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REMETRIZATION THEOREMS FOR INFINITE FAMILIES OF NONLINEAR MAPPINGS, AND GENERALIZED JOINT SPECTRAL RADIUS

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Dedicated to Simeon Reich on the occasion of his jubilee

Abstract. Let $\mathcal{T}$ be a nonempty family of selfmaps of a metric space (X, d). Recently, Barnsley and Vince showed that if $\mathcal{T}$ is a compact (in the compact-open topology) family of affine selfmaps of the Euclidean space $(\mathbb{R}^m, d_e)$, then the following two statements are equivalent: there exists a metric ρ Lipschitz equivalent to d_e such that each mapping from $\mathcal{T}$ is a contraction; the joint spectral radius of the family $\mathcal{T}$ is less than one. In this paper we generalize the notion of the Rota–Strang joint spectral radius by attributing it to any uniformly lipschitzian family of nonlinear selfmaps of an arbitrary metric space, and we give an extension of the Barnsley–Vince theorem. We also establish four remetrization theorems for families of non-lipschitzian mappings, and present their applications in the theory of iterated function systems. Finally, as a consequence of the remetrization theorem, we obtain an extension of the Rota–Strang and Goebel formulae for the joint spectral radius.

1. Introduction

Let (X, d) be a complete metric space and $T_1, \ldots, T_N$ be continuous selfmaps of X. Then the system $((X, d); T_1, \ldots, T_N)$ is said to be an *iterated function system* (IFS). The following question is important in the theory of IFSs: when does there exist a metric ρ equivalent to d such that (X, ρ) is complete and each T_i, $i = 1, \ldots, N$, is a contraction with respect to ρ? In such a case, by Hutchinson's [11] theorem, there is a unique nonempty compact set $K \subseteq X$ such that $K = \bigcup_{i=1}^N T_i(K)$. The set K is called a *fractal in the sense of Hutchinson and Barnsley* [2] generated by the IFS $((X, d); T_1, \ldots, T_N)$. Recently, a partial answer to the above question was given by Barnsley et al. [1,3]. Recall that two metrics d and ρ on X are *Lipschitz equivalent* (see, e.g., [1]) if there exist positive constants α and β such that

$$\alpha d(x, y) \le \rho(x, y) \le \beta d(x, y) \quad \text{for any } x, y \in X. \tag{1.1}$$

Following [1] we say that an IFS $((X, d); T_1, \ldots, T_N)$ is *hyperbolic* if there exists a metric ρ Lipschitz equivalent to d such that each T_i is a contraction with respect to ρ. An IFS $((\mathbb{R}^m, d_e); T_1, \ldots, T_N)$ (d_e denotes the Euclidean metric) is called *affine* if each T_i is affine.

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To present the results of Barnsley et al. [1, 3] we also need the notion of the joint spectral radius introduced by Rota and Strang [14]. Let X be a Banach space and $\mathcal{T}$ be a nonempty bounded subset of $\mathcal{B}(X)$, the Banach space of all linear bounded operators. Then the *joint spectral radius* $r(\mathcal{T})$ of the family $\mathcal{T}$ is defined by

$$r(\mathcal{T}) := \lim_{n\to\infty} \sup\{\| T_1 \circ \cdots \circ T_n \|^{1/n}\colon T_1, \ldots, T_n \in \mathcal{T}\}.$$

This notion can also be attributed to a family of continuous affine mappings by considering the joint spectral radius of the set of linear factors of these mappings as done in [1] and [3].

In [1] the authors established a list of equivalent conditions for an affine IFS $((\mathbb{R}^m, d_e); T_1, \ldots, T_N)$ to be hyperbolic. In particular, they announced the following result: such an IFS is hyperbolic if and only if the joint spectral radius of the family $\{T_1, \ldots, T_N\}$ is less than one. Subsequently, this result was proved and extended in [3] to infinite affine IFSs $((\mathbb{R}^m, d_e); \mathcal{T})$, where $\mathcal{T}$ is assumed to be compact in the compact-open topology. In particular, this implies that the set of Lipschitz constants of all mappings from $\mathcal{T}$ is bounded (see Proposition 2.1).

In this paper we generalize the results of Barnsley et al. by considering *lipschitzian* IFSs $((X, d); \mathcal{T})$: here (X, d) is an arbitrary metric space, $\mathcal{T}$ is allowed to be infinite and all mappings from $\mathcal{T}$ are lipschitzian. Actually, we also use yet more general assumption on $\mathcal{T}$; namely, $\mathcal{T}$ is such that for some positive integer p, the set of Lipschitz constants of all compositions of p mappings from $\mathcal{T}$ is bounded. For such families $\mathcal{T}$, we establish four remetrization theorems giving necessary and sufficient conditions for the existence of an equivalent (uniformly equivalent or Lipschitz equivalent) metric ρ such that all mappings from $\mathcal{T}$ are lipschitzian with respect to ρ. We emphasize that the proofs given in [3] depend strongly on the linear structure of the Euclidean space $(\mathbb{R}^m, d_e)$. Our argument is completely different since our proofs deal with a nonlinear case.

Finally, we extend the notion of the joint spectral radius $r(\mathcal{T})$: it is possible to define it for any family $\mathcal{T}$ of selfmaps, which is uniformly lipschitzian, i.e., the set of Lipschitz constants of all these mappings is bounded. Then the inequality $r(\mathcal{T}) < 1$ is equivalent to the condition that for some $p \in \mathbb{N}$, the set of Lipschitz constants of all compositions of any p mappings from $\mathcal{T}$ is upper bounded by some constant less than one. This fact shows that our remetrization theorems given in the next section do extend results of Barnsley et al. [1, 3]. At last we obtain a generalization of the Rota–Strang formula for the joint spectral radius of a bounded family of linear continuous operators. This also extends Goebel's [7] (or see [8, p. 11]) formula for a generalized spectral radius of a lipschitzian selfmap of a metric space. Let us point out that though Goebel's result concerns a nonlinear mapping and his argument is different than that of Rota and Strang, it seems to be unclear how to modify his proof so that it would work for a family of mappings. Thus we had to use a different approach than both in [14] and [7].

2. Remetrization theorems for families of mappings

Given a selfmap T of a metric space (X, d), we denote by $L_d(T)$ the Lipschitz constant of T, i.e.,

$$L_d(T) := \sup\{d(Tx, Ty)/d(x, y) : x, y \in X,\ x \neq y\}.$$

(We allow $L_d(T)$ to be infinite.) If $\mathcal{T}$ is a nonempty family of selfmaps of X, then we set

$$L_d(\mathcal{T}) := \sup\{L_d(T) : T \in \mathcal{T}\}.$$

Moreover, for any $n \in \mathbb{N}$, we define

$$\mathcal{T}^n := \{T_1 \circ \cdots \circ T_n : T_1, \ldots, T_n \in \mathcal{T}\}.$$

Also, we set $\mathcal{T}^0 := \{\mathrm{Id}\}$, where Id is the identity mapping on X.

We start with a characterization of IFSs considered by Barnsley and Vince [3].

Proposition 2.1. *Let $((\mathbb{R}^m, d_e); \mathcal{T})$ be an affine IFS which is compact in the compact-open topology. Then there exists $T_0 \in \mathcal{T}$ such that $L_{d_e}(\mathcal{T}) = L_{d_e}(T_0)$. In particular, $L_{d_e}(\mathcal{T})$ is finite.*

Proof. It suffices to show that the mapping $T \mapsto L_{d_e}(T)$ is continuous from $\mathcal{T}$ with the compact-open topology to $(\mathbb{R}, d_e)$. Since in this case the compact-open topology is metrizable (see, e.g., [6, p. 332]), it is enough to prove that the above mapping is sequentially continuous. So let $T,\ T_n \in \mathcal{T}$ and $T_n \to T$. There exist $a_n, a \in \mathbb{R}^m$ and linear operators L_n and L such that $T_n = L_n + a_n$ and $T = L + a$. Since, in particular, $T_n 0 \to T0$, we get that $a_n \to a$. For any $x \in \mathbb{R}^m$, we have that $\| L_n x - Lx \| \le \| T_n x - Tx \| + \| a_n - a \|$. Hence, by compactness of the closed unit ball, we infer that $\| L_n - L \| \to 0$, so $\| L_n \| \to \| L \|$. To complete the proof it is enough to observe that $\| L_n \| = L_{d_e}(T_n)$ and $\| L \| = L_{d_e}(T)$. □

In the sequel we establish four remetrization theorems for a family $\mathcal{T}$ such that $L_d(\mathcal{T}^p) < \infty$ for some $p \in \mathbb{N}$. We start with a few lemmas which seem to be folklore results.

Lemma 2.2. *Let d and ρ be equivalent metrics on X such that for some $\alpha > 0$, $\rho(x, y) \ge \alpha d(x, y)$ for all $x, y \in X$. If (X, d) is complete, so is (X, ρ).*

Proof. Let (x_n) be a Cauchy sequence in (X, ρ). Since $d(x_n, x_m) \le (1/\alpha)\rho(x_n, x_m)$, we infer (x_n) is a Cauchy sequence in (X, d), so by completeness, $d(x_n, x) \to 0$ for some $x \in X$. Then $\rho(x_n, x) \to 0$ since the two metrics are equivalent. □

Let us recall that two metrics d and ρ on a set X are uniformly equivalent (see, e.g., [6, p. 321]) if the identity mapping on X is uniformly continuous from (X, d) onto (X, ρ), and from (X, ρ) onto (X, d).

Lemma 2.3. *Let d and ρ be equivalent metrics on X, and $\mathcal{T}$ be a nonempty family of selfmaps of X such that $L_\rho(\mathcal{T}) < \infty$. Then we have:*

1. *if there exists $\alpha > 0$ such that $\rho(x, y) \ge \alpha d(x, y)$ for all $x, y \in X$, then $\mathcal{T}$ is equicontinuous with respect to d;*
2. *if d and ρ are uniformly equivalent, then $\mathcal{T}$ is uniformly equicontinuous with respect to d;*

(3) *if d and ρ are Lipschitz equivalent with constants α, β as in (1.1), then*

$$L_d(\mathcal{T}) \le \frac{\beta}{\alpha} L_\rho(\mathcal{T}).$$

Proof. 1. Let $x_0 \in X$ and $T \in \mathcal{T}$. By hypothesis, for any $x \in X$,

$$d(Tx, Tx_0) \le \frac{1}{\alpha}\rho(Tx, Tx_0) \le \frac{L_\rho(\mathcal{T})}{\alpha}\rho(x, x_0). \tag{2.1}$$

If $L_\rho(\mathcal{T}) = 0$ then we are done. So let $L_\rho(\mathcal{T}) > 0$. Since d and ρ are equivalent, given $\varepsilon > 0$ there is $\delta > 0$ such that

$$\text{if} \quad d(x, x_0) < \delta, \quad \text{then} \quad \rho(x, x_0) < \frac{\alpha\varepsilon}{L_\rho(\mathcal{T})}.$$

Hence, by (2.1), we get that $d(Tx, Tx_0) < \varepsilon$ if $d(x, x_0) < \delta$. Since δ does not depend on T, we obtain that $\mathcal{T}$ is equicontinuous at x_0.

2. As in point 1, it suffices to consider the case when $L_\rho(\mathcal{T}) > 0$. By hypothesis, for any $\varepsilon > 0$, there is $\eta > 0$ such that if $u, v \in X$ and $\rho(u, v) < \eta$, then $d(u, v) < \varepsilon$. On the other hand, there is $\delta > 0$ such that if $x, y \in X$ and $d(x, y) < \delta$, then $\rho(x, y) < \eta / L_\rho(\mathcal{T})$. Hence, for such x, y, we get

$$\rho(Tx, Ty) \le L_\rho(\mathcal{T})\rho(x, y) < \eta$$

which yields $d(Tx, Ty) < \varepsilon$. So $\mathcal{T}$ is uniformly equicontinuous with respect to d.

3. Let $x, y \in X$ and $T \in \mathcal{T}$. By (1.1), we get

$$d(Tx, Ty) \le \frac{1}{\alpha}\rho(Tx, Ty) \le \frac{L_\rho(\mathcal{T})}{\alpha}\rho(x, y) \le \frac{\beta}{\alpha} L_\rho(\mathcal{T}) d(x, y),$$

and hence $L_d(\mathcal{T}) \le (\beta/\alpha) L_\rho(\mathcal{T})$. □

The following example shows that condition '$\rho \ge \alpha d$' in point 1 of Lemma 2.3 cannot be omitted. (Actually, it can be weakened by assuming that the identity mapping is uniformly continuous from (X, ρ) onto (X, d).)

Example 2.4. Set $X := [1, \infty)$, $d := d_e$, and for $x, y \in X$, $\rho(x, y) := |\ln x - \ln y|$. Clearly, ρ is a metric equivalent to d and (X, ρ) is complete. For $n \in \mathbb{N}$ and $x \in X$, define

$$T_n x := nx \quad \text{and} \quad \mathcal{T} := \{T_n : n \in \mathbb{N}\}.$$

Then T_n are isometries with respect to ρ, so $L_\rho(\mathcal{T}) = 1$, but $\mathcal{T}$ is not equicontinuous with respect to d at any point of X. Let us note that $\rho \le d$ by the mean value theorem.

We omit a simple proof of the following

Lemma 2.5. *Let $\mathcal{T}$ be a nonempty family of selfmaps of a metric space (X, d). If $\mathcal{T}$ is uniformly equicontinuous, so is $\mathcal{T}^n$ for any $n \in \mathbb{N}$.*

Theorem 2.6. *Let (X, d) be a [complete] metric space and $\mathcal{T}$ be a nonempty family of selfmaps of X such that for some $p \in \mathbb{N}$, $L_d(\mathcal{T}^p) < \infty$ and for any $x, y \in X$,*

$$\sup\left\{d(Tx, Ty) : T \in \bigcup_{k=0}^{p-1} \mathcal{T}^k\right\} < \infty. \tag{2.2}$$

The following statements are equivalent:

(i) *the family* $\bigcup_{k=0}^{p-1}\mathcal{T}^k$ *is equicontinuous;*

(ii) *for any* $\varepsilon > 0$*, there exists a [complete] metric* ρ *equivalent to* d *such that* $\rho \geq \alpha d$ *for some* $\alpha > 0$*, and*

$$L_\rho(\mathcal{T}) \leq \max\{\varepsilon, L_d(\mathcal{T}^p)^{1/p}\}.$$

Proof. (i)⇒(ii): For $\lambda > 0$ and $x, y \in X$, define

$$\begin{aligned}\rho_\lambda(x,y) \quad &:= \sup\{d(Tx,Ty) : T \in \mathcal{T}^{p-1}\} + \lambda \sup\{d(Tx,Ty) : T \in \mathcal{T}^{p-2}\} + \cdots \\ &\quad + \lambda^{p-2}\sup\{d(Tx,Ty) : T \in \mathcal{T}\} + \lambda^{p-1}d(x,y).\end{aligned} \tag{2.3}$$

By (2.2), ρ_λ is well-defined. It is easily seen that ρ_λ is a metric and condition (i) implies that ρ_λ and d are equivalent. Since $\rho_\lambda \geq \lambda^{p-1}d$, Lemma 2.2 implies that if (X, d) is complete, so is (X, ρ_λ). Observe that for any $S \in \mathcal{T}$,

$$\begin{aligned}\rho_\lambda(Sx,Sy) \quad &\leq \sup\{d(Tx,Ty) : T \in \mathcal{T}^{p}\} + \lambda \sup\{d(Tx,Ty) : T \in \mathcal{T}^{p-1}\} + \cdots \\ &\quad + \lambda^{p-2}\sup\{d(Tx,Ty) : T \in \mathcal{T}^2\} + \lambda^{p-1}\sup\{d(Tx,Ty) : T \in \mathcal{T}\} \\ &\leq L_d(\mathcal{T}^p)d(x,y) + \lambda\rho_\lambda(x,y) - \lambda^p d(x,y) \\ &= (L_d(\mathcal{T}^p) - \lambda^p)d(x,y) + \lambda\rho_\lambda(x,y),\end{aligned}$$

since $T \circ S \in \mathcal{T}^{p-i+1}$ for any $i = 1, \ldots, p$ and $T \in \mathcal{T}^{p-i}$. Now we consider the following two cases.

1. $L_d(\mathcal{T}^p) > 0$. Set $\lambda_0 := L_d(\mathcal{T}^p)^{1/p}$ and $\rho := \rho_{\lambda_0}$. Then $\rho(Sx, Sy) \leq \lambda_0\rho(x, y)$ which yields $L_\rho(\mathcal{T}) \leq L_d(\mathcal{T}^p)^{1/p}$.

2. $L_d(\mathcal{T}^p) = 0$. Then for any $\lambda > 0$ and $S \in \mathcal{T}$, $\rho_\lambda(Sx, Sy) \leq \lambda\rho_\lambda(x, y)$, so $L_{\rho_\lambda}(\mathcal{T}) \leq \lambda$.

Thus in both cases given $\varepsilon > 0$, there is a metric as in (ii).

(ii)⇒(i): Clearly, (ii) implies that $L_\rho(\mathcal{T}) < \infty$ and hence $L_\rho(\mathcal{T}^n) < \infty$ for any $n \in \mathbb{N}$ since $L_\rho(\mathcal{T}^n) \leq (L_\rho(\mathcal{T}))^n$. By Lemma 2.3 applied to $\mathcal{T}^n$, we get that each $\mathcal{T}^n$ is equicontinuous with respect to d. Hence the finite union $\bigcup_{k=0}^{p-1}\mathcal{T}^k$ is equicontinuous. □

Observe that, under the assumptions of Theorem 2.6, if a family $\mathcal{T}$ is finite, then (2.2) is satisfied. Moreover, in this case (i) is equivalent to the condition that each mapping from $\mathcal{T}$ is continuous since the family $\bigcup_{k=0}^{p-1}\mathcal{T}^k$ is finite. Thus Theorem 2.6 yields the following result which was proved in [13].

Corollary 2.7. *Let* (X, d) *be a [complete] metric space and* $\mathcal{T} := \{T_1, \ldots, \mathcal{T}_N\}$ *be a family of selfmaps of* X *such that for some* $p \in \mathbb{N}$, $L_d(\mathcal{T}^p) < \infty$*. The following statements are equivalent:*

(i) $T_1, \ldots, T_N$ *are continuous;*

(ii) *for any* $\varepsilon > 0$*, there exists a [complete] metric* ρ *equivalent to* d *such that* $\rho \geq \alpha d$ *for some* $\alpha > 0$*, and*

$$L_\rho(\mathcal{T}) \leq \max\{\varepsilon, L_d(\mathcal{T}^p)^{1/p}\}.$$

We give two examples to illustrate the assumptions of Theorem 2.6. First let us note that condition '$L_d(\mathcal{T}^p) < \infty$' implies that $\sup\{d(Tx, Ty) : T \in \mathcal{T}^p\} < \infty$ for any $x, y \in X$, but in general it does not imply (2.2) as shown in the following

Example 2.8. Set $X := [0, \infty)$, $d := d_e$ and for $n \in \mathbb{N}$,

$$T_n x := n - (n-1)x \quad \text{if} \quad x \in [0,1], \quad \text{and} \quad T_n x := 1 \quad \text{if} \quad x > 1.$$

Define $\mathcal{T} := \{T_n : n \in \mathbb{N}\}$. It is easily seen that for any $m, n \in \mathbb{N}$ and $x \in X$, $(T_m \circ T_n)x = 1$, so $L_d(\mathcal{T}^2) = 0$. On the other hand, if $x, y \in X$ and $x \neq y$, then $\sup\{d(Tx, Ty) : T \in \mathcal{T}\} < \infty$ if and only if $x, y \geq 1$, so (2.2) does not hold.

The next example shows that condition (2.2) cannot be simplified by assuming only that $\sup\{d(Tx, Ty) : T \in \mathcal{T}\} < \infty$ for any $x, y \in X$. Also, the same example illustrates that condition (i) in Theorem 2.6 is not equivalent to the condition '$\mathcal{T}$ is equicontinuous'.

Example 2.9. Set $X := \mathbb{R}$, $d := d_e$, and for $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$T_n x := x + n \quad \text{and} \quad T_0 x := x^2.$$

Let $\mathcal{T} := \{T_n : n \in \mathbb{N} \cup \{0\}\}$. Then for any $x, y \in \mathbb{R}$,

$$\sup\{|Tx - Ty| : T \in \mathcal{T}\} = \max\{|x - y|, |x^2 - y^2|\} < \infty,$$

but if $x \neq y$, then

$$\sup\{|Tx - Ty| : T \in \mathcal{T}^2\} \geq \sup\{|(T_0 \circ T_n)x - (T_0 \circ T_n)y| : n \in \mathbb{N}\} = \infty.$$

Moreover, it is easily seen that $\mathcal{T}$ is equicontinuous, but $\mathcal{T}^2$ is not equicontinuous. (Consider its subfamily $\{T_0 \circ T_n : n \in \mathbb{N}\}$.)

Our next remetrization theorem deals with a pair of uniformly equivalent metrics.

Theorem 2.10. *Let (X, d) be a metric space and $\mathcal{T}$ be a nonempty family of selfmaps of X such that for some $p \in \mathbb{N}$, $L_d(\mathcal{T}^p) < \infty$ and for any $x, y \in X$, (2.2) holds. The following statements are equivalent:*

(i) *$\mathcal{T}$ is uniformly equicontinuous;*
(ii) *for any $\varepsilon > 0$, there exists a metric ρ uniformly equivalent to d such that*

$$L_\rho(\mathcal{T}) \leq \max\{\varepsilon, L_d(\mathcal{T}^p)^{1/p}\}.$$

Proof. It suffices to modify the proof of Theorem 2.6.

(i)⇒(ii): For $\lambda > 0$, define the metric ρ_λ by (2.3). By Lemma 2.5, since $\mathcal{T}$ is uniformly equicontinuous, so is the family $\mathcal{T}^k$ for each $k = 0, \ldots, p-1$. This easily yields that d and ρ_λ are uniformly equivalent. Now it suffices to repeat the argument used in the proof of Theorem 2.6 ((i)⇒(ii)).

(ii)⇒(i): By (ii), $L_\rho(\mathcal{T}) < \infty$, so Lemma 2.3 (point 2) implies that (i) holds. □

It turns out that the formulation of Theorem 2.10 can be simplified if a metric space (X, d) is connected. In this case condition (2.2) can be dropped. To show that, we start with the following

Lemma 2.11. *Let $\mathcal{T}$ be a nonempty family of selfmaps of a connected metric space (X, d). If $\mathcal{T}$ is uniformly equicontinuous, then for any $x, y \in X$, $\sup\{d(Tx, Ty) : T \in \mathcal{T}\} < \infty$.*

Proof. By hypothesis, there exists $\delta > 0$ such that for any $x, y \in X$, if $d(x,y) < \delta$, then $d(Tx,Ty) < 1$ for any $T \in \mathcal{T}$. Fix $x_0, y_0 \in X$. Since (X,d) is connected, there exists a finite sequence $(x_i)_{i=0}^N$ such that $x_N = y_0$ and $d(x_{i-1}, x_i) < \delta$ for each $i \in \{1,\dots,N\}$ (see, e.g., [6, p. 442]). Hence we get that for any $T \in \mathcal{T}$,

$$d(Tx_0, Ty_0) \le \sum_{i=1}^{N} d(Tx_{i-1}, Tx_i) < N,$$

so $\sup\{d(Tx_0,Ty_0) : T \in \mathcal{T}\} < \infty$. □

Theorem 2.12. *Let (X,d) be a connected metric space and $\mathcal{T}$ be a nonempty family of selfmaps of X such that for some $p \in \mathbb{N}$, $L_d(\mathcal{T}^p) < \infty$. The following statements are equivalent:*

(i) *$\mathcal{T}$ is uniformly equicontinuous;*
(ii) *for any $\varepsilon > 0$, there exists a metric ρ uniformly equivalent to d such that*

$$L_\rho(\mathcal{T}) \le \max\{\varepsilon, L_d(\mathcal{T}^p)^{1/p}\}.$$

Proof. (i)⇒(ii): By Lemma 2.5, since $\mathcal{T}$ is uniformly equicontinuous, so is $\mathcal{T}^k$ for $k = 0,\dots,p-1$. Hence the family $\bigcup_{k=0}^{p-1}\mathcal{T}^k$ is uniformly equicontinuous, so Lemma 2.11 applied to this family implies that (2.2) holds. By Theorem 2.12, (ii) is satisfied.

Implication (ii)⇒(i) follows from Lemma 2.3 (point 2). □

Remark 2.13. Actually, the connectivity condition in Theorem 2.12 can be weakened: it suffices to assume that for any $\varepsilon > 0$, (X,d) is *ε-chainable* (see, e.g., [9, p. 19]), i.e., for any $x, y \in X$, there exists a finite sequence $(x_i)_{i=0}^N$ such that $x_0 = x$, $x_N = y$ and $d(x_{i-1}, x_i) < \varepsilon$. The latter condition, however, is equivalent to the connectivity if (X,d) is compact [6, p. 442].

We close this section with the remetrization theorem dealing with a pair of Lipschitz equivalent metrics.

Theorem 2.14. *Let (X,d) be a metric space and $\mathcal{T}$ be a nonempty family of selfmaps of X. The following statements are equivalent:*

(i) *$L_d(\mathcal{T}) < \infty$;*
(ii) *for any $p \in \mathbb{N}$ and $\varepsilon > 0$, there exists a metric ρ Lipschitz equivalent to d such that $L_\rho(\mathcal{T})$ is finite and*

$$L_\rho(\mathcal{T}) \le \max\{\varepsilon, L_d(\mathcal{T}^p)^{1/p}\}.$$

Proof. (i)⇒(ii): Let $p \in \mathbb{N}$. Since $L_d(\mathcal{T})$ is finite, so is $L_d(\mathcal{T}^k)$ for $k = 0,\dots,p-1$. Hence (2.2) holds for any $x, y \in X$, and by (2.3), we may define the metric ρ_λ for any $\lambda > 0$. It is easily seen that for any $x, y \in X$,

$$\lambda^{p-1} d(x,y) \le \rho_\lambda(x,y) \le (L_d(\mathcal{T}^{p-1}) + \lambda L_d(\mathcal{T}^{p-2}) + \cdots + \lambda^{p-2} L_d(\mathcal{T}) + \lambda^{p-1}) d(x,y),$$

so d and ρ_λ are Lipschitz equivalent. Now the same argument as in the proof of Theorem 2.6 ((i)⇒(ii)) shows that (ii) holds.

(ii)⇒(i): By (ii), $L_\rho(\mathcal{T}) < \infty$, so by Lemma 2.3 (point 3) we get that also $L_d(\mathcal{T})$ is finite. □

3. Generalized joint spectral radius and an extension of the Rota–Strang and Goebel formulas

Let (X,d) be a metric space and $\mathcal{T}$ be a nonempty family of selfmaps of X such that $L_d(\mathcal{T}) < \infty$. Set $a_n := L_d(\mathcal{T}^n)$ for $n \in \mathbb{N}$. Then each a_n is finite and it is easy to check that

$$a_{m+n} \le a_m a_n \quad \text{for any} \quad m,n \in \mathbb{N}.$$

We say then that (a_n) is *submultiplicative*. It is known that any submultiplicative sequence (a_n) of nonnegative reals has the property that the limit $\lim_{n\to\infty} a_n^{1/n}$ exists and

$$\lim_{n\to\infty} a_n^{1/n} \le a_m^{1/m} \quad \text{for any} \quad m \in \mathbb{N}.$$

(For a direct proof of this fact, see, e.g., [12]; the result can also be derived from [10, Lemma 4.7.1] applied to the sequence $(\log a_n)$.) In particular, the limit

$$r_d(\mathcal{T}) := \lim_{n\to\infty} L_d(\mathcal{T}^n)^{1/n}$$

exists and we call it the *generalized joint spectral radius* of $\mathcal{T}$ or just joint spectral radius of $\mathcal{T}$. Let us observe that if X is a Banach space and $\mathcal{T} \subseteq \mathcal{B}(X)$, then for any $T \in \mathcal{T}$, $L_d(T) = \| T \|$ (here d denotes the metric induced by the norm on X), so the inequality $L_d(\mathcal{T}) < \infty$ means that $\mathcal{T}$ is a bounded (with respect to the operator norm) subset of $\mathcal{B}(X)$. In this case $r_d(\mathcal{T})$ coincides with the Rota–Strang [14] joint spectral radius. On the other hand, if $\mathcal{T}$ is a singleton, say $\mathcal{T} = \{T\}$, then $r_d(\mathcal{T})$ is equal to the constant denoted in [8, p. 10] by $k_\infty(T)$. Goebel [7] (see also [8, p. 11]) proved the following formula for $k_\infty(T)$:

$$k_\infty(T) = \inf\{L_\rho(T) : \rho \text{ is Lipschitz equivalent to } d\}. \tag{3.1}$$

A similar formula was proved earlier by Rota and Strang [14] for the joint spectral radius $r(\mathcal{T})$ of a nonempty bounded family $\mathcal{T}$ of linear continuous operators on a Banach space X:

$$r(\mathcal{T}) = \inf\left\{ \sup_{T\in\mathcal{T}} \mathcal{N}(T) : \mathcal{N} \text{ is equivalent to the operator norm on } \mathcal{B}(X) \right\}. \tag{3.2}$$

In this section we give a common extension of both these results. We start with the following

Lemma 3.1. *Let (X,d) be a metric space and $\mathcal{T}$ be a nonempty family of selfmaps of X such that $L_d(\mathcal{T}) < \infty$. If a metric ρ is Lipschitz equivalent to d, then $r_d(\mathcal{T}) = r_\rho(\mathcal{T})$.*

Proof. By hypothesis, there exist $\alpha, \beta > 0$ as in (1.1). Clearly, $L_d(\mathcal{T}^n) < \infty$ for any $n \in \mathbb{N}$. By Lemma 2.3 (point 3) applied to $\mathcal{T}^n$, we infer that $L_d(\mathcal{T}^n) \le (\beta/\alpha)L_\rho(\mathcal{T}^n)$, and by interchanging the roles between d and ρ, $L_\rho(\mathcal{T}^n) \le (\beta/\alpha)L_d(\mathcal{T}^n)$. Hence we get that

$$\left(\frac{\alpha}{\beta}\right)^{1/n} L_d(\mathcal{T}^n)^{1/n} \le L_\rho(\mathcal{T}^n)^{1/n} \le \left(\frac{\beta}{\alpha}\right)^{1/n} L_d(\mathcal{T}^n)^{1/n},$$

so letting n tend to ∞ we obtain $r_d(\mathcal{T}) = r_\rho(\mathcal{T})$. □

Let us note that in the case when $\mathcal{T}$ is a singleton, Lemma 3.1 was proved by Goebel [7]. The case of a finite $\mathcal{T}$ was considered in [13].

Theorem 3.2. *Let (X, d) be a metric space and $\mathcal{T}$ be a nonempty family of selfmaps of X such that $L_d(\mathcal{T}) < \infty$. For $\lambda > 0$, let ρ_λ be defined by (2.3). Then*

$$r_d(\mathcal{T}) = \inf\{L_\rho(\mathcal{T}) : \rho \text{ is Lipschitz equivalent to } d\} = \inf_{\lambda>0} L_{\rho_\lambda}(\mathcal{T}).$$

Proof. Denote $a := \inf\{L_\rho(\mathcal{T}) : \rho \text{ is Lipschitz equivalent to } d\}$ and $b := \inf_{\lambda>0} L_{\rho_\lambda}(\mathcal{T})$. Let ρ be Lipschitz equivalent to d. By Lemma 3.1, $r_d(\mathcal{T}) = r_\rho(\mathcal{T})$. Hence, since $r_\rho(\mathcal{T}) \leq L_\rho(\mathcal{T}^n)^{1/n}$ for any $n \in \mathbb{N}$, we infer that $r_d(\mathcal{T}) \leq L_\rho(\mathcal{T})$. This shows that $r_d(\mathcal{T}) \leq a$. Since for any $\lambda > 0$, ρ_λ is Lipschitz equivalent to d, it is clear that $a \leq b$. Thus it suffices to prove that $b \leq r_d(\mathcal{T})$. We consider the following two cases.

1. $L_d(\mathcal{T}^p) = 0$ for some $p \in \mathbb{N}$. Then by Theorem 2.14, for any $\varepsilon > 0$, there exists a metric ρ Lipschitz equivalent to d such that $L_\rho(\mathcal{T}) \leq \varepsilon$. In fact, the proof of Theorem 2.6 shows that we may set $\rho := \rho_\varepsilon$. Hence $b = 0$, so $b \leq r_d(\mathcal{T})$.

2. $L_d(\mathcal{T}^n) > 0$ for all $n \in \mathbb{N}$. Then by Theorem 2.14, for any $n \in \mathbb{N}$, there exists a metric ρ Lipschitz equivalent to d such that $L_\rho(\mathcal{T}) \leq L_d(\mathcal{T}^n)^{1/n}$. Since ρ may be chosen from the family $\{\rho_\lambda : \lambda > 0\}$ as shown in the proof of Theorem 2.6, this implies that $b \leq L_d(\mathcal{T}^n)^{1/n}$. Hence, letting n tend to ∞ we get that $b \leq r_d(\mathcal{T})$. □

It is clear that Theorem 3.2 extends Goebel's formula (3.1). Now we show that it also extends the Rota–Strang formula (3.2). So let X be a Banach space and $\mathcal{T}$ be a bounded subset of $\mathcal{B}(X)$. Then for any $\lambda > 0$, ρ_λ (see (2.3)) is well-defined and it is easily seen that ρ_λ is induced by the norm $\| \cdot \|_\lambda$ on X defined by

$$\| x \|_\lambda := \sup\{\| Tx \| : T \in \mathcal{T}^{p-1}\} + \lambda \sup\{\| Tx \| : T \in \mathcal{T}^{p-2}\} + \cdots + \lambda^{p-1} \| x \| .$$

Every norm $\| \cdot \|_\lambda$ on X induces the norm $\| \cdot \|_\lambda$ on $\mathcal{B}(X)$ (we use the same notation for both norms), and $L_{\rho_\lambda}(T) = \| T \|_\lambda$ for $T \in \mathcal{T}$, so $L_{\rho_\lambda}(\mathcal{T}) = \sup_{T\in\mathcal{T}} \| T \|_\lambda$. Thus Theorem 3.2 yields that

$$r(\mathcal{T}) = \inf\left\{\sup_{T\in\mathcal{T}} \| T \|_\lambda : \lambda > 0\right\},$$

which implies that $r(\mathcal{T}) \leq c$, where c denotes the right side of (3.2). That $c \leq r(\mathcal{T})$ follows from the fact that if $\mathcal{N}$ is any norm equivalent to the operator norm on $\mathcal{B}(X)$, then by Lemma 3.1, $r(\mathcal{T}) = \lim_{n\to\infty}(\sup_{T\in\mathcal{T}^n} \mathcal{N}(T))^{1/n}$ and $r(\mathcal{T}) \leq (\sup_{T\in\mathcal{T}^n} \mathcal{N}(T))^{1/n}$ for any $n \in \mathbb{N}$ since the sequence $(\sup_{T\in\mathcal{T}^n} \mathcal{N}(T))_{n\in\mathbb{N}}$ is submultiplicative; in particular, $r(\mathcal{T}) \leq \sup_{T\in\mathcal{T}} \mathcal{N}(T)$.

Now we recall the result of Barnsley and Vince [3] mentioned in the introduction. (In fact, they established a list of five equivalent conditions.)

Theorem 3.3. *Let $((\mathbb{R}^m, d_e); \mathcal{T})$ be an affine IFS such that $\mathcal{T}$ is compact in the compact-open topology. The following statements are equivalent:*

- (i) *there exists a metric ρ Lipschitz equivalent to d_e such that each $T \in \mathcal{T}$ is a contraction with respect to ρ;*
- (ii) *$r(\mathcal{T}) < 1$.*

As an immediate consequence of Theorem 3.2, we obtain the following extension of Theorem 3.3.

Corollary 3.4. *Let (X, d) be a metric space and $\mathcal{T}$ be a nonempty family of selfmaps of X such that $L_d(\mathcal{T}) < \infty$. The following statements are equivalent:*

(i) *there exists a metric ρ Lipschitz equivalent to d such that $L_\rho(\mathcal{T}) < 1$;*
(ii) $r_d(\mathcal{T}) < 1$.

We show that indeed, Corollary 3.4 is a generalization of Theorem 3.3. So let $((\mathbb{R}^m, d_e); \mathcal{T})$ be as in Theorem 3.3. By Proposition 2.1, $L_{d_e}(\mathcal{T}) < \infty$. Thus implication (ii)⇒(i) of Theorem 3.3 follows directly from Corollary 3.4. We prove (i)⇒(ii) of Theorem 3.3. By Proposition 2.1, there exists $T_0 \in \mathcal{T}$ such that $L_\rho(\mathcal{T}) = L_\rho(T_0)$. Hence $L_\rho(\mathcal{T}) < 1$, so by Corollary 3.4, (ii) holds.

Now let us observe that since $r_d(\mathcal{T}) = \lim_{n\to\infty} L_d(\mathcal{T}^n)^{1/n}$ and $r_d(\mathcal{T}) \le L_d(\mathcal{T}^n)^{1/n}$ for any $n \in \mathbb{N}$, we have the following equivalence:

$$r_d(\mathcal{T}) < 1 \quad \text{if and only if} \quad L_d(\mathcal{T}^p) < 1 \quad \text{for some} \;\; p \in \mathbb{N}.$$

Thus if $\mathcal{T}$ is not a family of lipschitzian mappings, so that $r_d(\mathcal{T})$ cannot be defined, we may substitute condition '$L_d(\mathcal{T}^p) < 1$ for some $p \in \mathbb{N}$' for '$r_d(\mathcal{T}) < 1$'. For such families of mappings, we may obtain further extensions of Theorem 3.3 with the help of our remetrization theorems. To illustrate this fact, we present here only one of possible applications of the results of the previous section.

Corollary 3.5. *Let (X, d) be a connected metric space and $\mathcal{T}$ be a nonempty family of selfmaps such that for some $p \in \mathbb{N}$, $L_d(\mathcal{T}^p) < 1$. The following statements are equivalent:*

(i) *$\mathcal{T}$ is uniformly equicontinuous;*
(ii) *there exists a metric ρ uniformly equivalent to d such that $L_\rho(\mathcal{T}) < 1$.*

Proof. (i)⇒(ii): By Theorem 2.12, there exists a metric ρ uniformly equivalent to d such that

$$L_\rho(\mathcal{T}) \le \max\{1/2, L_d(\mathcal{T}^p)^{1/p}\} < 1,$$

so (ii) holds.

(ii)⇒(i) follows from Lemma 2.3 (point 2). □

Finally, we present yet another condition equivalent to the inequality $r(\mathcal{T}) < 1$ for affine IFSs. This extends the list of equivalent conditions in [3, Theorem 4].

Theorem 3.6. *Let $((\mathbb{R}^m, d_e); \mathcal{T})$ be an affine IFS such that $\mathcal{T}$ is compact in the compact-open topology. The following statements are equivalent:*

(i) $r(\mathcal{T}) < 1$;
(ii) *for any sequence (T_n) such that $T_n \in \mathcal{T}$ for $n \in \mathbb{N}$, $L_d(T_1 \circ \cdots \circ T_n) \to 0$.*

Proof. For any $T \in \mathcal{T}$, there is a unique $L \in \mathcal{B}(\mathbb{R}^m)$ and $a \in \mathbb{R}^m$ such that $Tx = Lx + a$ for any $x \in \mathbb{R}^m$. Thus we may define the mapping

$$F(T) := L \quad \text{for any} \;\; T \in \mathcal{T}.$$

The proof of Proposition 2.1 shows that F is continuous from $\mathcal{T}$ with the compact-open topology into $\mathcal{B}(\mathbb{R}^m)$. Thus the set $F(\mathcal{T})$ is compact with respect to the operator norm. By definition of $r(\mathcal{T})$, we have that $r(\mathcal{T}) = r(F(\mathcal{T}))$. Hence by [12, Theorem 5.1], (i) is equivalent to the condition

(iii) for any sequence (L_n) such that $L_n \in F(\mathcal{T})$ for $n \in \mathbb{N}$, $\lim_{n\to\infty} \| L_1 \circ \cdots \circ L_n \| = 0$.

Observe that for any sequence (T_n) elements of $\mathcal{T}$, if $L_n := F(T_n)$, then for any $n \in \mathbb{N}$, there is $b_n \in \mathbb{R}^m$ such that

$$(T_1 \circ \cdots \circ T_n)x = (L_1 \circ \cdots \circ L_n)x + b_n \quad \text{for all} \quad x \in \mathbb{R}^m,$$

which implies that $L_d(T_1 \circ \cdots \circ T_n) = \| L_1 \circ \cdots \circ L_n \|$. This easily yields the equivalence between (ii) and (iii), so the proof is completed. □

Theorem 3.6 is a generalization of [4, Theorem 4.1] (see also [5]) by Daubechies and Lagarias in which $\mathcal{T}$ is assumed to be a finite family of linear mappings.

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