Journal of Nonlinear and Convex Analysis
Volume 15, Number 2, 2014, 355–377

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

RAYS OF EQUIVALENT NORMS WITH THE FIXED POINT PROPERTY IN SOME NONREFLEXIVE BANACH SPACES

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Abstract. We consider a vector space X endowed with a family of seminorms $\rho = \{\rho_k(\cdot)\}_k$ which separates points and it is pointwise bounded. Define the norm $|x|_\rho := \sup_k \rho_k(x)$ for all $x \in X$. We study sufficient conditions over the family $\{\rho_k(\cdot)\}_k$ to obtain fixed point results for mappings which are nonexpansive for the $|\cdot|_\rho$-norm. We consider Banach spaces that are isomorphic to a one-direct sum of finite dimensional Banach spaces, as for instance ℓ_1, the Fourier-Stieltjes $B(G)$ for separable compact groups or $L_1(\mathcal{M})$ for $\mathcal{M}$ a finite atomic von Neumann algebra. We apply the above technique to obtain linear rays of equivalent norms with the fixed point property, which proves that certain linear structure can be found in the subset of all equivalent norms that verify the fixed point property. Some closed subspaces of $L_1[0,1]$ will also be considered. In case of function spaces $L_1(\mu)$ or noncommutative $L_1(\mathcal{M})$ spaces, where $\mathcal{M}$ is a finite von Neumann algebra, our technique provides rays of equivalent norms with the fixed point property for affine nonexpansive mappings. In fact, all the fixed point results obtained in this manuscript are extended to a more general class of mappings: the (L)-type mappings for which the nonexpansive mappings are a particular subclass.

1. Introduction

Metric Fixed Point Theory studies the existence of fixed points under conditions which depend on the metric and that are not invariant if we change the metric by an equivalent one. The most common aim of Renorming Theory is to find an equivalent norm which satisfies (or which does not satisfy) certain specific properties. Metric Fixed Point Theory and Renorming Theory have been usually studied as independent topics. However, in the last years, some interesting results have appeared which show the interrelation between both theories. In this manuscript we continue this field of research studying new connections between Metric Fixed Point Theory and Renorming Theory in some nonreflexive Banach spaces and for nonexpansive mappings.

Recall that a mapping T defined from a metric space M into M is said to be nonexpansive if $d(Tx,Ty) \leq d(x,y)$ for every $x, y \in M$. It is well known that the Contractive Mapping Principle fails for nonexpansive mappings. Actually, the first fixed point results for nonexpansive mappings appeared in 1965 in the setting of Banach spaces and they are strongly connected with the geometry of the space. Recall that a Banach space X is said to have the fixed point property (FPP) if

2010 *Mathematics Subject Classification.* 46B03, 47H09, 47H10.

Key words and phrases. Fixed point theory, nonexpansive mappings, affine mappings, condition (L), renorming theory.

The first author is partially supported by CONACyT (Mexico).

The second author is partially supported by MCIN, Grant MTM-2012-34847-C02-01 and Junta de Andalucía, Grant FQM-127 and P08-FQM-03543.

every nonexpansive mapping $T : C \to C$, where C is a closed convex bounded subset of X, has a fixed point. In this sense, every uniformly convex space or more generally every reflexive Banach space with normal structure verify the FPP [20]. Two important monographs which collect the advances in the development of the Fixed Point Theory for nonexpansive mappings are [13] and [21].

Notice that the nonexpansiveness of a mapping depends strongly on the norm which is considered in the Banach space. If we replace the original norm by an equivalent one, the set of nonexpansive mappings may change. In fact, the FPP is not preserved by isomorphisms which implies that the FPP is a condition that depends on the given norm in the Banach space.

For a long time it was unknown whether the fixed point property could imply reflexivity. This question was solved in 2008 by P.K. Lin [24] in a negative way: he proved that there exists an equivalent norm in ℓ_1 with the FPP. His result showed the first known nonreflexive Banach space with the FPP and opened new fields of research in Fixed Point Theory for nonexpansive mappings. We say that a Banach space X is FPP-renormable if there exists some equivalent norm p on X such that (X, p) satisfies the FPP. Although it is a long open question whether every reflexive Banach space satisfies the FPP, T. Domínguez-Benavides [5] proved that all reflexive Banach spaces are FPP-renormable, even in a dense way, that is, in [8] the authors proved that the subset of all renormings in a reflexive Banach space with the FPP is dense in the set of all equivalent norms on X. The case of nonreflexive Banach spaces is quite different. Firstly, there are nonreflexive (and nonseparable) Banach spaces which are not FPP-renormable: ℓ_∞, $\ell_1(\Gamma)$ and $c_0(\Gamma)$ (for Γ uncountable) cannot be renormed to have the FPP [21](Chapter 9). In the case of nonreflexive and separable Banach spaces, it is known that ℓ_1 fails the FPP but it is still FPP-renormable [24]. Whether the sequence space c_0 is FPP-renormable is unknown. Some other nonreflexive Banach spaces which are FPP-renormable can be found in [15]. However, the structure of the subset of all equivalent norms with the FPP in ℓ_1 is not well-understood. In this manuscript we obtain new equivalent norms in some nonreflexive Banach spaces verifying the FPP and we prove that the subset of such norms has certain linear structure, in the sense that it contains linear rays. These are new results in Fixed Point Theory since it is unknown whether the sum of two norms with the FPP keeps the FPP. In Section 4 we particularize our results for the sequence space ℓ_1 and for those Banach spaces that can be isomorphically written as a one-direct sum of finite dimensional Banach spaces, such as the Fourier-Stieltjes algebra $B(G)$ of a separable compact group G or $L_1(\mathcal{M})$ for a finite atomic von Neumann algebra on a separable Hilbert space. Some closed subspaces of $L_1[0, 1]$ such as the Bergman space will also be considered.

In Section 5 we use the techniques introduced in Section 3 in order to obtain some linear rays of equivalent norms in $L_1[0, 1]$ and, more generally in noncommutative $L_1(\mathcal{M})$ spaces, which verify the fixed point property for affine nonexpansive mappings.

All the renormings included in the papers [24], [25], [15], [16], [17] and [18] with the FPP can be derived from Theorem 3.1 and Theorem 3.5 in Section 3. We will prove that the applications achieved by using the techniques introduced in this manuscript go beyond those obtained in the previous articles.

More precisely, the fixed point results proved in this paper are given for (L)-type mappings, which form a class of mappings that includes the nonexpansive mappings as a particular subclass. This shows that all the equivalent norms given in [24], [25], [15], [16], [17] and [18] verify the fixed point property for mappings with the (L)-condition. The class of (L)-type mappings was defined in [28] and they contain strictly the class of nonexpansive mappings, mappings satisfying condition (C) of Suzuki [34], most of generalized nonexpansive mappings and certain mappings with the (E) condition introduced in [12].

This manuscript is organized as follows:

In Section 2 we give the notation and the definitions that will be used throughout the paper. Section 3 includes Theorem 3.1, which provides a fixed point result for mappings that verify the (L)-condition for norms defined as the supremum of a family of seminorms satisfying certain properties. Also Theorem 3.5 is proved in this section, which is the technical key to let us obtain open rays of equivalent norms with the fixed point property. In Section 4 we apply these results to produce new equivalent norms with the fixed point property in certain nonreflexive Banach spaces. As far as we know, we collect all known equivalent norms with the FPP in ℓ_1 as a consequence of Theorem 3.1. We add new examples and the existence of open rays of equivalent norms with the FPP in ℓ_1 and in some other nonreflexive Banach spaces, which are not isomorphic to ℓ_1. It will be remarkable the existence of many norms failing to have the FPP but that they can be considered as the initial points of different open rays composed of equivalent norms verifying the FPP. Finally, in Section 5 we will study how apply our results to $L_1(\mu)$ function spaces and to non-commutative L_1-spaces. In these spaces, we will obtain equivalent norms with the fixed point property for affine nonexpansive mappings and the existence of some open rays of equivalent norms with such property. We finish the paper with a short section of comments and some open problems connecting Renorming Theory with Fixed Point Theory for nonexpansive mappings

2. Preliminaries

We start this section by introducing some notation which will be used throughout the manuscript.

For a Banach space X we define by $\mathcal{P}(X)$ the set of all equivalent norms on X. The set $\mathcal{P}(X)$ can be endowed with the structure of a metric space by the distance [10]:

$$d(p,q) = \sup\{|p(x) - q(x)| : \|x\| \leq 1\}, \quad \text{if } p,q \in \mathcal{P}(X).$$

Moreover, $\mathcal{P}(X)$ is a convex cone, in the sense that $\lambda p_1 + \mu p_2 \in \mathcal{P}(X)$ if $p_1, p_2 \in \mathcal{P}(X)$ and $\lambda, \mu \geq 0$ with $\max\{\lambda, \mu\} > 0$. Given $p_1, p_2 \in \mathcal{P}(X)$ we say that the subset $\{p_1 + \lambda p_2 : \lambda > 0\}$ is a (linear) open ray contained in $\mathcal{P}(X)$ and that p_1 is its initial point.

By using Banach's Contraction Principle, it can be proved that every nonexpansive mapping has an approximate fixed point sequence $\{x_n\} \subset C$ whenever C is a convex closed bounded subset of X and $T : C \to C$. Recall that a sequence $\{x_n\}$ is called an approximate fixed point sequence (a.f.p.s.) for a mapping T if

$$\lim_n \|x_n - Tx_n\| = 0.$$

Notice that every subsequence of an a.f.p.s. is again an a.f.p.s.

In this manuscript we will obtain fixed point results for a family of mappings which contain the nonexpansive mappings as a particular subclass. Next we introduce the (L)-type mappings:

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space and C a subset of X. It is said that a mapping $T : C \to C$ satisfies condition (L), (or it is an (L)-type mapping), if it fulfills the following two properties:

(1) Every closed, convex, bounded, T-invariant and nonempty subset D of C contains an a.f.p.s.
(2) For any a.f.p.s. $\{x_n\}$ in C and each x belonging to C
$$\limsup_n \|x_n - Tx\| \leq \limsup_n \|x_n - x\|.$$

The definition of (L)-type mappings was introduced by E. Llorens-Fuster and E. Moreno-Gálvez in [28]. They prove that different classes of mappings which often appear in metric fixed point theory are (L)-type mappings. It is worth noticing that, whereas condition (1) is invariant by renorming, condition (2) strongly depends on the given norm and it can fail if we replace the norm by an equivalent one.

Nonexpansive mappings are the classical examples of (L)-type mappings. However, we can find in the literature different classes of mappings which satisfy condition (L) and are not nonexpansive in general.

The following definition was introduced by T. Suzuki in [34]:

Definition 2.2. Let C be a nonempty subset of a Banach space X. It is said that $T : C \to X$ satisfy condition (C) if
$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \quad \text{implies} \quad \|Tx - Ty\| \leq \|x - y\|,$$
for all $x, y \in C$.

Every nonexpansive mapping trivially satisfies condition (C). However, several examples of noncontinuous mappings satisfying condition (C) are given in [34].

Definition 2.3. A mapping $T : C \to X$ is called a generalized nonexpansive mapping if there exist $a, b, c \geq 0$ which satisfy $a + 2b + 2c \leq 1$ such that
$$\|Tx - Ty\| \leq a\|x - y\| + b(\|x - Tx\| + \|y - Ty\|) + c(\|x - Ty\| + \|y - Tx\|),$$
for all $x, y \in C$.

The following class of mappings was introduced in [12]:

Definition 2.4. For $\mu \geq 1$, a mapping $T : C \to X$ is said to satisfy condition (E_μ) on C if
$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|$$
for all $x, y \in C$. It is said that T satisfies condition (E) on C if T satisfies condition (E_μ) on C for some $\mu \geq 1$.

It can be proved that the class of (L)-type mappings contains strictly the class of nonexpansive mappings, mappings satisfying condition (C) of Suzuki, generalized nonexpansive mappings (if $b \neq 1/2$) and mappings satisfying condition (E) which in turn satisfy condition (1) in the definition of (L)-type mappings [28].

We say that a Banach space X has the fixed point property for (L)-type mappings (FPP for (L)-type mappings) if every self-mapping satisfying condition (L) has a fixed point when it is defined from a closed convex bounded subset of X. Obviously, FPP for (L)-type mappings implies the FPP.

The following remark, which easily follows from the definition, will be essential throughout the proof of the main theorem in the next section:

Remark 2.5. Let C be a closed convex bounded subset of a Banach space and $T : C \to C$ an (L)-type mapping. Let $\{x_n\}$ be an a.f.p.s. in C and $d > 0$. If the set

$$D = \left\{ x \in C : \limsup_n \|x_n - x\| \leq d \right\}$$

is nonempty, then D is a convex closed T-invariant subset of C.

For $x \in X$ and a bounded sequence $\{x_n\} \subset X$, the asymptotic radius of $\{x_n\}$ at x is defined as

$$r(\{x_n\}, x) := \limsup_n \|x_n - x\|.$$

For a subset $C \subset X$, the value

$$r(\{x_n\}, C) := \inf\{r(\{x_n\}, x) : \ x \in C\}$$

is called the asymptotic radius of $\{x_n\}$ relatively to C.

The following lemma holds for all Banach spaces and all (L)-type mappings. It can be proved in a similar way as the remark after Lemma 1 in [15]:

Lemma 2.6. *Let C be a nonempty closed convex bounded subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ an (L)-type mapping. If T is fixed point free, there exist some constant $a > 0$ and a convex closed T-invariant nonempty subset $D \subset C$ such that*

$$r(\{x_n\}, D) \geq a$$

for each a.f.p.s. $\{x_n\}$ in D.

We make the following observation:

Remark 2.7. Let D be the subset given in Lemma 2.6. If we assign to every a.f.p.s. $\{x_n\}$ in D, satisfying certain property P, a vector $\hat{x} \in X$ depending on the sequence, there holds

$$\inf \{r(\{x_n\}, \hat{x}) : \quad \{x_n\} \text{ is an a.f.p.s. in } D \text{ satisfying } P \ \} > 0.$$

Indeed, from Lemma 2.6, we have

$$a \leq \limsup_n \limsup_m \|x_n - x_m\| \leq 2 \limsup_n \|x_n - \hat{x}\|.$$

Hence $r(\{x_n\}, \hat{x}) \geq a/2$ for all $\{x_n\}$ an a.f.p.s. satisfying P in the subset D .

3. Technical results

This section contains the technical part of this manuscript. It consists mainly of two theorems. Theorem 3.1 is a general fixed point result for (L)-type mappings defined for a certain norm given by a family of seminorms. We will introduce the background setting and conditions (I), (II), (III), which have to be verified by the family of seminorms in order to assure the fixed point results. The arguments used in the proof are inspired in [24] and [15]. However, we will discover that the scope of its applications goes beyond the results obtained in [15], [16], [17], [18] and [24].

Given a family of seminorms with conditions (I), (II) and (III), in Theorem 3.5 we will prove how to obtain new families of seminorms in the same conditions. This will be the key point for obtaining rays of equivalent norms verifying the fixed point property.

Since the results are given for (L)-type mappings, they can be particularized for nonexpansive mappings, mappings with the condition (C) of Suzuki and for certain generalized nonexpansive mappings and mappings with condition (E) defined previously.

Our general framework will be the following:

Let X be a vector space. Assume that X can be endowed with a topology $\mathcal{T}$ for which the convergence is invariant by translations, in the sense that $x_n \xrightarrow{\mathcal{T}} x$ if and only if $x_n - x \xrightarrow{\mathcal{T}} 0$.

Assume that there is a family $\rho = \{\rho_k(\cdot) : k \in \mathbb{N}\}$ of seminorms on X which separates points and it is pointwise bounded. In this case the function

$$|x|_\rho := \sup_k \rho_k(x)$$

defines a norm on X. Assume that $(X, |\cdot|_\rho)$ is complete and that the family of seminorms satisfy the following properties:

(I) There exists a positive sequence (δ_k) with $\lim_k \delta_k = 1$ such that for all $k \in \mathbb{N}$ there holds

$$\limsup_n \rho_k(x_n) + \rho_k(x) \leq \delta_k \limsup_n \rho_k(x_n + x)$$

whenever $x \in X$ and $\{x_n\}$ is a ρ_k-bounded sequence with $x_n \xrightarrow{\mathcal{T}} 0$.

(II) There exist two sequences (α_k) and (β_k) satisfying $0 \leq \alpha_k \leq \beta_k < 1$ with $\lim_k \alpha_k = 1$ such that for all $k \in \mathbb{N}$

$$\alpha_k \limsup_n |x_n|_\rho \leq \limsup_n \rho_k(x_n) \leq \beta_k \limsup_n |x_n|_\rho,$$

whenever $\{x_n\}$ is a ρ_k-bounded sequence with $x_n \xrightarrow{\mathcal{T}} 0$.

(III) There exists some $\alpha > 1$ such that for every $\mathcal{T}$-null sequence $\{x_n\}$ there holds

$$\limsup_k \rho_k(x_0) \leq \frac{\limsup_n |x_n|_\rho}{\alpha}$$

for some $x_0 \in \overline{co}^{|\cdot|_\rho}(\{x_n\})$.

Theorem 3.1. *Under the above conditions the following holds: Let C be a convex, $|\cdot|_\rho$-closed, $|\cdot|_\rho$-bounded subset of X and $T: C \to C$ a mapping which satisfies condition (L). Then T has a fixed point whenever T has some $\mathcal{T}$-convergent approximate fixed point sequence in every $|\cdot|_\rho$-closed convex T-invariant subset of C.*

Notice that condition (L) lets us assure the existence of approximate fixed point sequences in every $|\cdot|_\rho$-closed convex T invariant subset of C. In the absence of compactness hypotheses, it can not be deduced one of these sequences to be $\mathcal{T}$-convergent.

Before going to the proof we need the following previous lemmas:

Lemma 3.2. *Under the above conditions, the following inequality holds: if $\{x_n\}$ and $\{y_n\}$ are $|\cdot|_\rho$-bounded sequences in X such that $x_n \xrightarrow{\mathcal{T}} x$ and $y_n \xrightarrow{\mathcal{T}} y$ then*

$$\limsup_m \limsup_n |x_n - y_m|_\rho \geq \limsup_n |x_n - x|_\rho + \limsup_m |y_m - y|_\rho.$$

Proof.

$$\begin{aligned}
\limsup_m \limsup_n |x_n - y_m|_\rho &\geq \limsup_m \limsup_n \rho_k(x_n - y_m) \\
&\geq \frac{1}{\delta_k} \limsup_m \left[\limsup_n \rho_k(x_n - x) + \rho_k(x - y_m) \right] \qquad \text{(by (I))} \\
&= \frac{1}{\delta_k} \left[\limsup_n \rho_k(x_n - x) + \limsup_m \rho_k(x - y_m) \right] \\
&\geq \frac{1}{\delta_k} \left[\limsup_n \rho_k(x_n - x) + \frac{1}{\delta_k} \left(\limsup_m \rho_k(y - y_m) + \rho_k(x - y) \right) \right] \quad \text{(by (I))} \\
&\geq \frac{1}{\delta_k} \limsup_n \rho_k(x_n - x) + \frac{1}{\delta_k^2} \limsup_m \rho_k(y_m - y) \\
&\geq \frac{\alpha_k}{\delta_k} \limsup_n |x_n - x|_\rho + \frac{\alpha_k}{\delta_k^2} \limsup_m |y_m - y|_\rho \qquad \text{(by (II))}
\end{aligned}$$

Taking limit when k goes to infinity we have

$$\limsup_m \limsup_n |x_n - y_m|_\rho \geq \limsup_n |x_n - x|_\rho + \limsup_m |y_m - y|_\rho.$$

□

Lemma 3.3. *Under the above conditions, let C be a convex, $|\cdot|_\rho$-closed, $|\cdot|_\rho$-bounded subset of X and $T: C \to C$ an (L)-type mapping. If T is fixed point free, let D be as in Lemma 2.6. If K is any closed convex T-invariant subset of D and*

$$r = \inf \left\{ \limsup_n |x_n - x|_\rho : \{x_n\} \subset K \textit{ is an a.f.p.s. and } x_n \xrightarrow{\mathcal{T}} x \right\},$$

then $r > 0$ and for every a.f.p.s. $\{x_n\} \subset K$ which is $\mathcal{T}$-convergent and for every $z \in K$ we have

$$\limsup_n |x_n - z|_\rho \geq 2r.$$

Proof. For a sequence $\{x_n\}$ we define the property P as being $\mathcal{T}$-convergent and consider $\hat{x}$ as its $\mathcal{T}$-limit (see Remark 2.7). This implies that $r > 0$. Now using

Lemma 3.2 and the same arguments as in the proof of Lemma 3 in [15] we obtain the result. □

Now we prove Theorem 3.1.

Proof. Assume the contrary, that is, there exist a convex bounded subset C of $(X, |\cdot|_\rho)$ and an (L)-type mapping $T : C \to C$ without fixed points.

We introduce the following notation: for a $|\cdot|_\rho$-closed convex T-invariant subset K of C we define

$$A(K) = \{\{x_n\} \subset K : \{x_n\} \text{ is an a.f.p.s } \mathcal{T}\text{-convergent to some } x \in X\}.$$

Notice that $A(K)$ always contains an approximate fixed point sequence by hypotheses.

Let D be as in the conclusion of Lemma 2.6.

Take

$$c = \inf\left\{\limsup_n |x_n - x|_\rho : \{x_n\} \in A(D),\ x_n \xrightarrow{\mathcal{T}} x\right\},$$

which is strictly greater than cero from Lemma 3.3.

We choose some constants m, $\varepsilon_1 > 0$ such that

$$2\varepsilon_1 + \frac{c+\varepsilon_1}{\alpha} < m < c. \tag{3.1}$$

Take an a.f.p.s. $\{x_n\} \subset D$ such that $x_n \xrightarrow{\mathcal{T}} x$ and $\limsup |x_n - x|_\rho < c + \varepsilon_1$. Now, by translation we can suppose that $x = 0$.

Define

$$K = \left\{z \in D : \limsup_n |x_n - z|_\rho \le 2c + 2\varepsilon_1\right\}.$$

Notice that K is nonempty, closed, convex, bounded and T-invariant. In fact, there exists $n_0 \in \mathbb{N}$ such that $x_n \in K$ for $n \ge n_0$.

Define

$$d_k := \delta_k^2(2c + 2\epsilon_1) - \alpha_k(\delta_k + 1)c \tag{3.2}$$

Notice that (d_k) is a bounded sequence and $\lim_k d_k = 2\epsilon_1$.

Let us prove that for all $\{y_n\} \in A(K)$ with $y_n \xrightarrow{\mathcal{T}} y$ there holds

$$\rho_k(y) \le d_k \tag{3.3}$$

for all $k \in \mathbb{N}$. Indeed, in this case

$$\begin{aligned}
2c + 2\varepsilon_1 &\ge \limsup_m \limsup_n |x_n - y_m|_\rho \ge \limsup_m \limsup_n \rho_k(x_n - y_m) \\
&\ge \limsup_m \delta_k^{-1}\left(\limsup_n \rho_k(x_n) + \rho_k(y_m)\right) \qquad \text{(by (I))} \\
&\ge \delta_k^{-1}\limsup_n \rho_k(x_n) + \delta_k^{-2}\left(\limsup_m \rho_k(y_m - y) + \rho_k(y)\right) \qquad \text{(by (I))} \\
&\ge \delta_k^{-1}\alpha_k \limsup_n |x_n|_\rho + \delta_k^{-2}\left(\alpha_k \limsup_m |y_m - y|_\rho + \rho_k(y)\right) \qquad \text{(by (II))} \\
&\ge \delta_k^{-1}\alpha_k c + \delta_k^{-2}\left(\alpha_k c + \rho_k(y)\right),
\end{aligned}$$

which implies that $\rho_k(y) \le d_k$ for all $k \in \mathbb{N}$.

Define r by

$$r := \inf\left\{\limsup_n |y_n - y|_\rho : \{y_n\} \in A(K),\ y_n \xrightarrow{\mathcal{T}} y\right\}.$$

From definition of r we have

$$c \le r \le \limsup_n |x_n|_\rho < c + \varepsilon_1 \tag{3.4}$$

In what follows we will prove the existence of a $\mathcal{T}$-convergent sequence $\{y_n\} \subset K$ which is an a.f.p.s. and

$$r(\{y_n\}, K) < 2r$$

which is a contradiction with Lemma 3.3.

By (III) there exists some $x_0 \in \overline{co}^{|\cdot|_\rho}(\{x_n\}) \subset K$ such that

$$\limsup_k \rho_k(x_0) \le \frac{\limsup_n |x_n|_\rho}{\alpha} < \frac{c + \varepsilon_1}{\alpha} \tag{3.5}$$

From (3.1) we can choose k_0 such that

$$d_k + \rho_k(x_0) < m \tag{3.6}$$

for all $k \ge k_0$.

Since the set K is bounded, and so is the sequence d_k defined in (3.2), there is a constant $P > 0$ such that

$$\rho_k(x_0 - y) \le P \tag{3.7}$$

whenever y is the $\mathcal{T}$-limit of any a.f.p.s. in K.

Define $\beta_0 := \max\{\beta_k : k = 1, \dots, k_0\}$ which is strictly less than 1.

Take some $\lambda \in (0, 1)$ such that

$$\beta_0(2 - \lambda)r + \lambda P < 2r,$$

what is possible because $\lim_{\lambda \to 0^+} \beta_0(2 - \lambda)r + \lambda P = 2\beta_0 r < 2r$.

Moreover

$$(2 - \lambda)r + \lambda m = 2r - \lambda(r - m) < 2r$$

since $m < c \le r$.

Therefore, we can find $\varepsilon_2 > 0$ such that

$$\beta_0(2 - \lambda)(r + \varepsilon_2) + \lambda P < 2r. \tag{3.8}$$

$$(2 - \lambda)(r + \varepsilon_2) + \lambda m < 2r, \tag{3.9}$$

Set

$$M := \max\{(2 - \lambda)(r + \varepsilon_2) + \lambda m, \beta_0(2 - \lambda)(r + \varepsilon_2) + \lambda P\},$$

which is strictly less than $2r$.

Take $(y_n) \subset K$ an a.f.p.s. such that $y_n \xrightarrow{\mathcal{T}} y$ and

$$\limsup_n |y_n - y|_\rho < r + \varepsilon_2. \tag{3.10}$$

Take $N_0 \in \mathbb{N}$ such that $|y_n - y|_\rho < r + \varepsilon_2$ for all $n \ge N_0$.

Moreover, using (II), we have

$$\limsup_n \rho_k(y_n - y) < \beta_0(r + \varepsilon_2), \qquad \text{for } k \in \{1, \dots, k_0\}.$$

Therefore we can find $N_1 \geq N_0$ such that

$$\rho_k(y_n - y) \leq \beta_0(r + \varepsilon_2) \tag{3.11}$$

for all $n \geq N_1$ and $k = 1, \ldots, k_0$.

Define the vector

$$z = \lambda x_0 + (1-\lambda) y_{N_1}$$

which belongs to K, because K is convex.

We are going to prove that $\limsup_n |y_n - z|_\rho \leq M$. In order to do this, we will check that for all $k \in \mathbb{N}$ and all $n \geq N_1$ we have

$$\rho_k(y_n - z) \leq M.$$

Notice that

$$y_n - z = y_n - y - (1-\lambda)(y_{N_1} - y) - \lambda(x_0 - y).$$

Fix $n \geq N_1$. We split the proof into two cases:
Case 1: $k > k_0$.

$$\begin{aligned} \rho_k(y_n - z) &\leq \rho_k(y_n - y) + (1-\lambda)\rho_k(y_{N_1} - y) + \lambda\left[\rho_k(x_0) + \rho_k(y)\right] \\ &\leq |y_n - y|_\rho + (1-\lambda)|y_{N_1} - y|_\rho + \lambda\left[\rho_k(x_0) + \rho_k(y)\right] \\ &\qquad\qquad\qquad \text{(by definition of } |\cdot|_\rho) \\ &< (2-\lambda)(r+\varepsilon_2) + \lambda\left[\rho_k(x_0) + d_k\right] \qquad \text{(by 3.10 and 3.3)} \\ &< (2-\lambda)(r+\varepsilon_2) + \lambda m \qquad \text{(from 3.6)} \\ &< 2r \qquad \text{(from 3.9)} \end{aligned}$$

Case 2: $k \leq k_0$.

$$\begin{aligned} \rho_k(y_n - z) &\leq \rho_k(y_n - y) + (1-\lambda)\rho_k(y_{N_1} - y) + \lambda\rho_k(x_0 - y) \\ &\leq \beta_0(2-\lambda)(r+\varepsilon_2) + \lambda\rho_k(x_0 - y) \qquad \text{(from 3.11)} \\ &\leq \beta_0(2-\lambda)(r+\varepsilon_2) + \lambda P \qquad \text{(by 3.7)} \\ &< 2r \qquad \text{(by 3.8)} \end{aligned}$$

Then $\rho_k(y_n - z) \leq M$ for all $k \in \mathbb{N}$ and for all $n \geq N_1$. This implies that $|y_n - z|_\rho \leq M$ for all $n \geq N_1$. Therefore

$$\limsup_n |y_n - z|_\rho \leq M < 2r,$$

which is a contradiction with Lemma 3.3 and this finishes the proof. □

First notice that Theorem 3.1 is an strict improvement of of Theorem 1 in [15] if we define the family of seminorms by $\rho_k(x) := \gamma_k R_k(x)$ in the notation introduced in [15]. Conditions (I), (II), (III) introduced in this manuscript are less restrictive than conditions included in [15]. Therefore, all the equivalent norms with the FPP for some nonreflexive Banach spaces studied in [15] are included in the framework of Theorem 3.1. Moreover, in the next section we will obtain new families of equivalent norms with the FPP that can not be derived from [15]. Theorem 3.1 also let us include previous examples obtained by P.K. Lin in [25] as a consequence of a new theorem with sufficient conditions so that a renorming in ℓ_1 has a FPP. We will explain the details in the next section.

Also, Theorem 3.1 lets extend the fixed point results for nonexpansive mappings in the above articles to the setting of mappings with the (L) condition.

In the remaining of this section we will prove another technical result which will let us obtain new scopes of Theorem 3.1 which are not included in [15].

We introduce the following definition:

Definition 3.4. Let X be a Banach space endowed with a topology $\mathcal{T}$. Let $p \in \mathcal{P}(X)$. We say that the norm p has the $\mathcal{T}(*)$ condition if the following equality is satisfied:

$$\limsup_n p(x_n + x) = \limsup_n p(x_n) + p(x)$$

for every $\mathcal{T}$-null (norm)-bounded sequence $\{x_n\}$ and for all $x \in X$.

The condition $\mathcal{T}(*)$ and the following theorem will be the key tool to generate open rays with the FPP in the following two sections.

Theorem 3.5. *Let X be a Banach space endowed with an equivalent norm given by $|x|_\rho = \sup_k \rho_k(x)$ where $\{\rho_k(\cdot)\}_k$ is a sequence of seminorms satisfying conditions (I), (II), (III) with respect to a topology $\mathcal{T}$ introduced previously. Let $p \in \mathcal{P}(X)$ satisfying the $\mathcal{T}(*)$ condition. If λ is any positive constant, the equivalent norm*

$$p(\cdot) + \lambda|\cdot|_\rho.$$

can be defined through a family of seminorms which again verify conditions (I), (II) and (III).

Proof. Without loss of generality, we can assume that $\lambda = 1$. Notice that for all $x \in X$ we can write

$$p(x) + |x|_\rho = p(x) + \sup_k \rho_k(x) = \sup_k (p(x) + \rho_k(x))$$

Define the family of seminorms $\rho' = \{\rho'_k(\cdot)\}_k$ given by

$$\rho'_k(x) := p(x) + \rho_k(x)$$

for all $x \in X$ and for all $k \in \mathbb{N}$. Using the notation introduced in the previous section we can write

$$|\cdot|_{\rho'} = \sup_k \rho'_k(\cdot) = p(\cdot) + |\cdot|_\rho.$$

Since $|\cdot|_\rho$ satisfies (I), (II) and (III) of Section 3, there exist some constants δ_k, α_k, β_k, α according to the notation of the above properties. We will use the corresponding symbols δ'_k, α'_k, β'_k, α' for the new seminorms $\rho'_k(\cdot)$.

Let us check that the family $\{\rho'_k(\cdot)\}_k$ again verifies condition (I), (II) and (III).

(I): It is easy to check that condition (I) holds for $\delta'_k = \delta_k$.

(II): Fix $k \in \mathbb{N}$ and take $\{x_n\}$ a $\mathcal{T}$-null sequence. We can extract a subsequence $\{x_{n_s}\}_s$ such that $\lim_s |x_{n_s}|_{\rho'} = \limsup_n |x_n|_{\rho'}$ and $\lim_s p(x_{n_s})$, $\lim_s |x_{n_s}|_\rho$, $\lim_s \rho_k(x_{n_s})$ exist. Hence

$$\begin{aligned}\limsup_n \rho'_k(x_n) &\geq \limsup_s \rho'_k(x_{n_s}) = \lim_s p(x_{n_s}) + \lim_s \rho_k(x_{n_s}) \\ &\geq \lim_s p(x_{n_s}) + \alpha_k \lim_s \rho_k(x_{n_s})\end{aligned}$$

$$\geq \alpha_k \left[\lim_s p(x_{n_s}) + \lim_s |x_{n_s}|_\rho\right]$$
$$= \alpha_k \lim_s |x_{n_s}|_{\rho'} = \alpha_k \limsup_n |x_n|_{\rho'}.$$

On the other hand, define $a = \inf\{|x|_\rho : p(x) = 1\}$ which is strictly greater than zero because both norms are equivalent. This implies that $ap(x) \leq |x|_\rho$ and $p(x) \leq \frac{1}{1+a}|x|_{\rho'}$ for all $x \in X$. Let $\{x_n\}$ be a $\mathcal{T}$-null sequence and $\{x_{n_s}\}$ a subsequence such that $\limsup_n \rho'_k(x_n) = \lim_s \rho'_k(x_{n_s})$ and $\lim_s p(x_{n_s})$, $\lim_s \rho_k(x_{n_s})$, $\lim_s |x_{n_s}|_\rho$ exist. Therefore

$$\begin{aligned}
\limsup_n \rho'_k(x_n) &= \lim_s \rho'_k(x_{n_s}) = \lim_s p(x_{n_s}) + \lim_s \rho_k(x_{n_s}) \\
&\leq \lim_s p(x_{n_s}) + \beta_k \lim_s |x_{n_s}|_\rho \\
&= \beta_k \left[\lim_s p(x_{n_s}) + \lim_s |x_{n_s}|_\rho\right] + (1-\beta_k) \lim_s p(x_{n_s}) \\
&\leq \beta_k \lim_s |x_{n_s}|_{\rho'} + (1-\beta_k) \frac{1}{1+a} \lim_s |x_{n_s}|_{\rho'} \\
&\leq \left[\beta_k + (1-\beta_k)\frac{1}{1+a}\right] \limsup_n |x_n|_{\rho'}
\end{aligned}$$

We can take $\alpha'_k = \alpha_k$, $\beta'_k = \beta_k + (1-\beta_k)\frac{1}{1+a} < 1$ for all $k \in \mathbb{N}$ and condition (II) holds.

(III): Let $\{x_n\}_n$ be a $\mathcal{T}$-null sequence. Take the constant a as above and consider $\epsilon > 0$ with

$$\left[\frac{1}{\alpha} + \left[(1+\epsilon) - \frac{1}{\alpha}\right]\frac{1}{1+a}\right] < 1.$$

Choose a subsequence $\{x_{n_s}\}$ such that $\lim_s p(x_{n_s})$ and $\lim_s |x_{n_s}|_\rho$ exist.

Let s_0 be such that $p(x_{n_s}) \leq (1+\epsilon)\lim_s p(x_{n_s})$ for all $s \geq s_0$. For the sequence $\{x_{n_s}\}_{s\geq s_0}$ take $x_0 \in \overline{co}^{|\cdot|_\rho}(\{x_{n_s}\}_{s\geq s_0}) \subset \overline{co}^{|\cdot|_\rho}(\{x_n\})$ satisfying condition (III) for the family of seminorms $\{\rho_k(\cdot)\}_k$. Then

$$\begin{aligned}
\limsup_k \rho'_k(x_0) &= p(x_0) + \limsup_k \rho_k(x_0) \\
&\leq (1+\epsilon) \lim_s p(x_{n_s}) + \frac{\lim_s |x_{n_s}|_\rho}{\alpha} \\
&= \frac{1}{\alpha} \lim_s |x_{n_s}|_{\rho'} + \left[(1+\epsilon) - \frac{1}{\alpha}\right] \lim_s p(x_{n_s}) \\
&\leq \frac{1}{\alpha} \limsup_n |x_n|_{\rho'} + \left[(1+\epsilon) - \frac{1}{\alpha}\right] \limsup_n p(x_n) \\
&\leq \left[\frac{1}{\alpha} + \left[(1+\epsilon) - \frac{1}{\alpha}\right]\frac{1}{1+a}\right] \limsup_n |x_n|_{\rho'}
\end{aligned}$$

From the above, there exists some $\alpha' > 1$ satisfying condition (III) for $\{\rho'_k(\cdot)\}_k$ and the proof is complete.

□

4. Applications: Equivalent norms and rays of equivalent norms with the fixed point property

In this section we will apply the fixed point theorems obtained in Section 3 for (L)-type mappings to several classes of nonreflexive Banach spaces. Firstly we introduce some notation and obtain some general results.

If X is endowed with a topology $\mathcal{T}$ we denote by

$$\mathcal{P}_{\mathcal{T}(*)}(X) := \{p \in \mathcal{P}(X) : \ (X, p) \text{ verifies the } \mathcal{T}(*) \text{ condition}\}$$

Since nonexpansive mappings are the classical examples of mappings satisfying condition (L), we will state the following theorems and examples for such class of mappings although they are valid in the more general context.

Set

$$\mathcal{P}_{FPP}(X) := \{p \in \mathcal{P}(X) : \ (X, p) \text{ has the FPP}\}.$$

Notice that $\mathcal{P}_{\mathcal{T}(*)}(X)$ could be empty. Also the set $\mathcal{P}_{FPP}(X)$ could be empty. The last happens for ℓ_∞, $\ell_1(\Gamma)$ and $c_0(\Gamma)$ for Γ uncountable, that is, these spaces cannot be renormed to satisfy the FPP because it can be proved that every equivalent norm contains either an asymptotically isometric copy of ℓ_1 or c_0 [21](Chapter 9). However, whenever X is a reflexive Banach space, it is known that $\mathcal{P}_{FPP}(X)$ is nonempty [5] and dense in $\mathcal{P}(X)$ for the topology induced by the above metric (in fact, it is of the second category) [8].

We emphasize the following remarkable fact:

Lemma 4.1. *For every Banach space, $\mathcal{P}_{FPP}(X) \cap \mathcal{P}_{\mathcal{T}(*)}(X) = \emptyset$.*

Proof. The proof of the above assertion cames from Proposition 1 in [11], where it is shown that if a Banach space $(X, \|\cdot\|)$ contains a normalized sequence $\{x_n\}$ such that

$$\limsup_n \|x_n + y\| = \limsup_n \|x_n\| + \|y\|$$

for every y belonging to the span$\{x_n : n \in \mathbb{N}\}$, and the span$\{x_n : n \in \mathbb{N}\}$ is infinite dimensional, then the sequence $\{x_n\}$ contains a subsequence which spans an asymptotically isometric copy of ℓ_1 and therefore $(X, \|\cdot\|)$ fails the FPP [21] (Chapter 9). □

In the following statement we summarize two important consequences of the results obtained in the previous sections.

Corollary 4.2. *Let X be a Banach space which can be endowed with a topology $\mathcal{T}$ and an equivalent norm $|\cdot|_\rho$ given by a family of seminorms $\rho = \{\rho_k(\cdot)\}_k$ satisfying conditions (I), (II) and (III) introduced in Section 3. Assume that every (norm) bounded sequence has a $\mathcal{T}$-convergent subsequence. Then every $p \in \mathcal{P}_{\mathcal{T}(*)}(X)$ is the initial point of an open ray of equivalent norms contained in $\mathcal{P}_{FPP}(X)$.*

Proof. The proof is direct by using Theorem 3.1 and Theorem 3.5, since for every $p \in \mathcal{P}_{\mathcal{T}(*)}(X)$ and for every $\lambda > 0$ the family of equivalent norms

$$p(\cdot) + \lambda|\cdot|_\rho$$

has the FPP for every $\lambda > 0$. □

Next we give some examples of different classes of nonreflexive Banach spaces where Corollary 4.2 can be applied. We will use Theorem 3.1 and Theorem 3.5 to obtain new equivalent norms with the FPP.

We will start by considering the sequence Banach space ℓ_1. Next we will study Banach spaces which can be written as a one direct sum of finite dimensional Banach spaces. Examples of such spaces are the Fourier-Stieltjes algebras for separable compact groups and the preduals of a finite atomic von Neumann algebras. Also some subspaces of $L_1[0,1]$ will be considered.

We denote by $\|\cdot\|_1$ the usual norm in ℓ_1. It is a classical result that $(\ell_1, \|\cdot\|_1)$ fails to satisfy the FPP. In ℓ_1 we can consider the topology $\mathcal{T}$ as the weak* topology $\sigma(\ell_1, c_0)$. The unit ball of ℓ_1 is $\mathcal{T}$-sequentially compact so every (norm) bounded sequence has a weak*-convergent subsequence.

Example 4.3. In 2008, P.K. Lin proved that there is an equivalent norm in ℓ_1 which satisfies the FPP [24]. This norm can be defined as follows:

Take any sequence $(\gamma_k)_k \subset (0,1)$ with $\lim_k \gamma_k = 1$ and for $x = \{x(n)\}_n \in \ell_1$ define

$$|||x||| := \sup_k \gamma_k \sum_{n=k}^{\infty} |x(n)|.$$

Notice that the $|||\cdot|||$-norm belongs to $\mathcal{P}(\ell_1)$ and satisfies conditions of Theorem 3.1 if we define a family of the seminorms given by

$$\rho_k(x) := \gamma_k \sum_{n=k}^{\infty} |x(n)|.$$

In this case, taking $\delta_k = 1, \alpha_k = \beta_k = \gamma_k$ for all $k \in \mathbb{N}$, conditions (I) and (II) are satisfied. Moreover, the constant $\alpha > 1$ in condition (III) can be chosen as any positive constant, since $\lim_k \rho_k(x) = 0$ for all $x \in \ell_1$. Therefore $(\ell_1, |||\cdot|||)$ is a renorming of ℓ_1 which has the fixed point property for nonexpansive mappings and, more generally, for (L)-type mappings (see also [27]).

In what follows we set $P_0(x) := 0$ and $P_k(x) = \sum_{n=1}^{k} x(n)e_n$ if $x = \{x(n)\}_n$ for $x \in \ell_1$. The following equivalent norms in ℓ_1 are inspired in the examples introduced in [25], where the author shows that they satisfy the FPP.

Example 4.4. Consider a sequence of norms $v_k(\cdot)$ on the finite dimensional space $\mathbb{R}^k$ for $k \in \mathbb{N}$. Define

$$\rho_k(x) := \gamma_k \left[\sum_{n=k}^{\infty} |x(n)| + v_{k-1}(P_{k-1}(x)) \right].$$

Set

$$c_k := \sup\{v_k(P_k(x)) : \|P_k(x)\|_1 = 1\}$$

which implies that $v_k(P_k(x)) \leq c_k \|P_k(x)\|_1$ for all $x \in \ell_1$. Consider $c_0 = \limsup_k c_k$ and assume that $c_0 < \gamma_1$. Then the corresponding $|\cdot|_\rho$ norms belong to $\mathcal{P}(\ell_1)$ and satisfy the conditions of Theorem 3.1.

Proof. Indeed, notice that if $\{x_n\}$ is a weak* null sequence and $k \in \mathbb{N}$, then

$$\limsup_n v_{k-1}(P_{k-1}(x_n)) = 0$$

which implies that (I) holds. Moreover,

$$\limsup_n \rho_k(x_n) = \gamma_k \limsup \|x_n\|_1 = \limsup |x_n|_\rho$$

so property (II) is satisfied with $\alpha_k = \beta_k = \gamma_k$ for all $k \in \mathbb{N}$. Let us check that property (III) also holds.

Notice that for any $x \in \ell_1$,

$$\limsup_k \rho_k(x) = \limsup_k v_k(P_k(x)) \leq \limsup_k c_k \|P_k(x)\|_1 = c_0 \|x\|_1$$

With this notation

$$\limsup_k \rho_k(x) \leq \limsup_k \frac{c_k}{\gamma_1} \gamma_1 \|x\|_1 \leq \frac{c_0}{\gamma_1} |x|_\rho$$

Take $\epsilon > 0$ such that $(1+\epsilon)\frac{c_0}{\gamma_1} < 1$ and let $\{x_n\}$ be a weak* null sequence. Take n_0 such that $|x_{n_0}|_\rho \leq (1+\epsilon) \limsup_n |x_n|_\rho$. Therefore for such n_0,

$$\limsup_k \rho_k(x_{n_0}) \leq \frac{c_0}{\gamma_1} |x_{n_0}|_\rho \leq (1+\epsilon) \frac{c_0}{\gamma_1} \limsup_n |x_n|_\rho$$

as we wanted to check. □

Adapting the notation of the examples included in [25] to our setting, we can check that they are particular cases of the above family of equivalent norms in $\mathcal{P}(\ell_1)$.

We can enlarge the set of equivalent norms in ℓ_1 which are in the conditions of Theorem 3.1 considering the norms introduced in the following example:

Example 4.5. Take $(q_k)_k \subset (1, +\infty)$ with $\lim_k q_k = 1$ and $(\gamma_k)_k \subset (0,1)$ with $\lim_k \gamma_k = 1$. Define the seminorms

$$\rho_k(x) = \gamma_k \left[\|(I - P_k)(x)\|_1^{q_k} + \|P_k(cx)\|_1^{q_k} \right]^{1/q_k}$$

for some $c < 1$ and for all $k \in \mathbb{N}$, where by I we denote the identity operator and $P_k(\cdot)$ was defined previously. If we define as usual

$$|x|_\rho := \sup_k \rho_k(x),$$

the norm $|\cdot|_\rho$ belongs to $\mathcal{P}(\ell_1)$ and satisfies the conditions in Theorem 3.1.

Proof. Indeed, it is not difficult to check that (I) is satisfied taking $\delta_k = 2^{\frac{q_k - 1}{q_k}}$. On the other hand, if $\{x_n\}$ is weakly* null, then $\limsup_n \rho_k(x_k) = \gamma_k \limsup_n \|x_n\|_1$ and $\limsup_n |x_n|_\rho = \limsup_n \|x_n\|_1$ so (II) holds with $\alpha_k = \beta_k = \gamma_k$ for all $k \in \mathbb{N}$.

Take $\epsilon > 0$ with $c(1+\epsilon) < 1$. Consider $\{x_n\}$ a weak* null sequence and n_0 such that

$$\|x_{n_0}\|_1 \leq (1+\epsilon) \limsup_n \|x_n\|_1 = (1+\epsilon) \limsup_k |x_n|_\rho.$$

Therefore

$$\limsup_k \rho_k(x_{n_0}) = c\|x_{n_0}\|_1 \leq c(1+\epsilon) \limsup_k |x_n|_\rho$$

and (III) holds. □

More examples of equivalent norms in ℓ_1 given by seminorms verifying Theorem 3 with $\delta_k \neq 1$ in condition (I) can be found in [2]

Now, as a consequence of Lemma 4.1 and Corollary 4.2 we can deduce that the set of equivalent norms in ℓ_1 satisfying the FPP contains rays whose initial points are renomings which fail to have the FPP:

Corollary 4.6. *Let $p(\cdot)$ be any equivalent norm in ℓ_1 satisfying the $w^\star(*)$ condition, that is*

$$\limsup_n p(x_n + x) = \limsup_n p(x_n) + p(x)$$

for every weak*-null sequence $\{x_n\}$ and for all $x \in \ell_1$. Then (X, p) fails to have the FPP. However, p is the initial point of several open rays of equivalent norms with the FPP.

Typical examples of norms in ℓ_1 which satisfy the $w^\star(*)$ condition are the usual $\|\cdot\|_1$-norm and every equivalent norm which separates disjoint supports. They fail to have the FPP themselves, but it is remarkable that they provide different families of open rays composed of equivalent norms with the FPP.

Notice that Theorem 2.1 in [18] is a very particular case of the above result by using P.K. Lin's norm introduced in Example 4.3. Also in [18] the reader can find more examples of equivalent norms in ℓ_1 satisfying the $w^\star(*)$ condition and which do not separate disjoint supports.

Next, let us check how Theorem 3.1 and Theorem 3.5 can be applied to more general classes of nonreflexive Banach spaces.

Corollary 4.7. *Let (X_n) be a sequence of finite dimensional Banach spaces. Define the one-direct sum of (X_n) as*

$$X = \oplus_1 \sum_n X_n := \left\{ x = (x_n) : x_n \in X_n, \|x\| = \sum_n \|x_n\|_{X_n} < +\infty \right\}.$$

Let $\mathcal{T}$ be the the weak* topology generated by the predual

$$X_* = \left\{ x = (x_n) : x_n \in X, \lim_n \|x_n\|_{X_n} = 0, \|x\| = \sup_n \|x\|_{X_n} \right\}.$$

Then every equivalent norm satisfying the $\mathcal{T}(*)$ condition fails the FPP and it is the initial point of different families of open rays composed of equivalent norms with the FPP.

Proof. Notice that the original norm in X verifies the $\mathcal{T}(*)$ condition.

Consider $(\gamma_k)_k$ any sequence in $(0,1)$ with $\lim_k \gamma_k = 1$ and define $|x|_\rho := \sup_k \rho_k(x)$, where

$$\rho_k(x) = \gamma_k \sum_{n=k}^{\infty} \|x_n\|_{X_n}$$

if $x = (x_n)$ with $x_n \in X_n$ for all $n \in \mathbb{N}$. Now it is easy to check that $\{\rho_k(\cdot)\}_k$ verifies Theorem 3.1. Then $(X, |\cdot|_\rho)$ has the FPP and $p + \lambda|\cdot|_\rho \in \mathcal{P}_{FPP}(X)$ for every $p \in \mathcal{P}_{\mathcal{T}(*)}(X)$ and $\lambda > 0$. Similar rays can be constructed following the ideas in Examples 4.4 and 4.5. $\square$

Example 4.8. Fix any $p > 1$. We can apply the above corollary to the Banach space $X = \oplus_1 \sum_n \ell_p^n$.

It is worth mentioning that the above space is a nonreflexive Banach space which is not isomorphic to any subspace of ℓ_1 (this can be checked by comparing the type and cotype of both Banach spaces).

Example 4.9. Let G be a separable compact group and $B(G)$ its Fourier-Stieltjes algebra. Then the set of equivalent norms which satisfy the FPP for (L)-type mappings contains rays.

Proof. Using the arguments in the proof of [23, Lemma 3.1] and [14, Corollary 6.9] we deduce that $B(G)$ can be written as

$$B(G) = \oplus_1 \sum_{n=1}^{\infty} \mathcal{T}(H_n),$$

where H_n is a finite dimensional Hilbert space and $\mathcal{T}(H_n)$ is the trace class operators on H_n. □

Notice that the Fourier-Stieltjes algebra $B(G)$ endowed with its original norm fails to have the FPP unless that G is a finite group [22].

Example 4.10 ([16])**.** Let $\mathcal{M}$ be any finite atomic von Neumann algebra. Then its predual $L_1(\mathcal{M})$ can be endowed with rays of norms with the FPP for (L)-type mappings.

Proof. The proof is again based on the fact that $L_1(\mathcal{M})$ is isomorphic to a one-direct sum of finite dimensional Banach spaces whenever $\mathcal{M}$ is a finite atomic von Neumann algebra (see Proposition 2.2 in [33]). □

We now consider some closed subspaces of $L_1(\mu)$-spaces which we know that can be renormed to have the FPP:

Let (Σ, Ω, μ) be a σ-finite measure. Let $\|\cdot\|_1$ the usual norm in $L_1(\mu)C$ given by $\|x\|_1 = \int_\Omega |x| \, d\mu$. It is well-known that $(L_1(\mu), \|\cdot\|_1)$ fails the FPP. In fact, it is an open question whether there is some equivalent $L_1(\mu)$ which satisfies the FPP.

Corollary 4.11. *Let (Σ, Ω, μ) be a σ-finite measure space and X a closed subspace of $L_1(\mu)$. If the unit ball of X is relatively compact for the topology of the local convergence in measure, then the set $\mathcal{P}_{FPP}(X)$ contains rays. In particular, the usual $\|\cdot\|_1$ norm in X in the initial point of different open rays in $\mathcal{P}_{FPP}(X)$.*

Proof. In Section 5 of [15] it is proved that $L_1(\mu)$ can be endowed with a family of seminorms that verify conditions (I), (II) and (III) of Theorem 3.1. If we restrict these seminorms to the subspace X and define the corresponding $|\cdot|_\rho$ norm, the space $(X, |\cdot|_\rho)$ verifies the FPP since Theorem 3.1 holds for the topology $\mathcal{T}$ of the local convergence in measure. Since $\|\cdot\|_1$ verifies the $\mathcal{T}(*)$ condition for this topology, according to Theorem 3.5, $\|\cdot\|_1$ is the initial point of some open rays with the FPP. We can modify the semimorns introduced in [15] and obtain different open rays of norms in $\mathcal{P}_{FPP}(X)$ with initial point the $\|\cdot\|_1$ norm. □

Example 4.12. Let $\mathbb{D}$ denote the open unit disc. The Bergman space $L_a(\mathbb{D})$ is the subspace of $L_1(\mathbb{D})$ of all analytic functions on $\mathbb{D}$. J. Lindenstrauss and A. Pelczynski [26] proved that the Bergman space and the sequence space ℓ_1 are isomorphic, which implies that $L_a(\mathbb{D})$ is FPP-renormable by P.K. Lin's result [24]. However, these authors did not give an explicit definition of the above isomorphism. In fact, it turns out to be a difficult problem to find a system of functions which is a basis in $L_a(\mathbb{D})$ equivalent to the unit vector basis in ℓ_1 (see Notes and Remarks in Chapter III.A of [35] and the references therein). This makes specially difficult to give an specific renorming in the Bergman space with the FPP through P.K. Lin's norm.

On the other hand, the Bergman space is a dual Banach space and for bounded sequences weak* convergence is equivalent to uniform convergence on compact sets [30]. This shows that the weak* topology is finer than the topology of convergence in measure on the unit ball of $L_a(\mathbb{D})$ and consequently these two topologies coincide on $B_{L_a(\mathbb{D})}$. Therefore, by the family of seminorms introduced in Section 5 of [15] and Theorem 3.5, for every sequences $(\gamma_k)_k, (\delta_k)_k \subset (0,1)$ with $\lim_k \gamma_k = 1$, $\lim_k \delta_k = 0$, the norms

$$\int_{\mathbb{D}} |x|\ dm + \lambda \sup_k \gamma_k \sup \left\{ \int_E |x|\ dm :\ m(E) < \delta_k \right\}$$

gives an open ray of equivalent norms with the FPP.

In Example 6 of [15] and the references therein, we can find new subspaces of $L_1(\mu)$ that are not isomorphic to any subspace of ℓ_1, and for which we can construct rays of equivalent norms verifying the FPP following the same seminorms as above.

5. Rays of equivalent norms with the fixed point property for affine mappings

The above arguments to find equivalent norms with the fixed point property fail in case of the Banach space $L_1[0,1]$ or more generally in case of $L_1(\mu)$ when (Ω, Σ, μ) is any σ-finite measure space. In $L_1(\mu)$ we can consider the topology $\mathcal{T}$ as the topology of the local convergence in measure (clm), or the topology of the convergence in measure (cm) in case that the measure is finite. The usual $\|\cdot\|_1$ norm satisfies the $\mathcal{T}(*)$ condition and a family of seminorms with properties (I), (II) and (III) introduced in Section 3 can be given [15]. The problem arises when we try to find $\mathcal{T}$-convergent approximate fixed point sequences, since the unit ball of $L_1(\mu)$ is not $\mathcal{T}$-sequentially compact. We can overcome this situation if we assume that the mapping T is affine. Recall that a mapping T defined from a convex subset into itself is affine if

$$T(\lambda x + (1-\lambda)y) = \lambda T(x) + (1-\lambda)T(y)$$

for all $x, y \in C$ and $\lambda \in [0,1]$. Affine mappings play an important role in fixed point theory. For instance, they let characterize weakly compactness in Banach spaces through fixed point results. In [7] it is proved that a convex bounded subset C of a Banach space is weakly compact if and only if for every closed convex subset $K \subset C$ and for every affine continuous mapping $T: K \to K$, there exists a fixed point. In fact, the continuity condition can be replaced by nonexpansiveness whenever $X = L_1[0,1]$ or more generally whenever X is an L-embedded Banach space [7], [9]. Notice that the affine condition in the previous characterization of weakly compactness can not be omitted [1].

A Banach space is said to have the affine fixed point property (A-FPP) if every affine nonexpansive mapping defined from a closed convex bounded subset into itself has a fixed point. It is well-known that $L_1(\mu)$ fails the A-FPP and the same holds for every Banach space which contains an asymptotically isometric copy of ℓ_1 [21] (Chapter 9). It is proved in [17] that for the function space $L_1(\mu)$ and more generally, for the the predual of any finite von Neumann algebra $L_1(\mathcal{M})$, there exists an equivalent norm with the A-FPP. In this section we will check that we can extend these renormings and obtain rays of equivalent norms with the affine fixed point property. Notice that all the fixed point results obtained in this section also hold for affine mappings which verify the (L) condition.

Let $\mathcal{M}$ be a finite von Neumann algebra on a separable Hilbert space and let τ be a finite normal faithful trace on $\mathcal{M}$. Let $L_1(\mathcal{M})$ be the corresponding non-commutative L_1-space with its usual norm $\|x\|_1 = \tau(|x|)$, where by $|x|$ we denote the absolute value of the operator $x \in L_1(\mathcal{M})$. For definition and examples of non-commutative L_1-spaces, the reader can consult [32], [16] and the references therein. Since every commutative von Neumann algebra is finite, the function spaces $L_1(\mu)$ for (Ω, Σ, μ) any σ-finite measure are particular cases of non-commutative L_1 Banach spaces associated to a finite von Neumann algebra. Another example of such space is $L_1(\mathcal{R})$, $\mathcal{R}$ being the hyperfinite II_1 factor. Notice that $L_1(\mathcal{R})$ contains strictly $L_1[0,1]$ as an isometric subspace [33].

In [16] and [17] the following family of seminorms is considered:

$$\rho_1(x) = \gamma_1 \|x\|_1 = \gamma_1 \tau(|x|)$$

and for $k \geq 2$

$$\rho_k(x) = \gamma_k \sup \left\{ \|xp\|_1 : p \in \mathcal{P}(\mathcal{M}), \tau(p) < \frac{1}{k} \right\},$$

where (γ_k) is any sequence in $(0,1)$ with $\lim_k \gamma_k = 1$ and $\mathcal{P}(\mathcal{M})$ here denotes the collection of all orthogonal projections in $\mathcal{M}$. These seminorms verify properties (I), (II) and (III) of Theorem 3.1 (see [16]) when we consider the topology $\mathcal{T}$ as the measure topology in $L_1(\mathcal{M})$ which has, as a fundamental system of neighborhoods of zero, the following sets:

$$N(\varepsilon, \delta) = \left\{ x \in \mathcal{M} : \exists p \in \mathcal{P}(\mathcal{M}) \text{ such that } \|xp\|_\infty \leq \varepsilon \text{ and } \tau(p^\perp) \leq \delta \right\},$$

where ε and δ are positive real numbers [29].

We consider the following extension of Komlós Theorem due to N. Randrianantoanina [31] (Proposition 3.11).

Theorem 5.1. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra and suppose that $\{x_n\}$ is a bounded sequence in $L_1(\mathcal{M})$. Then there exists a subsequence $\{g_n\} \subset \{x_n\}$ and a vector $x \in L_1(\mathcal{M})$ such that for every further subsequence $\{h_n\} \subset \{g_n\}$,*

$$\frac{1}{n} \sum_{i=1}^{n} h_i \to_n x \qquad \textit{with respect to the measure topology.}$$

If the mapping T is affine, it is easy to check that the sequence formed by the arithmetic means of an approximate fixed point sequence $\{x_n\}$ is again an approximate fixed point sequence. Hence, the hypotheses of Theorem 3.1 are satisfied when the mapping T is an affine nonexpansive mapping, or more generally an affine (L)-type mapping. Therefore, similar statements can be deduced.

Since every renorming which contains an asymptotically isometric copy of ℓ_1 also fails the A-FPP, we can deduce that every equivalent norm in $L_1(\mathcal{M})$ with the $\mathcal{T}(*)$ condition fails the A-FPP, whereas it provides a ray of equivalent norms with the A-FPP. This assures that the set of equivalent norms verifying the A-FPP is dense in the set of equivalent norms with the $\mathcal{T}(*)$ condition.

Corollary 5.2. *Let $\mathcal{M}$ be a finite von Neumann algebra and $L_1(\mathcal{M})$ its predual. Let $(\gamma_k)_k \subset (0,1)$ with $\lim_k \gamma_k = 1$. Then for every equivalent norm $\|\cdot\|$ verifying the $\mathcal{T}(*)$ condition, in particular for the usual $\|\cdot\|_1$ norm, the equivalent norms defined by*

$$\|\cdot\| + \lambda \sup_k \gamma_k \sup\left\{\|xp\|_1 : p \in \mathcal{P}(\mathcal{M}), \tau(p) < \frac{1}{k}\right\}$$

for $\lambda > 0$ form an open ray of equivalent norms with the A-FPP, and more generally with the fixed point property for affine (L)-type mappings.

In particular for function spaces we obtain

Corollary 5.3. *For every σ-finite measure space (Ω, Σ, μ), the function space $L_1(\mu)$ can be renormed to have the A-FPP and every norm satisfying the clm$(*)$ condition fails to have the A-FPP but it is the initial point of an open ray formed by equivalent norms with the A-FPP.*

When the measure space is finite and nonatomic, the previous renorming can be rewritten by using maximal functions. Indeed,

$$\begin{aligned} \sup\{\textstyle\int_A |f| d\mu : \ \mu(A) < 1/k\} &= \sup\{\textstyle\int_A |f| d\mu : \ \mu(A) = 1/k\} \\ &= \textstyle\int_0^{1/k} f^*(s) ds = \frac{1}{k} f^{**}\left(\frac{1}{k}\right), \end{aligned}$$

where f^* denotes the decreasing rearrangement of the function f and f^{**} is the maximal function defined by $f^{**}(t) = \dfrac{1}{t}\displaystyle\int_0^t f^*(s)ds$ for $t > 0$ (see [3, Chapter 2], Lemma 2.5.). In particular, in $L_1[0,1]$, taking $\gamma_k = 1 - 1/2k$ for every $k \geq 1$, the equivalent norms

$$||f||_1 + \lambda \sup_{k\geq 1}\left(1 - \frac{1}{2k}\right)\frac{1}{k} f^{**}\left(\frac{1}{k}\right).$$

forms an open ray of equivalent norms in $L_1[0,1]$ with the fixed point property for affine (L)-type mappings (and the same holds if we replace the sequence $\{1/k\}$ by any sequence tending to zero).

Concluding remarks and some open problems:

In the last years new results have been published which interrelate Metric Fixed Point Theory for nonexpansive mappings with Renorming Theory in Banach spaces. We would like to point out some open problems in this line of research:

1. We do not know if the affine condition can be dropped from the statement of Corollary 5.2. In fact it is an open problem if there exists some equivalent norm in $L_1[0,1]$ with the fixed point property for nonexpansive mappings. Notice that we can not argue as in the nonseparable case $\ell_1(\Gamma)$ where it is proved that every renorming in $\ell_1(\Gamma)$ contains an asymptotically isometric copy of ℓ_1. Since $L_1[0,1]$ and more generally $L_1(\mathcal{M})$ are separable Banach spaces, we know that there exist some equivalent norms without asymptotically isometric copy of ℓ_1. In fact it can be proved that for all separable Banach spaces, the subset of all equivalent norms which fail to have an asymptotically isometric copy ℓ_1 is of the second category in $\mathcal{P}(X)$ and that always there exists an equivalent norm without asymptotically isometric copies of both c_0 and ℓ_1 [19].
2. Notice that as "close" as we like from the usual $\|\cdot\|_1$ norm in ℓ_1 we can find a renorming with the FPP (take γ_1 close to one in the P.K. Lin's norm). Also as close as we like from a norm verifying the FPP in ℓ_1 we can find an equivalent norm failing to have such property [6]. This implies that the subset $\mathcal{P}_{FPP}(\ell_1)$ is neither open nor closed in $\mathcal{P}(\ell_1)$ with respect to the topology induced by the above metric. Actually not much is known about the topological structure of $\mathcal{P}_{FPP}(\ell_1)$. In [6] it is proved that the complementary of $\mathcal{P}_{FPP}(\ell_1)$, that is, the subset of all equivalent norms which fail to have the FPP, is dense in $\mathcal{P}(\ell_1)$. We do not know whether $\mathcal{P}_{FPP}(\ell_1)$ is dense in the set of all equivalen norms in ℓ_1. The same question could be raised for every separable nonreflexive Banach space.

   More generally, is every norm in $\mathcal{P}(\ell_1)$ the initial point of an open ray of equivalent norms with the FPP? This is true is we consider rays of norms failing to have an asymptotically isometric copy of ℓ_1, that is, every norm in $\mathcal{P}(\ell_1)$ is the initial point of an open ray of norms in $\mathcal{P}(\ell_1)$ which fail to have an asymptotically isometric copy of ℓ_1 [19] so we still could have some chance in proving that this ray is composed of norms satisfying the FPP.

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Manuscript received May 6, 2013
revised August 23, 2013

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