Journal of Nonlinear and Convex Analysis
Volume 15, Number 2, 2014, 325–344

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

PROJECTED DYNAMICAL SYSTEMS ON HILBERT SPACES

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This work is dedicated to Professor Simeon Reich.

ABSTRACT. In this paper we present some simple solvability results applicable to the study of the theory of projected dynamical systems. In particular, we study the existence of solutions to a variational inequality using the existence of critical points of a projected dynamical system on a Hilbert space, as well as some results about the existence of periodic orbits for the above projected dynamical systems are also presented.

1. Introduction.

The theoretical study of projected dynamical systems (PDS for short) started in the early 90s. The original motivation for the study of a (PDS) is found in finite dimensional equilibrium problems. The equilibrium is an important state considered in Physics, Engineering sciences and Economics etc (see [16]). For instance, the problem to find the equilibrium of an economical system is exactly the problem to find the solutions of a variational inequality (see [13]). On the other hand, for many equilibrium problems in Elasticity, Fluid Mechanics and Engineering it is necessary to develop the theory of projected dynamical systems in an infinite dimensional Hilbert space. The solutions of a variational inequality yields a static information about an equilibrium state. In many problems in Economics, in Mechanics etc. we are interested to know the evolution of equilibrium with respect to the parameter "time".

When we associate to a variational inequality a local projected dynamical system [14], the critical points (i.e., the equilibrium points) of this system are exactly the solutions of the variational inequality and therefore are the equilibrium points of considered practical problems (see [15, 23]).

In this paper we present some simple solvability results applicable to the study of the theory of projected dynamical systems. Moreover, as in [11], we give sufficient conditions to solve the question of existence of periodic orbits.

2. Preliminaries

Let $(H, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and let D be a nonempty closed convex subset of H.

2010 *Mathematics Subject Classification.* 34A60, 34C25, 34C27, 47H10, 47H15.

Key words and phrases. Variational inequalities, monotone operator, Cauchy problem, fixed point, strong solution, projected dynamical systems.

The research of the first author has been partially supported by MTM 2012-34847-C02-02. The second author has been partially supported by CONACyT and SEP-CONACyT grant 102380.

Given a mapping $f : D \to H$, the *variational inequality* defined by f and D is

$$VI(f,D) : \begin{cases} \text{find } x_0 \in D \text{ such that} \\ \langle f(x_0), y - x_0 \rangle \geq 0, \text{ for all } y \in D. \end{cases} \tag{2.1}$$

We refer the reader to [26], for background material on theory of variational inequalities.

Given a point $x \in D$ recall that the closed convex cone

$$N_D(x) := \{\xi \in H : \langle \xi, y - x \rangle \leq 0, \text{ for all } y \in D\},$$

is called the *normal cone* of D at the point $x \in D$.

As usual, denote $P_D : H \to D$ the metric projection operator. For each $x \in H$, $P_D(x)$ is the unique point of D such that

$$\|x - P_D(x)\| \leq \|x - y\|, \text{ for all } y \in D.$$

The *directional derivate* of the metric projection operator P_D at any point $x \in D$ and at any arbitrary direction $v \in H$ exists [31]. We denote

$$\Pi_D(x, v) := \lim_{\delta \to 0^+} \frac{P_D(x + \delta v) - x}{\delta}.$$

Given the problem $VI(f,D)$, we consider the ordinary differential equation

$$\begin{cases} u'(t) = \Pi_D(u(t); -f(u(t))) \\ \text{where } u : [0, +\infty) \to H. \end{cases} \tag{2.2}$$

The projected dynamical system defined by the mapping f on the set D (PDS(f,D) for short), is defined as the mapping $\phi : D \times \mathbb{R}^+ \to D$ where $\phi(x,t) = \phi_x(t)$ solves the following initial value problem associated to Equation (2.2), that is,

$$\begin{cases} \phi_x'(t) = \Pi_D(\phi_x(t); -f(\phi_x(t))) \\ \phi_x(0) = x_0 \in D. \end{cases} \tag{2.3}$$

The importance of the notion of PDS for the study of variational inequalities is related to the following result.

Proposition 2.1 ([23]). *A point $x_0 \in D$ is an equilibrium point of the $PDS(f,D)$, i.e. $\Pi_D(x_0, -f(x_0)) = 0$ if, and only if, x_0 is a solution of the $VI(f,D)$.*

In the infinite dimensional case, we must study the existence of solution of Problem (2.3) and the stability of stationary points of $PDS(f,D)$ using the following differential inclusion

$$u'(t) + N_D(u(t)) \ni -f(u(t)), \tag{2.4}$$

and considering its *viable solutions*.

Generally, if T is a positive real number or $T = +\infty$ a function $u : [0,T) \to D$ is said to be a viable solution of (2.4) if u is absolute continuous on compact subsets of $[0,T)$ and it satisfies

$$u'(t) + N_D(u(t)) \ni -f(u(t))$$

for almost all $t \in [0,T)$.

Then, the initial value problem (2.3) consists of finding the *slow* solution (the viable solution of minimal norm) to the differential inclusion (2.4) under the initial condition $u(0) = x_0$.

Proposition 2.2 ([23]). *A point $x_0 \in D$ is a solution of the $VI(f, D)$ if and only if, x_0 is a equilibrium point of the (2.4), that is $0 \in -f(x_0) - N_D(x_0)$.*

The study of existence results for the initial value problem (2.3) in infinite dimensional Hilbert spaces has difficulties since Π_D can have discontinuities. However in [11], the following existence theorem was presented.

Theorem 2.3. *Let H be a real Hilbert space, $D \subseteq H$ is a nonempty closed convex subset and $f : D \to H$ is a lipschitzian mapping. Then Eq. (2.3) has a solution on $[0, +\infty[$.*

Let X be a real Banach space and X^* its topological dual. Let $B_r(x) := \{y \in X : \|x - y\| \leq r\}$ and $S_r(x) := \{y \in X : \|x - y\| = r\}$. Denote by $C(0, T; X)$ the space of X-valued continuous functions on $[0, T]$ with the norm $\|u\|_\infty = \sup\{\|u(t)\| \ : \ t \in [0, T]\}$, and given $z \in X$ a set $W_r(z) := \{u \in C(0, T; X) : \ \|u - z\|_\infty \leq r\}$. Finally, denote by $L^1(0, T; X)$ the space of X-valued Bochner integrable function on $[0, T]$ with the norm $\|u\|_1 = \int_0^T \|u(t)\| dt$.

If $x \in X$, we shall denote by $J(x)$ the normalized duality mapping at x defined by $J(x) := \{j \in X^* : \ j(x) = \|x\|^2, \ \|j\| = \|x\|\}$. We shall often use the mappings $\langle \cdot, \cdot \rangle_+, \langle \cdot, \cdot \rangle_- : X \times X \to \mathbb{R}$ defined by $\langle y, x \rangle_+ := \max\{j(y) : \ j \in J(x)\}$ and $\langle y, x \rangle_- := \min\{j(y) : \ j \in J(x)\}$.

Notice that if $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space, then $\langle \cdot, \cdot \rangle_+ = \langle \cdot, \cdot \rangle_- = \langle \cdot, \cdot \rangle$.

A mapping $A : D(A) \subseteq X \to 2^X$ will be called an operator on X. The domain of A is denoted by $D(A)$ and its range by $\mathcal{R}(A)$. An operator A on X is said to be *accretive* if and only if, $\langle u - v, x - y \rangle_+ \geq 0$ for every $(x, u), (y, v) \in A$.

If, in addition, $\mathcal{R}(I + \lambda A)$, is for one, then for all, $\lambda > 0$, precisely X, hence A is called m-accretive. We say that A satisfies the range condition if $\overline{D(A)} \subseteq \bigcap_{\lambda > 0} \mathcal{R}(I + \lambda A)$. Accretive operators were introduced by F.E. Browder [9] and T. Kato [24] independently. In this sense, it is interesting to notice that the concept of m-accretive operator and the concept of maximal monotone operator coincide on Hilbert spaces (see, for instance [10]). Those accretive operators which have the range condition play an important role in the study of nonlinear partial differential equations.

If $f \in L^1(0, T, X)$ and we consider the Cauchy problem

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \ t \in (0, T), \\ u(0) = x_0 \in \overline{D(A)}, \end{cases} \tag{2.5}$$

where A is accretive with the range condition on X. It is well known that (2.5) has a unique integral solution in the sense of Bénilan [6] *i.e.,* there exists a unique continuous function $u : [0, T] \to \overline{D(A)}$ such that $u(0) = x_0$, and moreover, for each $(x, y) \in A$ and $0 \le s \le t \le T$, we have

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \le 2 \int_s^t \langle f(\tau) - y, u(\tau) - x\rangle_+ d\tau. \tag{2.6}$$

It is also well known that (2.6) yields the inequality

$$\|u(t) - x\| \le \|x_0 - x\| + \int_0^t \|f(s) - y\| ds,$$

for all $(x, y) \in A$ and $0 \le t \le T$.

If u, v are integral solutions of $u'(t) + A(u(t)) \ni f(t)$ and $v'(t) + A(v(t)) \ni g(t)$, respectively, with $f, g \in L^1(0, T, X)$, then

- $\|u(t) - v(t)\| \le \|u(0) - v(0)\| + \int_0^t \|f(s) - g(s)\| ds$.
- $\|u(t) - v(t)\| \le \|u(s) - v(s)\| + \int_s^t \langle f(\tau) - g(\tau), \frac{u(\tau)-v(\tau)}{\|u(\tau)-v(\tau)\|}\rangle_+ d\tau$.

A *strong solution* of Problem (2.5) is a function $u \in W^{1,\infty}(0, T; X)$, *i.e.*, u is a locally absolutely continuous and differentiable almost everywhere, such that $u'(t) + A(u(t)) \ni f(t)$ for almost all $t \in [0, T]$ and $u(0) = x_0$.

Concerning the existence of strong solutions, the following theorem is known (see [7] and page 133 of [4]).

Theorem 2.4. *If X is a Banach space with the Radon-Nikodym property, $A : D(A) \subseteq X \to 2^X$ is an m-accretive operator, and $f \in BV(0, T; X)$, i.e., f is a function of bounded variation on $[0, T]$, then Problem (2.5) has a unique strong solution whenever $x_0 \in D(A)$.*

When we work on a Hilbert space the above results can be summarized in the following theorem:

Corollary 2.5. *Let H be a real Hilbert space, $A : D(A) \subseteq H \to 2^H$ is a maximal monotone operator, and $f \in BV(0, T; X)$, i.e., f is a function of bounded variation on $[0, T]$, then Problem (2.5) has a unique strong solution whenever $x_0 \in D(A)$.*

We refer the reader to [4, 7, 10, 32], for background material on accretivity.

We are also going to work with Bellman's inequality (see [32]).

Lemma 2.6. *If $x : [t_0, T] \to \mathbb{R}$ is a continuous function, $x_0 \in \mathbb{R}$, $k \in L^1_{loc}(t_0, T; \mathbb{R}^+)$, and*

$$x(t) \le x_0 + \int_{t_0}^t k(s)x(s)ds$$

for each $t \in [t_0, T[$, then

$$x(t) \le x_0 e^{\int_{t_0}^t k(s)ds}$$

for all $t \in [t_0, T[$.

We now go to fixed point theory.

Definition 2.7. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{B}(X)$ the family of bounded subsets of X. By a measure of non-compactness on X, we mean a function $\Phi : \mathcal{B}(X) \to \mathbb{R}^+$ satisfying:

(1) $\Phi(\Omega) = 0$ if and only if Ω is relatively compact in X
(2) $\Phi(\overline{\Omega}) = \Phi(\Omega)$,
(3) $\Phi(\text{conv}(\Omega)) = \Phi(\Omega)$, for all bounded subsets $\Omega \in \mathcal{B}(X)$, where conv denotes the convex hull of Ω
(4) for any subsets $\Omega_1, \Omega_2 \in \mathcal{B}(E)$ we have
$$\Omega_1 \subseteq \Omega_2 \Longrightarrow \Phi(\Omega_1) \leq \Phi(\Omega_2),$$
(5) $\Phi(\Omega_1 \cup \Omega_2) = \max\{\Phi(\Omega_1), \Phi(\Omega_2)\}$, $\Omega_1, \Omega_2 \in \mathcal{B}(X)$,
(6) $\Phi(\lambda\Omega) = |\lambda|\Phi(\Omega)$ for all $\lambda \in \mathbb{R}$ and $\Omega \in \mathcal{B}(X)$,
(7) $\Phi(\Omega_1 + \Omega_2) \leq \Phi(\Omega_1) + \Phi(\Omega_2)$.

The most important examples of measures of noncompactness are the *Kuratowski measure of noncompactness* (or set measure of noncompactness)

$$\alpha(\Omega) = \inf\{r > 0 \,:\, \Omega \text{ may be covered by finitely many sets of diameter} \leq r\},$$

and the *Hausdorff measure of noncompactness* (or ball measure of noncompactness)

$$\beta(\Omega) = \inf\{r > 0 \,:\, \text{ there exists a finite } r\text{-net for } \Omega \text{ in } X\}.$$

A detailed account of theory and applications of measures of noncompactness may be found in the monographs [1, 3] (see also [2]).

Definition 2.8. Let Φ be a measure of non-compactness on X and let D be a nonempty subset of X. A mapping $T : D \to X$ is said to be a $\Phi - k$-set contraction, $k \in (0, 1]$, if T is continuous and if, for all bounded subsets C of D, $\Phi(T(C)) \leq k\Phi(C)$. T is said to be Φ-condensing if T is continuous and $\Phi(T(A)) < \Phi(A)$ for every bounded subset A of D with $\Phi(A) > 0$.

The following theorems will be the key in the proof of some of our results. The first one was proved by Sadowskii [30] in 1967. In 1955 Darbo [12] proved the same result for $\Phi - k$-set contractions, $k < 1$. Such mappings are obviously Φ-condensing. The second one is a sharpening of the first one and it is due to W.V. Petryshyn [29].

Theorem 2.9 (Darbo-Sadovskii)**.** *Suppose M is a nonempty bounded closed and convex subset of a Banach space X and suppose $T : M \to M$ is Φ-condensing. Then T has a fixed point.*

Theorem 2.10 (Petryshyn)**.** *Let C be a closed, convex subset of a Banach space X such that $0 \in C$. Consider $T : C \to C$ a Φ-condensing mapping. If there exists $r > 0$ such that $Tx \neq \lambda x$ for any $\lambda > 1$ whenever $x \in C \cap S_r(0)$. Then T has a fixed point in C.*

- A mapping $T : D(T) \subseteq X \to X$ is said to be nonexpansive if the inequality $\|T(x) - T(y)\| \leq \|x - y\|$ holds for every $x, y \in D(T)$. Recall that a Banach space X is said to have the fixed point property for nonexpansive mappings

(FPP for short) if for each nonempty bounded closed and convex subset C of X, every nonexpansive self-mapping T has a fixed point (see [21, 25]).

- The mapping T is said to be pseudocontractive if for every $x, y \in D(T)$ and for all $r > 0$, the inequality

$$\|x - y\| \leq \|(1+r)(x-y) + r(Ty - Tx)\|$$

holds. Pseudocontractive mappings are easily seen to be more general than nonexpansive ones. The interest in these mappings also stems from the fact that they are firmly connected to the well known class of accretive mappings. Specifically, T is pseudocontractive if and only if $I - T$ is accretive, where I is the identity mapping.

We say that the mapping $T : D(T) \to X$ is weakly inward on $D(T)$ if

$$\lim_{\lambda \to 0^+} d((1-\lambda)x + \lambda T(x), D(T)) = 0,$$

for all $x \in D(T)$. Such condition is always weaker than the assumption of T mapping the boundary of $D(T)$ into $D(T)$. Recall that if $A : D(A) \to X$ is a continuous accretive mapping, $D(A)$ is convex and closed and $I - A$ is weakly inward on $D(A)$, then A has the range condition, (see [27]).

3. Fixed point theory on unbounded domains

Let C be a nonempty subset of X and let $T : C \to X$ be a mapping. A sequence (x_n) in C is said to be an almost fixed point sequence for T whenever $\lim_{n\to\infty} \|T(x_n) - x_n\| = 0$.

In [19] the authors considered several fixed point results for nonlinear mappings with unbounded domains in terms of a function $G : X \times X \to \mathbb{R}$ under the following assumptions:

(g1) $G(\lambda x, y) \leq \lambda G(x, y)$ for every $x, y \in X$ and every $\lambda > 0$,
(g2) there exists $R > 0$ such that $G(x, x) > 0$ for any $x \in X$ with $\|x\| \geq R$,
(g3) $G(x + y, z) \leq G(x, z) + G(y, z)$ for any $x, y, z \in X$,
(g4) for each $y \in X$ there exists $t > 0$ (depending on y) such that if $\|x\| \geq t$ then $|G(y, x)| < G(x, x)$.

Next, we shall show that assumption (g1) along with (g3) implies the following property

(g1') $G(\lambda x, y) = \lambda G(x, y)$ for every $x, y \in X$ and every $\lambda > 0$.

Lemma 3.1. *Let X be a real Banach space and suppose $G : X \times X \to \mathbb{R}$ satisfies conditions (g1) and (g3). Then G satisfies the condition (g1').*

Proof. Let $x, y \in X$ and $\lambda > 0$. First we suppose $0 < \lambda \leq 1$. We have that

$$G(x, y) \leq G(\lambda x, y) + G\big((1-\lambda)x, y\big) \leq \lambda G(x, y) + (1-\lambda)G(x, y) = G(x, y)$$

and thus $G(x, y) = G(\lambda x, y) + G\big((1-\lambda)x, y\big)$, that is

$$G(\lambda x, y) = G(x, y) - G\big((1-\lambda)x, y\big) \tag{3.1}$$

On the other hand, since $G\big((1-\lambda)x, y\big) \le (1-\lambda)G(x,y)$ we obtain that $-G\big((1-\lambda)x, y\big) \ge -(1-\lambda)G(x,y)$. By this and by (3.1) we get that

$$G(\lambda x, y) = G(x,y) - G\big((1-\lambda)x, y\big) \ge G(x,y) - (1-\lambda)G(x,y) = \lambda G(x,y).$$

Consequently we conclude that $G(\lambda x, y) = \lambda G(x,y)$. Now we suppose $\lambda > 1$. Define $\mu = 1/\lambda$ and $z = \lambda x$. By the above we get that $G(\mu z, y) = \mu G(z,y)$, that is, $G(x,y) = (1/\lambda)G(\lambda x, y)$ and then $\lambda G(x,y) = G\big(\lambda x, y\big)$. □

Proposition 3.2. *Let C be a closed convex and unbounded subset of a Banach space X and let $T : C \to C$ be an Φ-condensing mapping. Suppose there exist $R > 0$ and $G : X \times X \to \mathbb{R}$ satisfying conditions (g1') and*

(g2') $G(x,x) > 0$ *for any* $x \in S_R(0)$.

If there exists $x_0 \in C$ such that $G(T(x) - x_0, x - x_0) \le G(x - x_0, x - x_0)$ for all $x \in C \cap S_R(x_0)$ then T has a fixed point.

Proof. Let $F : C \cap B_R(x_0) \to C \cap B_R(x_0)$ be the mapping given by

$$F(x) = \begin{cases} T(x), & \text{if } \ \|T(x) - x_0\| \le R, \\ \frac{R}{\|T(x)-x_0\|}\, T(x) + \big(1 - \frac{R}{\|T(x)-x_0\|}\big)\, x_0, & \text{if } \ \|T(x) - x_0\| > R. \end{cases}$$

Clearly F is continuous. Let K be a subset of $C \cap B_R(x_0)$ such that $\Phi(K) > 0$. Define $K_1 = \{x \in K : \|T(x) - x_0\| \le R\}$ and $K_2 = \{x \in K : \|T(x) - x_0\| > R\}$. It is clear that $F(K_1) = T(K_1)$ and thus $\Phi\big(F(K_1)\big) = \Phi\big(T(K_1)\big)$. Since for every $x \in K_2$

$$F(x) = \frac{R}{\|T(x) - x_0\|}\, T(x) + \left(1 - \frac{R}{\|T(x) - x_0\|}\right) x_0 \in \mathit{conv}\big(T(K_2) \cup \{x_0\}\big)$$

we have $F(K_2) \subset \mathit{conv}\big(T(K_2) \cup \{x_0\}\big)$. Hence $\Phi\big(F(K_2)\big) \le \Phi\big(T(K_2)\big)$ and then

$$\begin{aligned}\Phi\big(F(K)\big) &= \Phi\big(F(K_1) \cup F(K_2)\big) = \max\big\{\Phi\big(F(K_1)\big), \Phi\big(F(K_2)\big)\big\} \\ &\le \max\big\{\Phi\big(T(K_1)\big), \Phi\big(T(K_2)\big)\big\} < \max\big\{\Phi(K_1), \Phi(K_2)\big\} = \Phi(K).\end{aligned}$$

Consequently F is Φ-condensing and thus, by Theorem 2.9, there exists $x_1 \in C \cap B_R(x_0)$ such that $F(x_1) = x_1$. Now we will prove that $\|T(x_1) - x_0\| \le R$. Assume that $\|T(x_1) - x_0\| > R$. We have that

$$x_1 = F(x_1) = \frac{R}{\|T(x_1) - x_0\|} T(x_1) + \left(1 - \frac{R}{\|T(x_1) - x_0\|}\right) x_0$$

and then $x_1 - x_0 = \frac{R}{\|T(x_1)-x_0\|}(T(x_1) - x_0)$. Consequently $x_1 \in C \cap S_R(x_0)$ and thus

$$\begin{aligned} G(x_1 - x_0, x_1 - x_0) \ &\ge G\big(T(x_1) - x_0, x_1 - x_0\big) \\ &= G\left(\tfrac{\|T(x_1)-x_0\|}{R}(x_1 - x_0), x_1 - x_0\right) \\ &= \tfrac{\|T(x_1)-x_0\|}{R} G(x_1 - x_0, x_1 - x_0) \\ &> G(x_1 - x_0, x_1 - x_0) \end{aligned}$$

which is not possible. Then $\|T(x_1) - x_0\| \le R$ and consequently $T(x_0) = F(x_0) = x_0$. □

The relationship between the above result and Theorem 2.10 can be seen in the following result.

Lemma 3.3. *Let C be a subset of a Banach space X and let $T : C \to C$ be a mapping. The following statements are equivalent:*

(a) *There exist $R > 0$, $x_0 \in C$ and $G : X \times X \to \mathbb{R}$ satisfying conditions (g1') and (g2') such that $G(T(x) - x_0, x - x_0) \leq G(x - x_0, x - x_0)$ for all $x \in C \cap S_R(x_0)$.*

(b) *There exist $R > 0$ and $x_0 \in C$ such that $T(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for all $x \in C \cap S_R(x_0)$.*

Proof. First we will prove (a) $\Rightarrow$ (b). Let $R > 0$, x_0 and G be as condition (a) and suppose there exist $x \in C \bigcap S_R(x_0)$ and $\lambda > 1$ such that $T(x) - x_0 = \lambda(x - x_0)$. Since G satisfies conditions (g1') and (g2') we have $G\big(T(x) - x_0, x - x_0\big) = G(\lambda(x - x_0), x - x_0) = \lambda G(x - x_0, x - x_0) > G(x - x_0, x - x_0)$, which is a contradiction. To prove (b) $\Rightarrow$ (a) define $G : X \times X \to \mathbb{R}$ by

$$G(x, y) = \begin{cases} \lambda, & \text{if } x = \lambda y \text{ for some } \lambda > 0 \text{ and } y \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

It is easy to see that G satisfies conditions (g1') and (g2') (in fact G also satisfies conditions (g1)-g(4)). Clearly for every $x \in C \cap S_R(x_0)$ we have that $G\big(T(x) - x_0, x - x_0\big) = 0$ or $G\big(T(x) - x_0, x - x_0\big) = \lambda$ for some $\lambda \leq 1$ and thus $G\big(T(x) - x_0, x - x_0\big) \leq 1 = G(x - x_0, x - x_0)$ for every $x \in C \cap S_R(x_0)$. ☐

Proposition 3.4. *Let C be a closed convex and unbounded subset of a Banach space X and let $T : C \to X$ be a weakly inward on C continuous pseudocontractive mapping. Suppose there exist $R > 0$ and $G : X \times X \to \mathbb{R}$ satisfying conditions (g1)-(g4). If $G(T(x), x) \leq G(x, x)$ for all $x \in C$ with $\|x\| \geq R$ then T has a bounded almost fixed point sequence.*

Proof. Let $A : C \to X$ be the mapping $A = I - T$. Since C is convex and closed and T is a continuous pseudocontractive mapping weakly inward on C, then A is a continuous accretive with the range condition, that is, $C \subset \bigcap_{\lambda > 0} R(I + \lambda A)$ (see [27]). Fix $\xi_0 \in C$. For each positive integer n there exists $x_n \in C$ such that $\xi_0 = x_n + nA(x_n)$. We will prove that (x_n) is a bounded sequence. Indeed, otherwise there exists a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| \to \infty$. By (g4) we have that there exists $t > 0$ such that if $\|x\| \geq t$ then $|G(\xi_0, x)| < G(x, x)$. Without loss of generality we may assume that $\|x_{n_k}\| \geq \max\{R, t\}$ for all positive integer k. It is easy to see that

$$x_{n_k} = \frac{n_k}{n_k + 1}\big(x_{n_k} - A(x_{n_k})\big) + \frac{1}{n_k + 1}\xi_0$$

and then

$$\begin{aligned} 0 < G(x_{n_k}, x_{n_k}) &= G\big(\tfrac{n_k}{n_k+1}\big(x_{n_k} - A(x_{n_k})\big) + \tfrac{1}{n_k+1}\xi_0, x_{n_k}\big) \\ &\leq \tfrac{n_k}{n_k+1} G\big(x_{n_k} - A(x_{n_k}), x_{n_k}\big) + \tfrac{1}{n_k+1} G(\xi_0, x_{n_k}) \\ &= \tfrac{n_k}{n_k+1} G\big(T(x_{n_k}), x_{n_k}\big) + \tfrac{1}{n_k+1} G(\xi_0, x_{n_k}) \\ &< \tfrac{n_k}{n_k+1} G\big(x_{n_k}, x_{n_k}\big) + \tfrac{1}{n_k+1} G(x_{n_k}, x_{n_k}) = G(x_{n_k}, x_{n_k}). \end{aligned}$$

This is a contradiction which proves our claim. Finally, since $A(x_n) = \frac{1}{n}(\xi_0 - x_n)$ we get that $\|A(x_n)\| \to 0$, that is, $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$. ☐

Proposition 3.5. *Let C be a closed convex and unbounded subset of a Banach space X and let $T : C \to X$ be a weakly inward on C continuous pseudocontractive mapping. Suppose there exist $R > 0$, $x_0 \in C$ and $G : X \times X \to \mathbb{R}$ satisfying conditions (g1) and (g2') such that $G(T(x) - x_0, x - x_0) \leq G(x - x_0, x - x_0)$ for all $x \in C \cap S_R(x_0)$. Then T has a bounded almost fixed point sequence.*

Proof. Let $A = I - T$ as in Proposition 3.4 and for each $\lambda > 0$ let $x_\lambda \in C$ be such that $x_0 = x_\lambda + \lambda A(x_\lambda)$. Denoting the resolvent of A by $J_\lambda := (I + \lambda A)^{-1}$ we will prove that $\{J_\lambda(x_0) : \lambda > 0\}$ is bounded. Indeed, otherwise there exists $\lambda_1 > 0$ such that $\|J_{\lambda_1}(x_0) - x_0\| > R$. Since for every $x \in \bigcap_{\lambda>0} R(I + \lambda A)$ the mapping $\lambda \to J_\lambda x$ is continuous we have that the mapping $f : [0, +\infty[\to [0, +\infty[$ given by $f(\lambda) = \|J_\lambda(x_0) - x_0\|$ is continuous. Since $f(0) = 0$ and $f(\lambda_1) > R$ we obtain that there exists $\lambda_2 \in [0, +\infty[$ such that $\|J_{\lambda_2}(x_0) - x_0\| = R$, that is, $\|x_{\lambda_2} - x_0\| = R$. Since $x_{\lambda_2} = \frac{\lambda_2}{\lambda_2+1}\big(x_{\lambda_2} - A(x_{\lambda_2})\big) + \frac{1}{\lambda_2+1}x_0$ we get that

$$\begin{aligned} 0 < G(x_{\lambda_2} - x_0, x_{\lambda_2} - x_0) \quad &= G\big(\tfrac{\lambda_2}{\lambda_2+1}\big(x_{\lambda_2} - A(x_{\lambda_2}) - x_0\big), x_{\lambda_2} - x_0\big) \\ &\leq \tfrac{\lambda_2}{\lambda_2+1} G\big(T(x_{\lambda_2}) - x_0, x_{\lambda_2} - x_0\big) \\ &\leq \tfrac{\lambda_2}{\lambda_2+1} G\big(x_{\lambda_2} - x_0, x_{\lambda_2} - x_0\big) < G(x_{\lambda_2} - x_0, x_{\lambda_2} - x_0). \end{aligned}$$

This is a contradiction which proves our claim. Since $x_n = J_n(x_0)$ for each positive integer n, we obtain by the above that (x_n) is a bounded sequence. Finally, since $A(x_n) = \frac{1}{n}(x_0 - x_n)$ we get that $\|A(x_n)\| \to 0$, that is, $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$. □

Lemma 3.6. *Let (w_n) be a bounded sequence in a Banach space X. Let $G : X \times X \to \mathbb{R}$ be the mapping defined by $G(x, y) = \limsup_n \langle x, y - w_n\rangle_+$ for $x, y \in X$. Then G satisfies conditions (g1)-(g4).*

Proof. It is clear that G satisfies conditions (g1) and (g3). Let $x \in X$. For each positive integer n there exists $j_n \in J(x - w_n)$ such that $\langle x, x - w_n\rangle_+ = j_n(x)$. Since $j_n(-w_n) \leq \|j_n\|\|w_n\| = \|x - w_n\|\|w_n\|$ we have that

$$\langle x, x - w_n\rangle_+ = j_n(x) = j_n(x - w_n) + j_n(w_n) \geq \|x - w_n\|^2 - \|x - w_n\|\|w_n\|.$$

Let $M > 0$ be such that $\|w_n\| \leq M$. We have for every n that $\|x - w_n\|(\|x - w_n\| - \|w_n\|) \geq (\|x\| - M)(\|x\| - 2M)$. Let $R > 2M$. We obtain that $\langle x, x - w_n\rangle_+ \geq (R - M)(R - 2M) > 0$ for every $x \in X$ with $\|x\| \geq R$ and consequently G satisfies (g2). Now we will prove (g4). Let $y \in X$ and define $t = \|y\| + 2M$. Since for each positive integer n we have that $\|x - w_n\| - \|w_n\| \geq \|x\| - 2M \geq \|y\|$ for every $x \in X$ with $\|x\| \geq t$, we get that

$$|\langle y, x - w_n\rangle_+| \leq \|y\|\|x - w_n\| \leq (\|x - w_n\| - \|w_n\|)\|x - w_n\| \leq \langle x, x - w_n\rangle_+$$

and thus $|G(y, x)| \leq G(x, x)$ for every $x \in X$ with $\|x\| \geq t$. □

Definition 3.7. We say that a mapping $T : D(T) \to X$ is strong pseudocontractive if for all $x, y \in C$ we have that

$$\langle\, (I - T)(x) - (I - T)(y), x - y\,\rangle_- \geq 0.$$

It is clear that every nonexpansive mapping is strong pseudocontractive and every strong pseudocontractive mapping is pseudocontractive. If X is a smooth Banach space then $\langle\cdot,\cdot\rangle_+ = \langle\cdot,\cdot\rangle_-$ and consequently strong pseudocontractive mappings and pseudocontractive mappings defined in X coincide.

Proposition 3.8. *Let C be a closed convex and unbounded subset of a Banach space X and let $T : C \to X$ be a strong pseudocontractive mapping with a bounded almost fixed point sequence (x_n) in C. Then for each $x_0 \in C$ there exist $R > 0$ and a mapping $G : X \times X \to \mathbb{R}$ satisfying conditions (g1)-(g4) such that $G(T(x) - x_0, x - x_0) \leq G(x - x_0, x - x_0)$ for all $x \in C$.*

Proof. Let $x_0 \in C$ and define $G : X \times X \to \mathbb{R}$ by $G(x, y) = \limsup_n \langle x, y + x_0 - x_n\rangle_+$ for $x, y \in X$. Taking $w_n = x_n - x_0$ in Lemma 3.6 we have that G satisfies conditions (g1)-(g4). Let $x \in C$. For each positive integer n there exists $j_n \in J(x - x_n)$ such that $\langle T(x) - x_0, x - x_n\rangle_+ = j_n\big(T(x) - x_0\big)$. We have that

$$\begin{aligned}
\langle T(x) - x_0, x - x_n\rangle_+ &= j_n\big(T(x) - x_0\big)\\
&= j_n\big(T(x) - T(x_n)\big) + j_n\big(T(x_n) - x_0\big)\\
&\leq j_n(x - x_n) + j_n\big(T(x_n) - x_0\big)\\
&= j_n\big(T(x_n) - x_n\big) + j_n(x - x_0)\\
&\leq \|j_n\|\|T(x_n) - x_n\| + \langle x - x_0, x - x_n\rangle_+\\
&= \|x - x_n\|\|T(x_n) - x_n\| + \langle x - x_0, x - x_n\rangle_+
\end{aligned}$$

and then

$$\begin{aligned}
G\big(T(x) - x_0, x - x_0\big) &= \limsup_n \langle T(x) - x_0, x - x_n\rangle_+\\
&\leq \limsup_n \langle x - x_0, x - x_n\rangle_+\\
&= G(x - x_0, x - x_0).
\end{aligned}$$

□

Summarizing the above results on pseudocontractive mappings, we have.

Theorem 3.9. *Let C be a closed convex and unbounded subset of a Banach space X and let $T : C \to X$ be a continuous strong pseudocontractive mapping weakly inward on C. The following statements are equivalent:*

(i) *There exists a bounded almost fixed point sequence for T in C.*
(ii) *For each $x_0 \in C$ there exist $R > 0$ and a mapping $G : X \times X \to \mathbb{R}$ satisfying conditions (g1)-(g4) such that $G(T(x) - x_0, x - x_0) \leq G(x - x_0, x - x_0)$ for all $x \in C$.*
(iii) *There exist $x_0 \in C$, $R > 0$ and a mapping $G : X \times X \to \mathbb{R}$ satisfying conditions (g1') and (g2') such that $G(T(x) - x_0, x - x_0) \leq G(x - x_0, x - x_0)$ for all $x \in C \cap S_R(x_0)$.*
(iv) *There exist $x_0 \in C$ and $R > 0$ such that $T(x) - x_0 \neq \lambda(x - x_0)$ for all $\lambda > 1$ and for all $x \in C \cap S_R(x_0)$.*

Remark 3.10. At this point it is interesting to notice that Theorem 3.9 says that, in some sense, Leray-Schauder's condition is the best one to guarantee the existence of a bounded almost fixed point sequence, among those which come defined by a function G satisfying either (g1)–(g4) or (g1')–(g2').

On the other hand, if X is a Banach space with the (FPP) then [17, Theorem 4.3] guarantees that if C is a nonempty closed convex subset of X and $T : C \to X$ is a continuous pseudocontractive mapping weakly inward on C and T admits a bounded almost fixed point sequence then T has a fixed point in C. This comment along with Theorem 3.9 allows us to obtain the relationship between the results showed here and those given in [19](for instance see [19, Theorem 4.1]).

4. Existence

Let D be a nonempty closed convex subset of real Hilbert space H. The function $I_D : H \to [0, +\infty]$ defined by

$$I_D(x) := \begin{cases} 0, \text{ if } x \in D, \\ +\infty \text{ if } x \in H \setminus D, \end{cases}$$

is called the indicator function of D. In [5], it is really seen that I_D is proper convex lower semi continuous function and its subdifferential $\partial I_D : H \to 2^H$ given by

$$\partial I_D(x) = \{\xi \in H : \langle \xi, y - x\rangle \leq I_D(y) - I_D(x), \text{ for all } y \in H\},$$

is clearly a maximal monotone operator on H where its effective domain is $D(\partial I_D) = D$. Moreover, it is easy to see that

$$\partial I_D(x) = N_D(x) \text{ for every } x \in D.$$

The above argument shows that differential inclusion (2.4) can be seen as follows:

$$\begin{cases} u'(t) + \partial I_D(u(t)) \ni -f(u(t)), \ t \in (0, T), \\ u(0) = x_0 \in D, \end{cases} \tag{4.1}$$

Theorem 4.1. *Problem 4.1 has a unique strong solution whenever $f : D \to H$ is a Lipschitzian mapping.*

Proof. It is well known that Problem 4.1 has a unique integral solution (see [20, Lemma 3.1]). This fact along with [8, Theorem 3.6] yields that the integral solution obtained is a strong solution since given a function $u \in C(0, T; D)$ it is clear that $f(u(.)) \in L^2(0, T; H)$. □

Theorem 4.2. *Under the following assumptions:*

(1) *For any $l > 0$ there exists a number $k_l \in \mathbb{R}^+$ such that $\|f(x) - f(y)\| \leq k_l\|x - y\|$ for every $x, y \in B_l(0)$,*
(2) *There exist $z \in D$ and $r > \|x_0 - z\|$ such that $\langle f(x), x - z\rangle \geq 0$, for each $x \in D \cap S_r(z)$.*

Equation (4.1) has a unique strong solution.

Proof. Let $r > 0$ be as in assumption (2). Let us introduce the following function

$$\rho(x) = \begin{cases} x, \text{ if } \|x - z\| \leq r \\ \frac{r}{\|x-z\|}x + (1 - \frac{r}{\|x-z\|})z, \text{ if } \|x - z\| \geq r \end{cases}$$

By assumption (1) it is not difficult to see that the function $f(\rho(.))$ is $2k_v$-lipschitzian, where $v := r + \|z\|$.

Therefore, by Theorem 4.1, the equation

$$\begin{cases} u'(t) + \partial I_D(u(t)) \ni f(\rho(u(t))), \ t \in (0, T) \\ u(0) = x_0, \end{cases} \tag{4.2}$$

has a unique strong solution u. Let us see that u lies in $W_r(z)$.

If $u \notin W_r(z)$ since u is a continuous function and $u(0) = x_0$, there exists $0 \leq t_0 < T$ such that $\|u(t) - z\| > r$ for every $t \in (t_0, t_0 + \delta)$ and $\|u(t_0) - z\| \leq r$.

Since $z \in D$, and $0 \in \partial I_D(z)$ we have that for $t \in (t_0, t_0 + \delta)$

$$\begin{array}{ll} r^2 < \|u(t) - z\|^2 & \leq \|u(t_0) - z\|^2 + 2\int_{t_0}^{t} \langle -f(\rho(u(\tau)) - 0, u(\tau) - z\rangle d\tau \\ & \leq \|u(t_0) - z\|^2 - 2\int_{t_0}^{t} \frac{\|u(\tau)-z\|}{r}\langle f(\rho(u(\tau)), \rho(u(\tau)) - z\rangle d\tau \\ & \leq \|u(t_0) - z\|^2 \\ & \leq r^2 \end{array},$$

which is a contradiction, thus $u \in W_r(z)$ and therefore $\rho(u(t)) = u(t)$ for all $t \in [0, T]$. This means that u is, in fact, a strong solution of Eq. (4.1).

Finally, let us see the uniqueness. Suppose that u, v are two integral solutions of Eq.(4.1). Since both functions are continuous there exists $\sigma > 0$ such that $u, v \in W_\sigma(0)$. Therefore,

$$\|u(t) - v(t)\| \leq \int_0^t \|f(u(\tau)) - f(v(\tau))\| d\tau \leq \int_0^t k_\sigma \|u(\tau) - v(\tau)\| d\tau,$$

Now, in order to obtain the uniqueness it is enough to apply Bellman's inequality. □

Remark 4.3. In Theorems 4.1 and 4.2 has been proved that Eq.(4.1) admits a unique strong solution. It is not difficult to see that such solution can be extended to $[0, +\infty)$. On the other hand, if in Theorem 4.2 we assume that there exist $z \in D$ and $r > 0$ such that $\langle f(x), x - z\rangle \geq 0$, for each $x \in D$ with $\|x - z\| \geq r$, then Problem (4.1) admits a unique strong solution for any intial date $x_0 \in D$.

Theorem 4.4. *Equation (4.1) has a unique strong solution whenever $f : D \to H$ is a monotone mapping with the range condition.*

Proof. Since ∂I_D is an m-accretive operator on H, it is clear that $A := \partial I_D + f$ is an accretive operator. Let us see that A satisfies the range condition. Indeed, let $x \in D$, since $f : D \to H$ has the range condition, there exists $y \in D$ such that $x = y + f(y)$. Moreover, since $0 \in \partial I_D(y)$then we infer that $x \in (I + A)(y)$. Now we can rewritten the Eq. (4.1) as follows:

$$\left\{ \begin{array}{l} u'(t) + A(u(t)) \ni 0, \ t \in (0, T), \\ u(0) = x_0 \in D, \end{array} \right. \tag{4.3}$$

Finally, since A is accretive with the range condition and we are working in a Hilbert space H, it is clear that the above problem admits a unique strong solution for $t \in (0, +\infty)$. □

Next, we will study under what conditions the Problem $VI(f, D)$ has a solution.

Theorem 4.5. *The variational inequality $VI(f, D)$ has a solution whenever $f : D \to H$ is monotone mapping with the range condition and there exist $R > 0$ and $z \in D$ such that $\langle f(x), x - z\rangle \geq 0$ for all $x \in D \cap S_R(z)$.*

Proof. By Proposition 2.2 it is enough to see that $0 \in \mathcal{R}(\partial I_D + f)$. In the proof of Theorem 4.4 we have seen that $A := \partial I_D + f$ is an accretive operator with the range condition on the Hilbert space H.

First, we will see that

$$\sup_{y \in A(x)} \langle x - y, x - z \rangle \leq \langle x, x - z \rangle \text{ whenever } x \in D \cap S_R(z), \tag{4.4}$$

holds.

In order to show Inequality (4.4), consider $x \in D \cap S_R(z)$ and let $\xi \in \partial I_D(x)$ such that $y = \xi + f(x)$. Then,

$$\langle x - y, x - z \rangle = \langle x - \xi - f(x), x - z \rangle,$$

Since $z \in D$ and $\xi \in \partial I_D(x)$ we have that $\langle \xi, z - x \rangle \leq 0$. Moreover, by hypothesis we know that $\langle f(x), x - z \rangle \geq 0$ for all $x \in D \cap S_R(z)$. Consequently,

$$\langle x - y, x - z \rangle = \langle x, x - z \rangle + \langle \xi, z - x \rangle - \langle f(x), x - z \rangle \leq \langle x, x - z \rangle,$$

this allows us to obtain our claim.

Second, since A is an accretive operator with the range condition, we know that $g := (I + A)^{-1} : D \to D$ is a nonexpansive mapping. Let us see that g has an almost fixed point sequence. Indeed,

Since $z \in D$, for each $\lambda > 0$ there exists $x_\lambda \in D$ and $y_\lambda \in A(x_\lambda)$ such that

$$z = x_\lambda + \lambda y_\lambda.$$

Hence, $z = (1 + \lambda)x_\lambda + \lambda(y_\lambda - x_\lambda)$. Consequently,

$$x_\lambda - y_\lambda = \frac{\lambda + 1}{\lambda} x_\lambda - \frac{1}{\lambda} z.$$

We claim that $\{x_\lambda : \ \lambda > 0\}$ is a bounded set. Otherwise, we can assume that there exists λ_1 such that $\|x_{\lambda_1} - z\| > R$. Since the function $\lambda \to (I + \lambda A)^{-1} z = x_\lambda$ is continuous, there exists $\lambda_0 \in (0, \lambda_1)$ such that $\|x_{\lambda_0} - z\| = R$. Therefore, we have

$$\begin{aligned} \langle x_{\lambda_0}, x_{\lambda_0} - z \rangle &= \langle \tfrac{\lambda_0}{1+\lambda_0}(x_{\lambda_0} - y_{\lambda_0}) + \tfrac{1}{1+\lambda_0} z, x_{\lambda_0} - z \rangle \\ &\leq \tfrac{\lambda_0}{1+\lambda_0} \langle x_{\lambda_0} - y_{\lambda_0}, x_{\lambda_0} - z \rangle + \tfrac{1}{\lambda_0 + 1} \langle z, x_{\lambda_0} - z \rangle. \end{aligned} \tag{4.5}$$

On the other hand, since $R^2 = \langle x_{\lambda_0} - z, x_{\lambda_0} - z \rangle$, we obtain that $\langle z, x_{\lambda_0} - z \rangle < \langle x_{\lambda_0}, x_{\lambda_0} - z \rangle$. Consequently

$$\langle x_{\lambda_0}, x_{\lambda_0} - z \rangle < \left(\frac{\lambda_0}{1 + \lambda_0} \langle x_{\lambda_0}, x_{\lambda_0} - z \rangle + \frac{1}{\lambda_0 + 1} \langle x_{\lambda_0}, x_{\lambda_0} - z \rangle \right) = \langle x_{\lambda_0}, x_{\lambda_0} - z \rangle \tag{4.6}$$

This is a contradiction which proves our claim.

Now, consider $\lambda = n \in \mathbb{N}$, then the sequence (x_n) is a bounded sequence, it is clear that (y_n) goes to 0 as n goes to infinity. Consider for each positive integer n, $w_n = x_n + y_n$. It is easy to see that $g(w_n) = x_n$ because $y_n \in A(x_n)$. In this case we obtain that

$$w_n - g(w_n) = y_n \to 0$$

This means that (w_n) is an almost fixed point sequence for g.

Finally, we may apply Remark 3.10 in order to conclude that g has a fixed point, which implies that $0 \in \mathcal{R}(A)$. $\square$

Remark 4.6. If $f : H \to H$ is a continuous monotone mapping, by [4, Theorem 3.2] we have that $A := \partial I_D + f$ is m-accretive on H. and thus, A has the range condition. Therefore, by theorem 4.4, we may conclude that Eq. (4.1) has also a unique strong solution and applying Theorem 4.5 that the variational inequality $VI(f, D)$ has a solution.

Definition 4.7. Let $\phi : H \to [0, +\infty)$ be a continuous function such that $\phi(0) = 0$, $\phi(x) > 0$ if $x \neq 0$ and which satisfies the following condition: For every sequence (x_n) in H such that $(\|x_n\|)$ is decreasing and $\phi(x_n) \to 0$ as $n \to \infty$ then $\|x_n\| \to 0$.

- An accretive operator $A : D(A) \subseteq H \to 2^H$ is said to be ϕ-accretive whenever there exists $z \in D(A)$ such that the inequality
$$\langle u, x - z\rangle \geq \phi(x - z), \quad \text{for all } (x, u) \in A \;\; \text{holds.}$$

Remark 4.8. In [18] was introduced the concept of operator ϕ-accretive at zero as follows: an accretive operator A is ϕ-accretive at zero if it satisfies the conditions of the above definition and $0 \in Az$. Using this concept in [18] was proved that if H is a Hilbert space and $A : D(A) \subseteq H \to 2^H$ is ϕ-accretive at zero with the range condition, then for each $x \in \overline{D(A)}$ the solution u_x of the problem

$$\begin{cases} u'(t) + A(u(t)) \ni 0, \; t \in (0, \infty), \\ u(0) = x \in \overline{D(A)}, \end{cases}$$

converges strongly to z as $t \to \infty$.

Theorem 4.9. *If f becomes ϕ-accretive with the range condition, then each strong solution of Problem 4.1 converges strongly as $t \to +\infty$ to the solution of $VI(f, D)$.*

Proof. Let z be the element in D such that $\langle f(x), x-z\rangle \geq \phi(x-z)$, for all $(x, f(x)) \in G(f)$. Let us see that $0 \in \partial I_D(z) + f(z)$. Since we are under the assumption of Theorem 4.5, there exists $w \in D$ such that $0 \in \partial I_D(w) + f(w)$.

First, we infer that $w = z$. Otherwise, since $-f(w) \in \partial I_D(w)$, we have

$$\begin{aligned} 0 &< \phi(w - z) \\ &\leq \langle f(w), w - z\rangle \\ &= \langle -f(w), z - w\rangle \\ &\leq 0. \end{aligned}$$

Which is a contradiction and therefore z is the unique element in D such that $0 \in \partial I_D(z) + f(z)$.

Second, we show that $\partial I_D + f$ is ϕ-accretive at zero. Indeed,

Consider $(x, u) \in \partial I_D + f$, this means that there exists $\xi \in \partial I_D(x)$ such that $u = \xi + f(x)$. Hence

$$\begin{aligned} \langle u, x - z\rangle &= \langle \xi + f(x), x - z\rangle \\ &= \langle \xi, x - z\rangle + \langle f(x), x - z\rangle \\ &\geq \langle f(x), x - z\rangle \\ &\geq \phi(x - z). \end{aligned}$$

Finally, since we know, by Theorem 4.4, that given a $x \in D$ there exists a unique strong solution $u_x : [0, +\infty) \to D$ of Problem 4.1. The above argument shows that $\partial I_D + f$ is ϕ-accretive at zero and thus we can apply [18, Theorem 8]. □

5. Periodic orbits

Consider Problem 4.1, when it admits a unique strong solution $u_{x_0}(\cdot) : [0, +\infty[\to D$. For $T > 0$ we can define the mapping $Q^T : D \to D$ by $Q^T(x) := u_x(T)$. Here, our stated goal, that of finding periodic orbits of a PDS, that is, to showing that Q^T has a fixed point for some $T > 0$.

Proposition 5.1. *Under condition of Theorem 4.2 the mapping $Q^T : D \to D$ is continuous.*

Proof. Let (x_k) be a sequence of elements of D such that $x_k \to x$ as $k \to \infty$. Since D is closed, then $x \in D$. Consider, $s := \max\{\|x_n\|, \|x\|, r\}$. Proof of Theorem 4.2 shows that if u_{x_n} and u_x are the solution of Problem 4.1 with initial datum x_n and x respectively, then $\|u_{x_n}(t)\| \vee \|u_x(t)\| \leq s$ for all $t \in [0, T]$. Then

$$\|u_{x_n}(T) - u_x(T)\| \leq \|x_n - x\| + \int_0^T \|f(u_{x_n}(\tau)) - f(u_x(\tau))\| d\tau.$$

Since f is locally lipschitzian, the above inequality yields

$$\|Q^T(x_n) - Q^T(x)\| = \|u_{x_n}(T) - u_x(T)\| \leq \|x_n - x\| + k_s \int_0^T \|u_{x_n}(\tau) - u_x(\tau)\| d\tau$$

Bellman's inequality deals

$$\|Q^T(x_n) - Q^T(x)\| \leq \|x_n - x\| e^{Tk_s},$$

which means that Q^T is a continuous mapping. □

Theorem 5.2. *Under condition of Theorem 4.2, if there exists $T > 0$ such that the mapping $Q^T : D \to D$ is Φ-condensing for some Φ measure of noncompactness, then Q^T has a fixed point.*

Proof. By hypothesis there exists $T > 0$ such that Q^T is Φ-condensing. We achieve the proof applying Proposition 3.2 and Lemma 3.3. Indeed, let us see that

$$Q^T(x) - z \neq \lambda(x - z) \quad \text{for all } \lambda > 1 \text{ and } x \in D \cap S_R(z). \tag{5.1}$$

Suppose that there exists $x \in D \cap S_R(z)$ such that Eq. (5.1) does not hold. In this case, there exists $\lambda > 1$ such that $u_x(T) - z = \lambda(x - z)$, which means, among other things, that $\|u_x(T) - z\| > R$.

Since $u_x(\cdot)$ is a continuous function and $\|u_x(0) - z\| = R$, we infer that there exists $\delta > 0$ such that $0 \leq T - \delta$ and $\|u(t) - z\| > R$ for every $t \in (T - \delta, T]$ and $\|u(T - \delta) - z\| \leq R$.

Consequently

$$\|u_x(T) - z\|^2 \leq \|u_x(T - \delta) - z\|^2 - 2 \int_{T-\delta}^T \langle f(u_x(\tau)), u_x(\tau) - z \rangle d\tau,$$

which implies that $\|u_x(T) - z\| \leq R$, and this is a contradiction. □

Remark 5.3. In Theorem 5.2 we give a sufficient condition for the existence of a periodic orbit of the Projected Dynamical System. This condition yields, of course, the existence of an almost periodic orbit in the sense of [22, Theorem 6]. Moreover, the assumptions of Theorem 5.2 are weaker than those given in [22, Theorem 6].

If we assume that H is a finite dimension Hilbert space we have the following result.

Corollary 5.4. *Under condition of Theorem 4.2 the mapping $Q^T : D \to D$ has a fixed point.*

Proof. Since the dimension of H is finite and by Proposition 5.1 Q^T is a continuous mapping, then Q^T is completely continuous (and hence Φ-condensing for any measure of noncompactness Φ). Thus, we obtain the result applying Theorem 5.2. □

Remark 5.5. In [11] the following open question is posed: Do periodic cycles for PDS exist in absence of monotony conditions? From the above two results we obtain an affirmative answer (also see [22]).

Theorem 5.6. *Let H be a real Hilbert space, $D \subset H$ a nonempty closed and convex subset and $f : D \to H$ a monotone mapping with range condition. If there exit $r > 0$, $T > 0$ and $z \in D$ such that $Q^T(x) - z \neq \lambda(x - z)$ for every $\lambda > 1$ and $x \in D \cap S_r(z)$, then exists a point in $x_0 \in D \cap B_r(z)$ such that $Q^T(x_0) = x_0$.*

Proof. First, we see that for every $T > 0$, the mapping $Q^T : D \to D$ is nonexpansive. Indeed, by proof of Theorem 4.4 we have that given $x, y \in D$ there exist u_x, u_y strong solution of problem

$$u'(t) + \partial I_D(u(t)) + f(u(t)) \ni 0,\ t \in (0, T),$$

with initial datum x, y respectively. Therefore,

$$\|Q^T(x) - Q^T(y)\| = \|u_x(T) - u_y(T)\| \leq \|x - y\| + \int_0^T 0 d\tau,$$

which means that Q^T is nonexpansive.

Second, if there exist $T > 0$ and $r > 0$ as in the hypothesis, by Theorem 3.9 we obtain that Q^T has a bounded almost fixed point sequence and thus invoking Remark 3.10 we achieve the conclusion. □

Theorem 5.7. *Let H be a real Hilbert space, $D \subset H$ a nonempty closed and convex subset and $f : D \to H$ a monotone mapping and lipschitz continuous. If there exit $r > 0$, $T > 0$ and $z \in D$ such that $Q^T(x) - z \neq \lambda(x - z)$ for every $\lambda > 1$ and for every $x \in D \cap S_r(z)$, then exists a point in $x_0 \in D \cap B_r(z)$ such that $Q^T(x_0) = x_0$.*

Proof. [11, Theorem 3.1] shows that $Q^T : D \to D$ is a continuous pseudocontractive mapping. This means that, if $T > 0$ is like in the hypothesis, then Q^T satisfies the assumptions of Theorem 3.9 and therefore the conclusion follows from Remark 3.10. □

Remark 5.8. In [11] is defined the following open question: Do the periodic cycles for PDS exist over constraint sets that do not satisfy the condition $0 \in \operatorname{int}(D)$? From The above two results we obtain an affirmative answer.

6. Formulation of a time-continuous migration model

6.1. **Formulation.** Next, we are going to develop the model on human migration which was introduced in [28]. Let there be N discrete locations in space.

Let $f(t) = (f_{12}(t), \dots, f_{1N}(t), f_{21}(t), \dots, f_{N(N-1)}(t))$ denote the vector flows of the population between locations, where $f_{ij}(t)$ represents the flow of population from location i to location j at time t. Let $p(t) = (p_1(t), .., p_N(t))$ denote the vector of population distribution at time t, where $p_i(t)$ is the population at location i at time t.

It is assumed that there is no population growth and consequently,

$$\sum_{i=1}^{N} p_i(t) = C, \quad \text{for all } t, \tag{6.1}$$

where $C > 0$ is a constant.

Also, let $u(p(t)) = (u_1(p(t)), u_2(p(t)), \dots, u_N(p(t)))$ denote the vector of utility functions, where $u_i(x)$ is the utility of locating in location i for a population distribution $x := (x_1, \dots, x_N)$. Finally, $c(f(t)) = (c_{ij}(f(t));\ i, j = 1, \dots, N,\ i \neq j)$ represents the vector of migration or transaction cost associated with migrating between locations, with $c_{ij}(y)$ denoting the cost associated with migrating between locations i and j for a flow y.

It is assumed that the rate of flow is directly related to the difference between the utility values minus the migration cost. More specifically, mathematically, the rate of change of migration flows may be expressed as:

$$\frac{df(t)}{dt} = \Pi_K(f(t), -F(p(t), f(t))) \tag{6.2}$$

where $\Pi_K(f(t), -F(p(t), f(t)))$ is the projection of the net gains in utility $-F(p(t), f(t))$ on $K := \{f(t) \geq 0\}$ at $f(t) \in K$, with component F_{ij} defined by

$$-F_{ij}(p(t), f(t)) = u_j(p(t)) - u_i(p(t)) - c_{ij}(f(t)). \tag{6.3}$$

Next, we must determine the relationship between the population distribution, $p(t)$, and migration flows, $f(t)$. The rate of change of the population at a location i must be equal to the difference between the inflow and outflow of that location. Moreover, the vector of population distribution must be nonnegative and, bounded by the no growth condition. Thus, the vector force field should be projected on the set $K_1 := \{p(t) = (p_1(t), \dots, p_N(t)) : \ \ p_i(t) \geq 0, \ \ \sum_{i=1}^{N} p_i(t) = C\}$, that is,

$$\frac{dp(t)}{dt} = \Pi_{K_1}(p(t), -G(f(t))) \tag{6.4}$$

where $-G(f(t)) = (\sum_{j=1}^{N}(-f_{ij}(t) + f_{ji}(t));\ i = 1, \dots, N,\ i \neq j)$.

By the above sections we know that (6.2 and 6.4) may be rewritten for determining $\overline{p}(t)$ and $\overline{f}(t) \geq 0$ such that

$$\frac{d(\overline{p}(t), \overline{f}(t))}{dt} + \partial I_{K_1 \times K}((\overline{p}(t), \overline{f}(t))) \ni -(G(\overline{f}(t)), F(\overline{p}(t), \overline{f}(t))). \tag{6.5}$$

The initial condition for the problem may be specified by the initial population distribution and the rate of migration.

6.2. **Development.** Consider the Hilbert space $H := \mathbb{R}^N \times \mathbb{R}^{N(N-1)}$. The elements of H will be represent by (x, y) where $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $y = (y_{ij} : \ i, j = 1, 2, \dots, N \ i \neq j) \in \mathbb{R}^{N(N-1)}$. Let D be the subset of H defined by $D := K_1 \times K$, where $K_1 := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N : \quad x_i \geq 0, \quad \sum_{i=1}^N x_i = C\}$ and $K := \{y = (y_{ij} : \ i \neq j) \in \mathbb{R}^{N(N-1)} \ : \ y_{ij} \geq 0\}$.

Now, consider $G : D \to \mathbb{R}^N$ defined by $G(x, y) = (\sum_{j=1}^N (y_{ij} - y_{ji}); i = 1, 2, \dots, N, j \neq i)$ and $F : D \to \mathbb{R}^{N(N-1)}$ defined by $F(x, y) = (F_{ij}(x, y) : i, j = 1, \dots, N, \ i \neq j)$ with $F_{ij}(x, y) = u_i(x) - u_j(x) + c_{ij}(y)$. This allows us to introduce the function $w : D \to H$ as $w(x, y) = (G(x, y), F(x, y))$.

Finally, consider $v : [0, T] \to H$ defined by $v(t) := (p(t), f(t))$. With the above comments the description of the model can be written as follows:

$$v'(t) + \partial I_D(v(t)) \ni -w(v(t)). \tag{6.6}$$

Theorem 6.1. *If $u : \mathbb{R}^N \to \mathbb{R}^N$ is an R-Lipschitzian mapping and $c : \mathbb{R}^{N(N-1)} \to \mathbb{R}^{N(N-1)}$ is a continuous monotone operator. Then Equation (6.6) has a unique strong solution for each initial data in the domain.*

Proof. As in the above paragraph, consider $w : H \to H$ defined by $w(x, y) = (G(x, y), F(x, y))$. It is easy to see that $G : H \to \mathbb{R}^N$ is a linear operator and then there exists $M > 0$ such that $\|G(x, y)\| \leq M\|(x, y)\|$.

On the other hand, we are assuming that $u : \mathbb{R}^N \to \mathbb{R}^N$ is an R-lipschitzian mapping, hence for any $x_1, x_2 \in \mathbb{R}^N$ we have that $\|u(x_1) - u(x_2)\| \leq R\|x_1 - x_2\|$.

Now, Let $(x_1, y_1), (x_2, y_2) \in H$. Then

$$\begin{aligned} \langle w(x_1, y_1) - w(x_2, y_2), (x_1 - x_2, y_1 - y_2)\rangle &= \langle G(x_1, y_1) - G(x_2, y_2), x_1 - x_2\rangle \\ &\quad + \langle F(x_1, y_1) - F(x_2, y_2), y_1 - y_2\rangle \\ &\geq -M\|(x_1 - x_2, y_1 - y_2)\|\|x_1 - x_2\| \\ &\quad -2R\|x_1 - x_2\|\|y_1 - y_2\| \\ &\quad + \langle c(y_1) - c(y_2), y_1 - y_2\rangle \end{aligned}$$

Having in mind that that c is a monotone operator, we obtain that

$$\begin{aligned} \langle w(x_1, y_1) - w(x_2, y_2), (x_1 - x_2, y_1 - y_2)\rangle &\geq -M\|(x_1 - x_2, y_1 - y_2)\|^2 \\ &\quad -2R\|x_1 - x_2\|\|y_1 - y_2\| \\ &\geq -(M + R)\|(x_1 - x_2, y_1 - y_2)\|^2 \end{aligned}. \tag{6.7}$$

Inequality (6.7) yields that $w : H \to H$ becomes $(M + R)$-accretive.

Since $w : H \to H$ is a continuous mapping, we infer, by [10, Theorem 4.12], that w is $(M + R)$-m-accretive operator on H.

Finally, it is clear (see [5, Theorem 2.6]), that $A = \partial I_D + w$ is a $(M + R)$-m-accretive operator on H and consequently by [5, Theorem 4.5] the differential inclusion

$$v'(t) + \partial I_D(v(t)) \ni -w(v(t))$$

admits a unique strong solution for each initial data in D. $\square$

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Manuscript received December 12, 2012
revised April 25, 2013

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