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A FURTHER GENERALIZATION OF NONEXPANSIVITY

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Dedicated to S. Reich thanking him for his kindness and his wide and deep contribution in Fixed Point Theory

Abstract. Many different generalizations of the notion of nonexpansivity have appeared in the last years. Recently T. Suzuki defined: let M be a metric space. A mapping $T : M \to M$ is said to satisfy condition (C) if $\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y)$. In this paper we extend this definition in the following way: a mapping $T : M \to M$ is of Suzuki type if there exists a nondecreasing function $\psi : (0, \infty) \to (0, \infty)$ such that $d(x, Tx) - \psi(d(x, Tx)) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y)$, for all $x, y \in M$. We prove the existence of a fixed point for this class of mappings under similar assumptions as those used for mappings satisfying condition (C). We also extend this definition and the corresponding fixed point results to the case of multivalued mappings. In the last section we show an example of a mapping of Suzuki type which does not satisfy any previously considered condition extending nonexpansivity.

1. Introduction

Let M be a metric space. A mapping $T : M \to M$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in M$. In 1965, F. Browder [4] proved that every nonexpansive mapping $T : C \to C$ has a fixed point when C is a bounded convex closed subset of a Hilbert space. This result was immediately extended to uniform convex spaces [5], [15] and to reflexive spaces with normal structure [21]. From then, a wide theory has been developed to extend Browder Theorem in several directions. On one side, many papers have appeared proving the validity of Browder Theorem in more general classes of spaces (see, for instance the monographs [2], [16], [18] and references therein). On another side, some authors have studied the existence of fixed points for multivalued nonexpansive mappings (see, for instance [9], [11],[29]). However, the results are weaker than the corresponding for singlevalued mappings. In fact, it is still unknown (see [10], [25]) if normal structure suffices to prove the existence of fixed points for multivalued nonexpansive mappings. Finally, some different generalizations of the notion of nonexpansivity have appeared in the last years and some fixed point results have been proved for these classes of mappings (see, for instance, [3], [17] , [26], [28]). One of the most relevant extension was defined by T. Suzuki in [27]:

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Definition 1.1. Let M be a metric space. A mapping $T : M \to M$ is said to satisfy condition (C) if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

A more general notion is defined in [14]:

Definition 1.2. Let M be a metric space. A mapping $T : M \to M$ is said to satisfy condition (C_λ) if for some $\lambda \in (0, 1)$

$$\lambda d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Some fixed point results for these classes of mappings have been proved in the above cited papers and some more recent papers (see [1], [7], [8], [12], [20]). In order to prove the existence of fixed points for mappings satisfying these conditions, the authors use that there exists a sequence of approximated fixed points. This fact is proved in [27] (Theorem 2) and [14] (Theorem 4) when the mapping T is defined in a convex bounded set of a Banach space by checking that $\mu T + (1 - \mu)I$ is an asymptotically regular mapping (for $\mu \in [1/2, 1)$ in the case of mappings satisfying condition (C) and $\mu \in [\lambda, 1)$ in the case of mappings satisfying condition (C_λ)). In this paper we will consider a more general class of mappings and we use a quite different approach to prove the existence of an approximated fixed point sequence. In section 2 we introduce this more general class of mappings which will be called *mappings of Suzuki type* and we prove that the minimal displacement for mappings of this class when defined on bounded convex sets is equal to zero. Section 3 is devoted to state fixed point results both for singlevalued and multivalued mappings of Suzuki type. Finally, in section 4 we include some examples and, in particular, an example of a mapping of Suzuki type which does not satisfy condition (C_λ) for any $\lambda \in (0, 1)$.

2. Preliminaries and previous lemmas

Let (X, d) be a metric space. In this paper we consider the following family of sets:

$P(X) = \{Y \subseteq X : Y \text{ is nonempty}\}$,
$P_{cl,b}(X) = \{Y \subseteq X : Y \text{ is nonempty, closed and bounded}\}$,
$P_{cp}(X) = \{Y \subseteq X : Y \text{ is nonempty and compact}\}$,
$P_{cp,cv}(X) = \{Y \subseteq X : Y \text{ is nonempty, compact and convex}\}$.

On $P_{cl,b}(X)$ we have the Hausdorff metric H given by

$$H(A, B) := \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}, \quad A, B \in P_{cl,b}(X),$$

where for $x \in X$ and $Y \subset X$, $d(x, Y) := \inf\{d(x, y) : y \in Y\}$ is the distance from the point x to the set Y.

A mapping $T : X \to P_{cl,b}(X)$ is said to be *continuous* on $x \in X$ (with respect to the Hausdorff metric H) if $H(Tx_n, Tx) \to 0_X$ whenever $x_n \to x$.

Let $K \in P_{cl,b}(X)$ and $\{T_n\}$ a sequence of mappings $T_n : K \to P_{cl,b}(X)$ such that for each $n \in \mathbb{N}$, $T_n(K)$ is a bounded set. We say that $\{T_n\}$ *converges uniformly* to another mapping $T_0 : K \to P_{cl,b}(X)$ if

$$\lim_n \sup\{H(T_n x, T_0 x) : x \in K\} = 0.$$

For a given mapping $T : K \to P(X)$, a sequence $\{x_n\}$ in K is called an *approximated fixed point sequence* (a.f.p.s., in short) provided $\lim_n d(x_n, Tx_n) = 0$.

Let K be a nonempty bounded closed subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. We denote $r(K, \{x_n\})$ and $A(K, \{x_n\})$ the *asymptotic radius* and the *asymptotic center* of $\{x_n\}$ relative to K, respectively, i.e.,

$$r(K, \{x_n\}) = \inf\{\limsup_n \|x_n - x\| : x \in K\};$$

$$A(K, \{x_n\}) = \{x \in K : \limsup_n \|x_n - x\| = r(K, \{x_n\})\}.$$

It is well known that $A(K, \{x_n\})$ is a nonempty weakly compact and convex set as K is, and consists of exactly one point whenever the space is *uniformly convex in every direction* (UCED).

The sequence $\{x_n\}$ is said to be *regular* with respect to K if each of its subsequences has the same asymptotic radius in K, and *asymptotically uniform* with respect to K if each of its subsequence has the same asymptotic center in K. It is well known (see lemma 15.2 in [16]) that any sequence in K contains a regular subsequence. Note that in a UCED Banach space every regular sequence with respect to a set is asymptotically uniform.

We now recall some properties of Banach spaces that will appear in the remainder of this paper:

- A Banach space X is said to have the *Opial property* if for every sequence $\{x_n\}$ in X weakly convergent to $x \in X$ one has that

$$\liminf_n \|x_n - x\| < \liminf_n \|x_n - y\|$$

  for every $y \in X$, $y \neq x$.
- A Banach space X is said to have *normal structure* if for each bounded, convex subset K of X with $\operatorname{diam}(K) > 0$, there exists a nondiametral point $p \in K$, that is a point $p \in K$ such that

$$\sup\{\|p - x\| : x \in K\} < \operatorname{diam}(K).$$

Next we introduce a new class of generalized nonexpansive mappings.

Definition 2.1. Let (X, d) be a metric space and $K \subset X$. We say that mapping $T : K \to X$ is of Suzuki type if there exists a nondecreasing function $\psi : (0, \infty) \to (0, \infty)$ such that

$$d(x, Tx) - \psi(d(x, Tx)) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in K$.

It is clear that a mapping of Suzuki type becomes a mapping satisfying condition (C) if $\psi(t) = t/2$ and condition (C_λ) if $\psi(t) = (1-\lambda)t$.

This definition can be adapted for multivalued mappings in the following way.

Definition 2.2. Let (X, d) be a metric space and $K \subset X$. A mapping $T : K \to P(X)$ is of Suzuki type if there exists a nondecreasing function $\psi : (0, \infty) \to (0, \infty)$, such that for each $x, y \in K$ and $u \in Tx$ with

$$d(x, u) - \psi(d(x, u)) \leq d(x, y),$$

there exits $v \in Ty$ such that

$$d(u, v) \leq d(x, y).$$

Condition (C) and condition (C_λ) were extended to the multivalued case (in different ways) by some authors and studied, among other papers, in [1], [12], [20] and [24]. It is in [12] and [24] where the multivalued version of (C_λ) coincides with the definition above for $\psi(t) = (1-\lambda)t$. Obviously, every nonexpansive mapping is of Suzuki type.

The class of mappings of Suzuki type on a closed, bounded and convex subset K of X shares with the class of nonexpansive mappings the existence of approximated fixed point sequences. Before stating this fact we shall need the lemma below. Although this lemma should be known, we include its proof by completeness.

Lemma 2.3. *Let X be a metric space, $K \in P_{cl,b}(X)$ and $\{T_n\}$ a sequence of mappings from K into $P_b(X)$ such that for each $n \in \mathbb{N}$, $T_n(K)$ is a bounded set. Suppose that $\{T_n\}$ converges uniformly to another mapping $T_0 : K \to P_b(X)$ and denote*

$$\delta_n = \inf\{d(x, T_n x) : x \in K\}, \quad n \geq 0.$$

Then, $\delta_0 = \lim_n \delta_n$.

Proof. For every $n \in \mathbb{N}$ choose $x_n \in M$ such that $d(x_n, T_n x_n) \leq \delta_n + 1/n$ and let $\epsilon_n = \sup\{H(T_n x, T_0 x) : x \in K\}$. We have

$$\delta_0 \leq d(x_n, T_0 x_n) \leq d(x_n, T_n x_n) + H(T_n x_n, T_0 x_n) \leq \delta_n + 1/n + \epsilon_n.$$

Thus $\delta_0 \leq \liminf_n \delta_n$.

Conversely, for an arbitrary $\epsilon > 0$ choose $x_0 \in M$ such that $d(x_0, T_0 x_0) \leq \delta_0 + \epsilon$. We note

$$H(T_n x_0, T_0 x_0) \leq \frac{1}{n}$$

for sufficiently large $n \in \mathbb{N}$. Hence

$$\delta_n \leq d(x_0, T_n x_0) \leq d(x_0, T_0 x_0) + H(T_0 x_0, T_n x_0) \leq \delta_0 + \epsilon + 1/n$$

which implies $\limsup \delta_n \leq \delta_0 + \epsilon$. Since ϵ is arbitrary we obtain the required equality. □

Lemma 2.4. *Let K be a closed, bounded and convex subset of a linear normed space X and $T : K \to P(K)$ a mapping of Suzuki type. Then,* $\inf\{d(x, Tx) : x \in K\} = 0$

Proof. By translation we can assume that $0 \in K$. Denote $M = \text{diam}K$. Assume, by contradiction, that $d = \inf\{d(x, Tx) : x \in K\} > 0$. By Lemma 2.3, there exists λ, $1 - \psi(d)/M < \lambda < 1$, such that $d_\lambda = \inf\{d(x, \lambda Tx) : x \in K\} > 0$. For $\epsilon < d_\lambda(1 - \lambda)$ choose $x_0 \in K$ such that $d(x_0, \lambda Tx_0) < d_\lambda + \epsilon$.

Let $y_0 \in Tx_0$. We have

$$\begin{aligned} \|x_0 - \lambda y_0\| &\geq \|x_0 - y_0\| - (1 - \lambda)M \\ &= (I - \psi)(\|x_0 - y_0\|) + \psi(\|x_0 - y_0\|) - (1 - \lambda)M \\ &\geq (I - \psi)(\|x_0 - y_0\|) + \psi(d) - (1 - \lambda)M \\ &\geq (I - \psi)(\|x_0 - y_0\|). \end{aligned}$$

Then, there exists $v \in T(\lambda y_0)$ such that $\|y_0 - v\| \leq \|x_0 - \lambda y_0\|$.

From above we have,

$$d_\lambda \leq d(\lambda y_0, \lambda T(\lambda y_0) \leq \|\lambda y_0 - \lambda v\| = \lambda\|y_0 - v\| \leq \lambda\|x_0 - \lambda y_0\|.$$

Since $y_0 \in Tx_0$ is arbitrary, we obtain the contradiction

$$d_\lambda \leq \lambda d(x_0, \lambda Tx_0) < \lambda(d_\lambda + \epsilon) < d_\lambda.$$

□

It is worth pointing out that the corresponding result to Lemma 2.3 for mappings which satisfy condition (C_λ) was proved in [14] and in [12] for multivalued mappings satisfying condition (C), but in a very different approach. In fact, they obtain an a.f.p.s for T by means of the Mann iteration scheme. However, this scheme cannot be used for mappings of Suzuki type.

In order to obtain some fixed point theorems for the class of mappings of Suzuki type we will need to assume that the graph of $I - T$ is *strongly demiclosedness at* 0_X.

Definition 2.5. Given a mapping $T : K \to P_{cl,b}(X)$, it is said that the graph of $I - T$ is strongly demiclosedness at 0_X if for every sequence $\{x_n\}$ in K strongly convergent to $x \in K$ such that $\lim_n d(x_n, Tx_n) = 0$ one has that $x \in Tx$.

Clearly the graph of $I - T$ is strongly demiclosedness at 0_X if the graph of $I - T$ is closed (in particular, if T is continuous) and from Lemma 7 in [27] it also holds if T is singlevalued and satisfies condition (C). For multivalued mappings satisfying condition (C) the same fact follows from Lemma 3.2 in [12].

3. Fixed point theorems for mappings of Suzuki type

3.1. **Singlevalued mappings.** The existence of an a.f.p.s. allows to use the method of asymptotic center for mappings of Suzuki type in order to prove some fixed point results which are similar to those which are known for mappings satisfying conditions (C) or (C_λ). For instance, it is easy to check that the proof of Theorem 4.7 in [23] also works for mappings of Suzuki type and so we can state:

Theorem 3.1. *Let K be a closed, convex, bounded subset of a Banach space X. Assume that $T : K \to K$ is a mapping of Suzuki type such that the graph of $I - T$ is strongly demiclosedness at 0_X. Then, at least one of the following statements is true:*

(1) *T has a fixed point,*
(2) *For any a.f.p.s. $\{x_n\}$ for T in K and each $x \in K$ we have* $\limsup_n \|x_n - Tx\| \leq \limsup_n \|x_n - x\|$.

To prove a fixed point result for singlevalued mappings we need the following result in [23]:

Theorem 3.2. *Let X be a Banach space with normal structure, K a weakly compact and convex subset of X and $T : K \to K$ a mapping which satisfies the following two conditions:*

(1) *If D is a nonempty, closed, convex and T-invariant subset of K, then there exists an a.f.p.s. for T in D.*
(2) *For any a.f.p.s. $\{y_n\}$ of T in K and each $x \in K$ there is a subsequence $\{x_n\}$ of $\{y_n\}$ such that* $\limsup_n \|x_n - Tx\| \leq \limsup_n \|x_n - x\|$.

Then, T has a fixed point.

From this theorem we easily obtain:

Theorem 3.3. *Let K be a weakly compact convex subset of a Banach space X with normal structure. If $T : K \to K$ is a mapping of Suzuki type such that the graph of $I - T$ is strongly demiclosedness at 0_X. Then, T has a fixed point.*

Proof. If T were fixed point free, then T would satisfy the condition (2) in Theorem 3.1. But, in this case Theorem 3.2 would imply that T has a fixed point. □

The assumption of the strongly demiclosedness of the graph of $I-T$ at 0_X cannot be removed. Indeed, consider the following example in [14]:

Example 3.4. Put $X = \mathbb{R}$ and $K = [-1/4, 1]$. Define a mapping $T : K \to K$ by

$$Tx = \begin{cases} 1 & \text{if } x = 0 \\ -(1/3)x & \text{if } x \in [-1/4, 0) \cup (0, 3/4] \\ 1 - x & \text{if } x \in [3/4, 1] \end{cases}$$

As proved in [14], this mapping is of Suzuki type for $\psi(t) = (1 - \lambda)t$ and $\lambda \in (\frac{3}{4}, 1)$ but it is fixed point free.

We recall that a Banach space is said to satisfy condition (D) if there exists $\lambda \in (0, 1)$ such that for any nonempty weakly compact convex subset E of X, any sequence $\{x_n\} \in E$ which is regular relative to E, and any sequence $\{y_n\}$ in $A(E, \{x_n\})$, which is regular relative to E we have

$$r(E, \{y_n\}) \leq \lambda r(E, \{x_n\}).$$

In [7] the following result is proved:

Theorem 3.5. *Let X be a Banach space satisfying property (D) and let E be a weakly compact convex subset of X. If $T : E \to E$ is a mapping satisfying condition (C), then T has a fixed point.*

Since every Banach space with property (D) has weak normal structure (see Theorem 3.3 in [6]), the above result (and all results which are cited in [7] as extended by the above theorem) is a consequence of Theorem 3.3 above.

3.2. **Multivalued mappings.** The following result is an extension of Lemma 1 of [14] to the class of multivalued mappings of Suzuki type.

Lemma 3.6. *Let K be a subset of a Banach space X and $T : K \to P_{cp}(X)$ a mapping of Suzuki type. Let $\{x_n\}$ be a bounded a.f.p.s. for T. Then, for all $x \in K$ such that* $\liminf_n \|x_n - x\| > 0$ *there exist a subsequence $\{z_n\}$ of $\{x_n\}$ and $u \in Tx$ such that*

$$\limsup_n \|z_n - u\| \leq \limsup_n \|z_n - x\|.$$

Proof. Since $\lim_n d(x_n, Tx_n) = 0$ and T is compact valued, we can find $y_n \in Tx_n$ such that $\lim_n \|x_n - y_n\| = 0$. Fix $x \in K$ such that $\liminf_n \|x_n - x\| > 0$, and put $\epsilon := (\frac{1}{2}) \liminf_n \|x_n - x\|$. Then we have

$$(I - \psi)(\|x_n - y_n\|) \leq \|x_n - y_n\| < \epsilon < \|x_n - x\|$$

for sufficiently large $n \in \mathbb{N}$. Since T is a Suzuki type mapping, there exists $u_n \in Tx$ such that $\|y_n - u_n\| \leq \|x_n - x\|$. Let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$ that converges to some $u \in Tx$. Then for each $k \in \mathbb{N}$

$$\|x_{n_k} - u_{n_k}\| \leq \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - u_{n_k}\| \leq \|x_{n_k} - y_{n_k}\| + \|x_{n_k} - x\|.$$

Taking the superior limit as $k \to \infty$ we obtain

$$\limsup_k \|x_{n_k} - u\| \leq \limsup_k \|x_{n_k} - x\|$$

If we denote by $\{z_n\}$ the sequence $\{x_{n_k}\}$ we have the desired result. □

We will appeal to the above lemma in order to give some fixed point results for multivalued mappings. As in the singlevalued case we have to assume that the graph of $I - T$ is strongly demiclosedness at 0_X.

Theorem 3.7. *Let K be a nonempty weakly compact convex subset of a Banach space X. Assume that $T : K \to P_{cp}(K)$ is a mapping of Suzuki type such that the graph of $I - T$ is strongly demiclosedness at 0_X. Suppose one of the following assumptions is satisfied*

a) $(X, \|\cdot\|)$ *has the Opial property.*
b) X *is UCED.*

Then, T has a fixed point.

Proof. From Lemma 2.4 there exists an a.f.p.s $\{x_n\}$ for T in K.

Case a): Without loss of generality, we may suppose that $x_n \rightharpoonup x \in K$. If $\{x_n\}$ admits a subsequence strongly convergent to x , it follows that $x \in Tx$. Otherwise, by Lemma 3.6 there exist a subsequence $\{z_n\}$ of $\{x_n\}$ and a point $u \in Tx$ such that

$$\limsup_n \|z_n - u\| \leq \limsup_n \|z_n - x\|.$$

Since $z_n \rightharpoonup x$, it follows from the Opial condition that $u = x \in Tx$ and the proof is complete.

Case b): Without loss of generality, we may assume that $\{x_n\}$ is regular with respect to K. Let x the unique point in the asymptotic center of $\{x_n\}$ in K. Following the same argument as above if $\{x_n\}$ admits a subsequence strongly convergent to x, then $x \in Tx$. Otherwise there exist a subsequence $\{z_n\}$ of $\{x_n\}$ and a point $u \in Tx$ such that

$$\limsup_n \|z_n - u\| \le \limsup_n \|z_n - x\|.$$

Taking into account that the asymptotic center of $\{z_n\}$ is precisely x, we obtain $u = x \in Tx$. □

Our next fixed point result for multivalued mappings of Suzuki type relies on the following proposition.

Proposition 3.8. *Let K be a nonempty closed, bounded and separable subset of a Banach space X. Assume that $T : K \to P_{cp}(X)$ is a mapping of Suzuki type such that the graph of $I-T$ is strongly demiclosedness at 0_X. Suppose that each sequence in K has a nonempty asymptotic center relative to K. Let $\{x_n\}$ be an a.f.p.s. for T. Then, there exists a subsequence $\{z_n\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that*

$$T(x) \cap A \neq \emptyset, \text{ for all } x \in A := A(K, (z_n)).$$

Proof. Since K is separable we can get a subsequence $(z_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ which is regular and asymptotically uniform with respect to K (see [16, p. 168]). Denote $r(K, (z_n))$ by r and take any $x \in A := A(K, (z_n))$. If $\{x_n\}$ admits a subsequence strongly convergent to x , it follows that $x \in Tx \cap A$. Otherwise, by Lemma 3.6, we obtain a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ and $u \in Tx$ such that

$$\limsup_k \|z_{n_k} - u\| \le \limsup_k \|z_{n_k} - x\| = r.$$

This shows that $u \in A$, and so $T(x) \cap A \neq \emptyset$. □

Now we are able to prove an analog of the Kirk-Massa theorem [22] for mappings of Suzuki type.

Theorem 3.9. *Let K be a closed, bounded, and convex subset of a Banach space X and $T : K \to P_{cp,cv}(K)$ be a continuous mapping of Suzuki type. Suppose that each sequence in K has a nonempty and compact asymptotic center relative to K. Then T has a fixed point.*

Proof. Since T is a self-mapping we can construct a closed convex subset of K which is separable and invariant under T (see [16, p. 168]). Then, we can suppose that K is separable. According to Lemma 2.4 and the previous proposition we can take a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that

$$T(x) \cap A \neq \emptyset, \text{ for all } x \in A := A(K, (x_n)).$$

Now we define the mapping $\tilde{T} : A \to P_{cp,cv}(A)$ by $\tilde{T}(x) = T(x) \cap A$. Since T is continuous, from Proposition 2.45 in [19] we know that the mapping $\tilde{T}$ is upper semi-continuous. Since $T(x) \cap A$ is a compact convex set we can apply the Kakutani-Bohnenblust-Karlin theorem (see [16], Theorem 18.12) to obtain a fixed point for $\tilde{T}$ and hence for T. □

Remark 3.10. When T is a single valued mapping satisfying condition (C), Dhompongsa et al. [7] have proved that the asymptotic center in K of an approximate fixed sequence for T is a T-invariant set. In Proposition 3.8 we can find an adaptation of this fact to the multivalued case (see also [12, Proposition 4.7]). Nevertheless, the T invariance of the asymptotic center does not hold even for multivalued nonexpansive mappings. This can be justified via the following example in [13].

Example 3.11. Let $(X, \|\cdot\|)$ be the Banach space $(\mathbb{R}^2, \|\cdot\|_\infty)$ where $\|\cdot\|_\infty$ is the sup norm. If

$$K := \{\ (x_1, x_2) \in \mathbb{R}^2 : |x_1| + |x_2| \leq 1\},$$

let $T : K \to P_{cp,cv}(K)$ be the mapping given by

$$T((x_1, x_2)) := \{x_1\} \times [|x_1| - 1, 1 - |x_1|].$$

Let H_∞ be the Hausdorff metric associated to the sup metric $d_\infty(x, y) = \|x - y\|_\infty$. Then T is H_∞-nonexpansive and the set of fixed points of T is K. Taking the a.f.p.s. sequence $\{x_n\} \equiv \{0_{\mathbb{R}^2}\}$ we obtain that $A(K, \{x_n\}) = \{(0, 0)\}$ which is not a T-invariant set.

Remark 3.12. Another natural extension of the Suzuki's condition (C_λ) for a multivalued mapping $T : K \to P(X)$ is the following: for all $x, y \in K$

$$\lambda d(x, T(x)) \leq \|x - y\| \Longrightarrow H(T(x), T(y)) \leq \|x - y\|.$$

It is not clear if a mapping satisfying this condition above also satisfies (C_λ) in our sense. However, if T takes compact values it is easy to see that this new condition implies condition (C_λ). This condition has been considered in [1] and [20] in order to extend some classic fixed point theorems for multivalued nonexpansive mappings to mappings satisfying condition (C) or (C_λ). In such results T is assumed to be compact valued and therefore our theorems are also generalizations of those appeared in [1] and [20]. Even more, our Theorem 3.9 improves and extends Theorem 4.8 in [12] and Theorem 3.5 in [20], where the uniform continuity of the mapping T is imposed.

Remark 3.13. Although some classical results of the fixed point theory for singlevalued mappings have been extended to multivalued mappings, many problems on the existence of fixed points remain unsolved in this setting. One of those problems was raised by Reich in 1983 [25]: *Given a weakly compact convex subset K of a Banach space X satisfying the fixed point property for (singlevalued) nonexpansive mappings, does every nonexpansive mapping $T : K \to P_{cp}(K)$ have a fixed point?* According to Kirk's theorem ([21]) a Banach space X with weak normal structure has the fixed point property for singlevalued nonexpansive mappings, so a particular case of Reich's question is: *if K has normal structure, does every nonexpansive mapping $T : K \to P_{cp}(K)$ have a fixed point?.*

It was proved in [6] that a nonexpansive mapping $T : K \to P_{cp,cv}(K)$ has a fixed point whenever the space X satisfies property (D). Since every Banach space with property (D) has weak normal structure (as we noted in Section 3.1), then a possible approach to the aforementioned problem is to study if properties implying weak normal structure also imply property (D). It should be mentioned that normal structure does not imply property (D), hence the problem of extending Kirk's theorem cannot be fully solved by this approach. The interested reader can find in [10] an exposition of the main results and current research directions in this subject.

It is shown in many cases that under suitable geometric properties of a Banach space the fixed point property for multivalued nonexpansive mappings actually implies the fixed point property for a strictly larger family of multivalued mappings (see [13] and references therein). For instance, in [13] the authors introduce some classes of multivalued nonexpansive type related to mappings of Suzuki type for which either Kirk-Massa Theorem or Kirk's Theorem are valid. However, it is a open problem if Kirk-Massa theorem holds for a mapping satisfying condition (C) without assuming the continuity of the mapping (see Theorem 4.8 in [12]).

4. Some examples

We will show in Example 4.2 a mapping of Suzuki type which does not satisfy any condition (C_λ) for $\lambda \in (0,1)$, but we need before the computations which appear in the following example:

Example 4.1. For $t \in [1/2, 1)$ define a mapping $T_t : [0,1] \to [0,1]$ by

$$T_t x = \begin{cases} tx & \text{if } x \in [0,1) \\ f(t) =: \frac{1}{2-t} - (1-t)^3 & \text{if } x = 1 \end{cases}$$

We will show that T_t is of Suzuki type for $\psi(s) = s^2$ but it does not satisfy the (C_λ)-condition for $t = \frac{10+2\lambda}{11+\lambda}$. We denote $\phi(s) = s - \psi(s) = s - s^2$. It is easy to check that $\phi(s + s^3) > s - s^2$ for every $s \in (0, 1/3]$ and ϕ is nondecreasing on this interval. Easy computations prove the following facts:

Fact 1. $|T_t(x) - T_t(y)| \le |x - y|$ for every $x, y \in [0,1)$.

Fact 2. $|T_t(x) - T_t(1)| \le |x - 1|$ if and only if $x \in \left[0, \frac{1}{2-t} + (1-t)^2\right]$.

The following claims will prove that T_t is a mapping of Suzuki type.

Claim 1. $\phi(|x - T_t x|) > 1 - x$ if $x \in [\frac{1}{2-t} + (1-t)^2, 1)$.

Claim 2. $\phi(|1 - T_t(1)|) > 1 - x$ if $x \in [\frac{1}{2-t} + (1-t)^2, 1)$.

Indeed, denote $s = \frac{1-t}{2-t}$. (Note that $s \in (0, 1/3]$). We have

$$\begin{aligned} \phi(|x - T_t x|) &= \phi((1-t)x) \ge \phi\left(\frac{1-t}{2-t} + (1-t)^3\right) \\ &= \phi\left(s + \frac{s^3}{(1-s)^3}\right) \ge \phi(s + s^3) > s - s^2 \ge s - \frac{s^2}{(1-s)^2} \\ &= \frac{1-t}{2-t} - (1-t)^2 \ge 1 - x. \end{aligned}$$

and so we have proved claim 1. The same argument proves claim 2.

Finally we will prove that T does not satisfy the (C_λ)-condition for $t = \frac{10+2\lambda}{11+\lambda}$. Indeed, denote again $s = \frac{1-t}{2-t} = \frac{1-\lambda}{12}$. Note that

$$\frac{s^2}{(1-s)^3} + \frac{s}{(1-s)^2} < 8s^2 + 4s < 12s = 1 - \lambda$$

because $s < 1/2$. Thus

$$\frac{s^3}{(1-s)^3} + \frac{s^2}{(1-s)^2} < s - \lambda s$$

which implies

$$\lambda s + \frac{s^3}{(1-s)^3} < s - \frac{s^2}{(1-s)^2}$$

and so

$$\lambda \left(s + \frac{s^3}{(1-s)^3} \right) < s - \frac{s^2}{(1-s)^2}.$$

Choose $x = \frac{1}{2-t} + (1-t)^2$. We have

$$1 - x = \frac{1-t}{2-t} - (1-t)^2 = s^2 - \frac{s^2}{(1-s)^2}$$

and

$$|x - T_t x| = (1-t)x = \frac{1-t}{2-t} + (1-t)^3 = s + \frac{s^3}{(1-s)^3}.$$

Thus, $\lambda|x - T_t x| < |1 - x|$. Since T_t is continuous on $[0, 1)$ there exists $x' > x$ such that the above inequality still holds, that is $\lambda|x' - T_t x'| < |1 - x'|$. However, by Fact 2 we have $|T_t(1) - T_t(x')| > |1 - x'|$ and T_t does not satisfy the condition (C_λ).

Example 4.2. Put $X = c_0$ and $K = \{(x_n) \in c_0 : 0 \le x_n \le 1; \sum_{n=1}^{\infty} x_n \le 1\}$. For $f(t)$ defined as in example 4.1 and a sequence $\{\lambda_n\} \in [1/2, 1)$ convergent to 1, define a mapping $T : K \to K$ by

$$T((x_n)) = \begin{cases} f(\lambda_k)e_k & \text{if } x = e_k \\ \\ (\lambda_n x_n) & \text{if } \|(x_n)\| < 1 \end{cases}$$

We will show that T is of Suzuki type and it does not satisfy any condition (C_λ) for $\lambda \in (0, 1)$. For short, we will write $T_k =: T_{\lambda_k}$ where T_{λ_k} is the mapping considered in Example 4.1, and $b_k =: \frac{1}{2-\lambda_k} + (1 - \lambda_k)^2$. It is clear that $\|Tx - Ty\| \le \|x - y\|$ if $\|x\| < 1$ and $\|y\| < 1$. It is also clear that $\|Te_k - Te_j\| < \|e_k - e_j\|$ for every $k \ne j$. Next, we study the case $\|x\| < 1$ and $\|y\| = 1$. In this case we will prove that T is of Suzuki type for $\psi(s) = s^2$.

1. Assume that $x_n \le b_n$ for all $n \in \mathbb{N}$. Then, $\|T((x_n)) - T(e_k)\| \le \|(x_n) - e_k\|$ for every $k \in \mathbb{N}$.

Indeed, since $x_k \le b_k$, by Fact 2 in Example 4.1 we have

$$\|T((x_n)) - T(e_k)\| \quad = \quad \max_{n \ne k} \{\lambda_n x_n, T_k(1) - T_k(x_k)\}$$

$$\leq \max_{n\neq k}\{x_n, 1-x_k\} = \|(x_n) - e_k\|.$$

2. Assume that there is a $j \in \mathbb{N}$ such that $x_j > b_j$ (note that this index j must be unique because $b_n > 1/2$ for every n). Denote $\phi(s) = s - \psi(s) = s - s^2$. In this case, since $x_n + x_j \leq 1$ for every $n \neq j$ we have

$$\phi(\|Te_j - e_j\|) = \phi(1 - T_j(1)) > 1 - x_j = \max\{1 - x_j, x_n\} = \|e_j - x\|$$

and

$$\phi(\|Tx - x\|) = \phi(\max_n\{(1-\lambda_n)x_n\}) \geq \phi((1-\lambda_j)x_j) > 1 - x_j = \|e_j - x\|$$

because $\phi(s)$ is a nondecreasing function when $s \leq 1/2$ and by Claim 1 in Example 4.1, we have $\phi(x_j - T_j(x_j)) > 1 - x_j$ when $x_j > b_j$.

Finally for $k \neq j$, having in mind that $x_k < b_k$, from Fact 2 in example 4.1, we obtain

$$\|Te_k - Tx\| = \max_{n\neq k}\{\lambda_n x_n, T_k(1) - T_k(x_k)\} \leq \max_{n\neq k}\{x_n, 1 - x_k\} = \|x - e_k\|.$$

We conclude the example showing that T does not satisfy any condition (C_λ) for $\lambda \in (0,1)$. Indeed, for a given $\lambda \in (0,1)$, choose $\lambda_k \geq \lambda$ and $t = \frac{10+2\lambda_k}{11+\lambda_k}$. From Example 4.1 we know that T_k does not satisfy condition (C_{λ_k}). Since T restricted to $\overline{\text{co}}\{e_k, 0\}$ becomes the mapping $T_k : [0,1] \to [0,1]$, it is clear that T fails to satisfy condition (C_{λ_k}) and so condition (C_λ).

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