self-mappings of C, the set $\operatorname{Fix}\mathcal{S}$ of common fixed points of $\mathcal{S}$ is a nonempty nonexpansive retract of C.*

Theorem 5.3. *Let C be a nonempty, convex and weakly compact subset of a Banach space X. Suppose also that C has the following property: If $T : C \to C$ is nonexpansive, then T has a fixed point in every nonempty, closed and convex and T-invariant subset of C. Then there exists a nonexpansive retraction R from C onto $\operatorname{Fix} T$ which satisfies:*

(i) $R \circ T = R$,

(ii) *every closed convex T-invariant subset of C is also R-invariant.*

The retraction R satisfying (i) and (ii) is called T-ergodic retraction. It is then natural that for a family $\mathcal{S}$ of nonexpansive self-mappings of C (C is a nonempty, convex and weakly compact subset of a Banach space X) a nonexpansive retraction R from C onto $\operatorname{Fix} \mathcal{S}$ which satisfies:

(i) $R \circ T = R$ for each $T \in \mathcal{S}$,

(ii) every closed convex $\mathcal{S}$-invariant subset of C is also R-invariant,

is called $\mathcal{S}$-ergodic retraction ([24]).

By a simple modification of the proof given in [7] we obtain the following generalization of Theorem 5.3.

Theorem 5.4. *Let C be a nonempty, convex and weakly compact subset of a Banach space X. Suppose also that C has the following property: If $T : C \to C$ is nonexpansive, then T has a fixed point in every nonempty, closed and convex and T-invariant subset of C. Let $\mathcal{S} = \{T_j : j = 1, 2, \dots, k\}$ be a finite family of commuting nonexpansive mappings from C to C. Then there exists a nonexpansive retraction R from C onto* $\operatorname{Fix} \mathcal{S}$ *which satisfies:*

(i) *$R \circ T_j = R$ for $j = 1, 2, \dots, k$,*

(ii) *every closed convex $\mathcal{S}$-invariant subset of C is also R-invariant.*

Now one can apply the latter theorem to the case of subsets C in Cartesian products of Banach spaces or use the result due to S. Saeidi (see Theorem 3.4 in [23]) and get.

Theorem 5.5. *Suppose X is uniformly convex, Y is a Banach space and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. Suppose also that the norm $\|\cdot\|_Z$ satisfies the condition $(*)$. Let C be a nonempty, convex and weakly compact subset of $X \oplus Y$ such that $P_2(C)$ is compact in Y and let $\mathcal{S} = \{T_\alpha\}_{\alpha \in A}$ be a family of commuting nonexpansive mappings from C to C. Then there exists a nonexpansive retraction R from C onto* $\operatorname{Fix} \mathcal{S}$ *which satisfies:*

(i) *$R \circ T_\alpha = R$ for $\alpha \in A$,*

(ii) *every closed convex $\mathcal{S}$-invariant subset of C is also R-invariant.*

We will also need the following definition.

Definition 5.6. Let X be a Banach space, C a nonempty subset of X, $\{x_\xi\}_{\xi \in I}$ a net in C and $T : C \to C$ a mapping. If

$$\lim_{\xi \in I} \|Tx_\xi - x_\xi\| = 0,$$

we say that $\{x_\xi\}$ is an approximate fixed point net (afpn) for T.

Suppose X is uniformly convex, Y is a Banach space and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. Suppose also that the norm $\|\cdot\|_Z$ satisfies the condition $(*)$. Let C be a nonempty, bounded, closed and convex subset of $X \oplus Y$ such that $P_2(C)$ is compact in Y, and let $T : C \to C$ be a uniformly continuous and asymptotically nonexpansive in the intermediate sense mapping. We now show that there exists a weakly convergent approximate fixed point net assigned to each (x, y) in C in such a way that the family of these nets is in some sense nonexpansive. Let us recall that we will apply the fixed ultranet $\{n_\xi\}$ of the sequence $\{n\}_{n\in\mathbb{N}}$ of positive integers, which we have used in the construction of the Banach space ultrapower $\widetilde{X\oplus Y}$ of $X\oplus Y$.

Theorem 5.7. *Suppose X is uniformly convex, Y is a Banach space and $\|\cdot\|_Z$ is a norm in $X\oplus Y$. Suppose also that the norm $\|\cdot\|_Z$ satisfies the condition $(*)$. Let C be a nonempty, bounded, closed and convex subset of $X\oplus Y$ such that $P_2(C)$ is compact in Y, and let $\mathcal{S}=\{T_\alpha : \alpha\in A\}$ be a family of commuting, uniformly continuous and asymptotically nonexpansive in the intermediate sense self-mappings of C. Then there exists a mapping*

$$C\ni(x,y)\to\{r_n(x,y)\}\in C^{\mathbb{N}}$$

such that

(i) *for each $(x,y)\in C$, the ultranet $\{r_{n_\xi}(x,y)\}$ is an approximate fixed point net (afpn) with respect to each T_α, that is,*

$$\lim_\xi \|T_\alpha(r_{n_\xi}(x,y))-r_{n_\xi}(x,y)\|_Z=0,$$

(ii) *each ultranet $\{r_{n_\xi}(x,y)\}$ is weakly convergent,*

(iii) *for all $(x,y),(x_1,y_1)\in C$ we have*

$$\lim_\xi \|r_{n_\xi}(x,y)-r_{n_\xi}(x_1,y_1)\|_Z\le\|(x,y)-(x_1,y_1)\|_Z,$$

(iv) *for each $(x,y)\in C$ and $\alpha\in A$,*

$$\lim_\xi \|r_{n_\xi}(T_\alpha(x,y))-r_{n_\xi}(x,y)\|_Z=0,$$

(v) *if D is a closed convex $\mathcal{S}$-invariant subset of C, then*

$$[\{r_n(x,y)\}]\in\widetilde{D}$$

and

$$w\text{-}\lim_\xi r_{n_\xi}(x,y)\in D$$

for each $(x,y)\in D$.

Proof. Our proof is partially based on the ideas given in the proof of Theorem 4.2 in the paper by Kirk, Yañez and Shin [20]. We consider the set $\widetilde{C}$ in a Banach space ultrapower $\widetilde{X\oplus Y}$ of $X\oplus Y$ (relative to the fixed ultranet $\{n_\xi\}$). As we know all mappings $\hat{T}_\alpha:\widetilde{C}\to\widetilde{C}$ are nonexpansive. Similarly mappings $\widetilde{T}_\alpha\circ\hat{T}_\alpha:\widetilde{C}\to\widetilde{C}$, $\alpha\in A$, are nonexpansive. It is easy to see that all mappings of these two types commute. Let $\mathcal{S}'$ consist of all these mappings. Hence, by Theorem 5.5, the common fixed point set of these mappings is nonempty and there exists a nonexpansive $\mathcal{S}'$-ergodic retraction

$$r:\widetilde{C}\to\operatorname{Fix}\mathcal{S}'=\bigcap_{\beta\in A}\left(\operatorname{Fix}\hat{T}_\beta\cap\operatorname{Fix}(\widetilde{T}_\beta\circ\hat{T}_\beta)\right).$$

Thus

$$r\circ\hat{T}_\alpha=r,$$
$$r\circ\widetilde{T}_\alpha\circ\hat{T}_\alpha=r$$

for $\alpha \in A$ and if $\widetilde{D}$ is a closed convex $\mathcal{S}'$-invariant subset of $\widetilde{C}$, then it is also r-invariant. Let us observe that for $\alpha \in A$ if

$$\widetilde{(x,y)} \in \operatorname{Fix} \mathcal{S}' = \bigcap_{\beta \in A} \Big(\operatorname{Fix} \hat{T}_\beta \cap \operatorname{Fix} (\widetilde{T}_\beta \circ \hat{T}_\beta) \Big),$$

then

$$\hat{T}_\alpha \widetilde{(x,y)} = (\widetilde{T}_\alpha \circ \hat{T}_\alpha) \widetilde{(x,y)} = \widetilde{(x,y)}$$

from which

$$\widetilde{T}_\alpha \widetilde{(x,y)} = \widetilde{(x,y)}.$$

This means that

$$\lim_\xi \|T_\alpha(x_{n_\xi}, y_{n_\xi}) - (x_{n_\xi}, y_{n_\xi})\|_Z = 0.$$

Next

$$\begin{aligned} r(\widetilde{(x,y)}) &= r(\hat{T}_\alpha \widetilde{(x,y)}) = r(\widetilde{T}_\alpha(\hat{T}_\alpha \widetilde{(x,y)})) \\ &= r(\hat{T}_\alpha(\widetilde{T}_\alpha \widetilde{(x,y)})) = r(\widetilde{T}_\alpha \widetilde{(x,y)}) \end{aligned}$$

for all $\widetilde{(x,y)} \in \widetilde{C}$ and $\alpha \in A$. Now taking the isometric mapping $i : C \to \dot{C} \subset \widetilde{C}$ given by $i(x,y) = (\dot{x,y})$ for $(x,y) \in C$, we get the mapping

$$r \circ i : C \to \operatorname{Fix} \mathcal{S}',$$

which is nonexpansive. We introduce the following notation

$$(r \circ i)(x,y) = r((\dot{x,y})) = [\{r_n(x,y)\}]$$

for each $(x,y) \in C$. Hence we have

$$\lim_\xi \|T_\alpha(r_{n_\xi}(x,y)) - r_{n_\xi}(x,y)\|_Z = 0$$

for $(x,y) \in C$, $\alpha \in A$; and

$$\lim_\xi \|r_{n_\xi}(x,y) - r_{n_\xi}(x_1,y_1)\|_Z \le \|(x,y) - (x_1,y_1)\|_Z$$

for all $(x,y), (x_1,y_1) \in C$. Additionally we get

$$\begin{aligned} [\{r_n(x,y)\}] &= r((\dot{x,y})) = r(\widetilde{T}_\alpha(\dot{x,y})) \\ &= r((T_\alpha (\dot{x,y}))) = [\{r_n(T_\alpha(x,y))\}], \end{aligned}$$

or equivalently,

$$\lim_\xi \|r_{n_\xi}(T_\alpha(x,y)) - r_{n_\xi}(x,y)\|_Z = 0$$

for all $(x,y) \in C$ and $\alpha \in A$. Since the net $\{r_{n_\xi}(x,y)\}$ is an ultranet and C is weakly compact, this ultranet $\{r_{n_\xi}(x,y)\}$ is weakly convergent to an element of C (for every $(x,y) \in C$). Next, if D is a closed convex $\mathcal{S}$-invariant subset of C then $\widetilde{D}$ is a closed convex $\mathcal{S}'$-invariant subset of $\widetilde{C}$. By $\mathcal{S}'$-ergodicity of r we have that $r(\widetilde{D}) \subset \widetilde{D}$. In particular, for $(x,y) \in D$ we find $[\{(u_{n,D}(x,y), v_{n,D}(x,y))\}]$ such that $(u_{n,D}(x,y), v_{n,D}(x,y)) \in D$ for each n and

$$r((\dot{x,y})) = [\{r_n(x,y)\}] = [\{(u_{n,D}(x,y), v_{n,D}(x,y))\}] \in \widetilde{D}.$$

Therefore

$$w\text{-}\lim_{\xi} r_{n_\xi}(x,y)) = w\text{-}\lim_{\xi} \big(u_{n_\xi,D}(x,y), v_{n_\xi,D}(x,y)\big) \in D.$$

Thus the mapping

$$C \ni (x,y) \to \{r_n(x,y)\} \in C^{\mathbb{N}}$$

enjoys the claimed properties. □

Remark 5.8. Properties (iv) and (v) describe, in some sense, a limit ergodic property of the mapping $C \ni (x,y) \to \{r_n(x,y)\} \in C^{\mathbb{N}}$ with respect to each T_α.

Now, using the family $\{r_{n_\xi}\}$ of mappings and demiclosedness principle we give an earlier announced construction of a nonexpansive retraction onto the fixed point set of commuting, uniformly continuous and asymptotically nonexpansive in the intermediate sense mappings.

Theorem 5.9. *Suppose X is uniformly convex, Y is a Banach space and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. Suppose also that the norm $\|\cdot\|_Z$ satisfies the condition $(*)$. Let C be a nonempty, convex and weakly compact subset of $X \oplus Y$ such that $P_2(C)$ is compact in Y and let $\mathcal{S} = \{T_\alpha : \alpha \in A\}$ be a family of commuting, uniformly continuous and asymptotically nonexpansive in the intermediate sense self-mappings of C. Then there exists a nonexpansive $\mathcal{S}$-ergodic retraction*

$$R : C \to \operatorname{Fix} \mathcal{S}.$$

Proof. Let (x,y) be a point in C. Taking the ultranet $\{r_{n_\xi}(x,y)\}$ from Theorem 5.7, which is weakly convergent and is an afpn with respect to each T_α, by the demiclosedness principle (Corollary 4.4), we see that the weak limit of this ultranet is a common fixed point of $\mathcal{S}$. Denote this limit by $R(x,y)$. Thus we have a mapping $R : C \to \operatorname{Fix} \mathcal{S}$, which by the lower semicontinuity of the norm (with respect to the weak topology) and by the properties (iii), (iv) and (v) in Theorem 5.7 is the claimed nonexpansive $\mathcal{S}$-ergodic retraction. □

As a direct consequence we get.

Corollary 5.10. *Suppose X is uniformly convex, Y has the Schur property and $\|\cdot\|_Z$ is a norm in $X \oplus Y$. Suppose also that the norm $\|\cdot\|_Z$ satisfies the condition $(*)$. Let C be a nonempty, convex and weakly compact subset of $X \oplus Y$ and let $\mathcal{S} = \{T_\alpha : \alpha \in A\}$ be a family of commuting, uniformly continuous and asymptotically nonexpansive in the intermediate sense self-mappings of C. Then there exists a nonexpansive $\mathcal{S}$-ergodic retraction*

$$R : C \to \operatorname{Fix} \mathcal{S}.$$

References

[1] A. G. Aksoy and M. A. Khamsi,*Nonstandard Methods in Fixed Point Theory*, Springer-Verlag, New York, 1990.

[2] C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces*, Studia Math. **17** (1958), 151–164.

[3] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.

[4] R. E. Bruck, *Nonexpansive retracts of Banach spaces*, Bull. Amer. Math. Soc. **76** (1970), 384–386 .

[5] R. E. Bruck, *Properties of fixed point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.

[6] R. E. Bruck, *A common fixed point theorem for a commuting family of nonexpansive mappings*, Pacific J. Math. **53** (1974), 59–71.

[7] R. E. Bruck, *Asymptotic behavior of nonexpansive mappings*, in Fixed Points and Nonexpansive Mappings (Cincinnati, Ohio, 1982), Contemp. Math., 18, Amer. Math. Soc., Providence, RI, 1983, pp. 1–47.

[8] R. E. Bruck , T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloq. Math. **LXV** (1993), 169–179.

[9] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

[10] T. Domínguez Benavides and P. Lorenzo Ramírez, *Structure of the fixed point set and common fixed points of asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **129** (2001), 3549–3557.

[11] J. García Falset, B. Sims and M. Smyth, *The demiclosedness principle for mappings of asymptotically nonexpansive type*, Houston J. Math. **22** (1996), 101–108.

[12] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[13] W. Kaczor, *A nonstandard proof of a generalized demiclosedness principle*, Ann. Univ. Mariae Curie Skodowska **59** (2005), 43–50.

[14] W. Kaczor, T. Kuczumow and S. Reich, *A mean ergodic theorem for mappings which are asymptotically nonexpansive in the intermediate sense*, Nonlinear Anal. **47** (2001), 2731–2742.

[15] W. Kaczor and S. Prus, *Asymptotical smoothness and its applications*, Bull. Austral. Math. Soc. **66** (2002), 405–418.

[16] W. Kaczor and J. Walczuk, *Demiclosedness principle and approximate fixed point sequences for asymptotically nonexpansive in the intermediate sense mappings*, in Fixed Point Theory and its Applications, Yokohama Publ., Yokohama, 2006, pp. 109–123.

[17] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory, Pure and Applied Mathematics (New York)*, John Willey & Sons, Inc., New York, 2001.

[18] M. A. Khamsi and B. Sims, *Ultra-methods in metric fixed point theory*, in Handbook of Metric Fixed Point Theory (W. A. Kirk and B. Sims, eds.), Kluwer Academic Publishers, Dordrecht, 2001, pp. 177-199.

[19] W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math. **17** (1974), 339–346.

[20] W. A. Kirk, C. M. Yañez and S. S. Shin, *Asymptotically nonexpansive mappings*, Nonlinear Anal. **33** (1998), 1-12.

[21] P.-K. Lin, *The Browder-Göhde property in product spaces*, Houston J. Math. **13** (1987), 235–240.

[22] B. Nanjaras and B. Panyanak, *Demiclosed principle for asymptotically nonexpansive mappings in CAT(0) spaces*, Fixed Point Theory Appl. 2010, Art. ID 268780, 14 pp.

[23] S. Saeidi, *Ergodic retractions for amenable semigroups in Banach spaces with normal structure*, Nonlinear Anal. **71** (2009), 2558–2563.

[24] S. Saeidi, *Ergodic retractions for families of asymptotically nonexpansive mappings*, Fixed Point Theory Appl. 2010, Art. ID 281362, 7 pp.

[25] B. Sims, *"Ultra"-techniques in Banach Space Theory*, Queen's Papers in Pure and Applied Math. 160, Queen's University, Kingston, Ontario, 1982.

[26] J. Stern, *Propriétés locales et ultrapuissances d'espaces de Banach*, Séminaire Maurey-Schwartz 1974-1975, Exp. Nos. VI et VII, École Polytech., Paris, 1975.

[27] H. K. Xu, *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Anal. **16** (1991), 1139–1146.

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W. Kaczor
Instytut Matematyki UMCS, 20-031 Lublin, Poland
E-mail address: wieslawa.kaczor@poczta.umcs.lublin.pl

J. Walczuk
Instytut Matematyki PWSZ, 22-400 Zamość, Poland
E-mail address: jwalczuk@op.pl