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LOCAL DUALITY IN LOEWNER EQUATIONS*

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Abstract. Among diversity of frameworks and constructions introduced in Loewner theory by different authors, one can distinguish two closely related but still different ways of reasoning, which colloquially may be described as "increasing" and "decreasing". In this paper we review in short the main types of (deterministic) Loewner evolution discussed in the literature and describe in detail the local duality between "increasing" and "decreasing" cases within the general unifying approach in Loewner theory proposed recently in [7, 8, 9]. In particular, we extend several results of [6], which deals with the chordal Loewner evolution, to this general setting. Although the duality is given by a simple change of the parameter, not all the results for the "decreasing" case can be obtained by mere translating the corresponding results for the "increasing" case. In particular, as a byproduct of establishing local duality between evolution families and their "decreasing" counterparts we obtain a new characterization of generalized Loewner chains.

1. Introduction

1.1. **Loewner theory in different frameworks.** Last decades there has been a great burst of interest to Loewner theory both in its deterministic variant, going back to Charles Loewner's seminal paper [31] of 1923, and more recent stochastic version by Odded Schramm [37].

Motivated by extremal problems for univalent functions, Loewner [31] considered univalent holomorphic functions $f(z) = z + a_2z^2 + \dots$ mapping the unit disk $\mathbb{D} := \{z : |z| < 1\}$ onto the complex plane $\mathbb{C}$ minus a Jordan arc Γ extending to infinity. Choose a (unique) parametrization $\gamma : [0, +\infty) \to \mathbb{C}$ of the Jordan arc Γ satisfying the following two conditions: (1) $\gamma(t) \to \infty$ as $t \to +\infty$; (2) $f_t'(0) = e^t$ for each $t \geq 0$, where f_t stands for the conformal mapping of $\mathbb{D}$ onto $\Omega_t := \mathbb{C} \setminus \gamma\big([t, +\infty)\big)$ normalized by $f_t(0) = 0$, $f_t'(0) > 0$. Loewner proved that for any Jordan arc Γ with such parametrization there exists a (unique) continuous function $k : [0, +\infty) \to$

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$\mathbb{T} := \{z : |z| = 1\}$ such that the family (f_t) satisfies the PDE

$$\frac{\partial f_t(z)}{\partial t} = z f_t'(z) \frac{1 + k(t)z}{1 - k(t)z}, \tag{1.1}$$

where $f_t'(z)$ stands for the partial derivative of $f_t(z)$ w.r.t. z. Actually, (f_t) is the unique solution to this PDE which is defined and univalent as a function of z for all $t \geq 0$ and all $z \in \mathbb{D}$. Moreover, for all $s \geq 0$,

$$f_s = \lim_{t \to +\infty} e^t \varphi_{s,t}, \tag{1.2}$$

where $\varphi_{s,t} := f_t^{-1} \circ f_s : \mathbb{D} \to \mathbb{D}$, $0 \leq s \leq t$, solves the characteristic ODE

$$\frac{d\varphi_{s,t}(z)}{dt} = -\varphi_{s,t}(z) \frac{1 + k(t)\varphi_{s,t}(z)}{1 - k(t)\varphi_{s,t}(z)}, \ t \geq s, \quad \varphi_{s,s}(z) = z, \ z \in \mathbb{D}. \tag{1.3}$$

He also proved that conversely, given a continuous $k : [0, +\infty) \to \mathbb{T}$, equation (1.3) and formula (1.2) define together, in a unique way, a family $(f_t)_{t \geq 0}$ of holomorphic univalent functions in $\mathbb{D}$ satisfying (1.1) and such that $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t$, although in this case f_0 need not be a conformal mapping onto the complement of a Jordan arc[1]. The differential equations (1.1) and (1.3) are known nowadays as the *Loewner PDE* and the *Loewner ODE*, respectively.

Later Kufarev [22] and Pommerenke [33, 34] extended Loewner's results for a more general class of families (f_t) by replacing the Schwarz kernel in the Loewner differential equations with an arbitrary holomorphic function of positive real part. In the theory that they constructed, a *(classical) Loewner chain* is a family $(f_t)_{t \geq 0}$ of univalent holomorphic functions $f_t : \mathbb{D} \to \mathbb{C}$ subject to the inclusion condition $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t$ and the normalization $f_t(z) = e^t(z + a_2(t)z^2 + \ldots)$ for all $t \geq 0$ and all $z \in \mathbb{D}$. A *driving term* or a *classical Herglotz function* is a function $p : \mathbb{D} \times [0, +\infty) \to \{z : \operatorname{Re} z > 0\}$ such that $p(z, \cdot)$ is measurable in $[0, +\infty)$ for all $z \in \mathbb{D}$ and $p(\cdot, t)$ is holomorphic in $\mathbb{D}$ with $p(0, t) = 1$ for all $t \geq 0$. It is known that for any classical Loewner chain (f_t) there exists a classical Herglotz function p, unique up to a null-set on the t-axis, such that f_t satisfies the *(classical) Loewner – Kufarev PDE* $\partial f_t(z)/\partial t = z f_t' p(z, t)$. In turn, the functions $\varphi_{s,t} := f_t^{-1} \circ f_s$, $0 \leq s \leq t$, satisfy the *Loewner – Kufarev* ODE $(d/dt)\varphi_{s,t}(z) = -\varphi_{s,t}(z) p\big(\varphi_{s,t}(z), t\big)$ with the initial condition $\varphi_{s,s} = \mathsf{id}_{\mathbb{D}}$, and again as in Loewner's original theory, the family (f_t) can be reconstructed using (1.2). Conversely, for any classical Herglotz function p there exists a unique classical Loewner chain (f_t) satisfying the Loewner – Kufarev PDE. This Loewner chain is given by (1.2), where $(\varphi_{s,t})$ is the unique solution to the Loewner – Kufarev ODE with the initial condition $\varphi_{s,s} = \mathsf{id}_{\mathbb{D}}$.

In modern literature the process described by the Loewner – Kufarev differential equations is referred to as the *radial Loewner evolution*. It involves holomorphic functions normalized at the origin. An analogous process for functions normalized at a boundary point was studied by Kufarev and his students [25], see also [38]. Consider the upper half-plane $\mathbb{H} := \{z : \operatorname{Im} z > 0\}$ with a slit Γ along a Jordan arc. "Slit" means here that Γ is the image of an injective continuous function $\gamma : [0, T] \to \mathbb{H} \cup \mathbb{R}$, $T > 0$, such that $\gamma\big([0, T)\big) \subset \mathbb{H}$ and $\gamma(T) \in \mathbb{R}$. For each $t \in [0, T]$ there exists

[1]The first example of such situation was discovered by Kufarev [24], who found a continuous function k in (1.3) for which f_0 maps $\mathbb{D}$ onto a half-plane.

a unique conformal mapping f_t of $\mathbb{H}$ onto $\Omega_t := \mathbb{H} \setminus \gamma\big([t, T)\big)$ normalized by its expansion at ∞, $f_t(z) = z + c(t)/z + \ldots$, $c(t) < 0$. Changing the parametrization of Γ, one may assume that $c(t) = 2(t - T)$ for all $t \in [0, T]$. Similarly, to Loewner's original setting, the functions f_t satisfy a PDE, and the functions $\varphi_{s,t} := f_t^{-1} \circ f_s$, $0 \le s \le t \le T$, satisfy the corresponding characteristic ODE. Namely,

$$\frac{\partial f_t(z)}{\partial t} = -f_t'(z) \frac{2}{\lambda(t) - z}, \tag{1.4}$$

$$\frac{d\varphi_{s,t}(z)}{dt} = \frac{2}{\lambda(t) - \varphi_{s,t}(z)}, \ t \in [s, T], \quad \varphi_{s,s}(z) = z, \ z \in \mathbb{H}, \tag{1.5}$$

where $\lambda : [0, T] \to \mathbb{R}$ is a continuous function determined in a unique way by the slit Γ. Conversely, for any such function λ there exists a unique family of holomorphic univalent functions $f_t : \mathbb{H} \to \mathbb{H}$, $f_t(z) = z - 2(T - t)/z + \ldots$ (as $z \to \infty$), $t \in [0, T]$, with $f_s(\mathbb{H}) \subset f_t(\mathbb{H})$ whenever $0 \le s \le t \le T$ satisfying PDE (1.4). In turn, the corresponding family $\varphi_{s,t} := f_t^{-1} \circ f_s$, $0 \le s \le t \le T$, satisfies (1.5). These two differential equations are known in the modern literature as the *chordal Loewner equations*[2]. Since $f_T = \mathsf{id}_{\mathbb{H}}$, formula (1.2) for the radial case is now replaced by the simpler relation $f_t = \varphi_{t,T}$ for all $t \in [0, T]$.

Another approach to the chordal Loewner equation, based on the reduction to the radial Loewner equation by means of a time-dependent conformal change of variable can be found in the book [1], which is also a good source for the classical radial Loewner theory and its application to extremal problems for univalent functions.

Essentially equivalent generalizations of the slit chordal Loewner evolution were given in [2, 3, 19]. In this case the Cauchy kernel in the right-hand side of (1.4) and (1.5) is replaced by a function with positive imaginary part, locally integrable in t and holomorphic in z with a particular regularity and normalization condition at ∞.

The complete analogy between the radial and chordal slit Loewner evolutions, described above, is broken by the fact that the former is defined for all $t \ge 0$, while the latter is limited in time to a compact interval. Consider the following simple construction. Reparametrize the slit Γ in $\mathbb{H}$ in such a way that $\gamma(0) \in \mathbb{R}$, $\gamma\big((0, T]\big) \subset \mathbb{H}$, and the functions f_t are normalized by $f_t(z) = z - 2t/z + \ldots$ This simple trick allows to consider chordal Loewner evolution defined for all $t \ge 0$. In this case the system of domains $\Omega_t := f_t(\mathbb{H})$ is decreasing. Accordingly, the right-hand side of equations (1.4) and (1.5) change the sign. This version of the chordal Loewner evolution appeared in the well-celebrated paper [37] of 2000 by Odded Schramm[3], who constructed a stochastic version of Loewner evolution (SLE) replacing the control function λ with the Brownian motion times a positive coefficient. This invention

[2] Up to our best knowledge these equations are due to P.P.Kufarev, with the chordal Loewner ODE for the first time mentioned in his paper [23] without the factor 2. This factor is not principal, but makes the radial and chordal Loewner ODEs to have the same asymptotic behaviour of the vector field near the pole on the boundary. This is convenient for the study of relationship between the geometry of the solutions and analytic properties of the control functions k and λ.

[3] In fact he also considered the radial case of the Loewner evolution, but the chordal case proved to be more important for applications in the context of SLE.

by Schramm proved to be very important in connection with its applications in Statistical Physics, see e.g. [27].

The deterministic chordal Loewner evolution underlying SLE is driven by the following initial value problem for the Loewner chordal ODE (with the "−" sign)

$$\frac{dw}{dt} = -\frac{2}{\lambda(t) - w}, \ t \geq 0, \quad w|_{t=0} = z \in \mathbb{H}. \tag{1.6}$$

The connection between the analogue of the classical Loewner chains in this setting and the above initial value problem is given by the following assertion, see[4], e.g., [27, Theorem 4.6, p. 93; Remark 4.10, p. 95],

Theorem A. *Let $\lambda : [0, +\infty) \to \mathbb{R}$ be a continuous function. Then the following assertions hold:*

(i) *For every $z \in \mathbb{H}$, there exists a unique maximal[5] solution $w(t) = w_z(t) \in \mathbb{H}$ to initial value problem* (1.6)*.*

(ii) *For every $t \geq 0$, the set Ω_t of all $z \in \mathbb{H}$ for which w_z is defined at the point t is a simply connected domain and the function $g_t : \Omega_t \to \mathbb{H}$ defined by $g_t(z) := w_z(t)$ for all $z \in \Omega_t$, is the unique conformal mapping of Ω_t onto $\mathbb{H}$ such that $g_t(z) - z \to 0$ as $z \to \infty$, $z \in \mathbb{H}$. Moreover, for each $t \geq 0$,*

$$g_t(z) = z + \frac{2t}{z} + O(1/|z|^2) \quad \text{as } z \to \infty, \ z \in \mathbb{H}.$$

(iii) *The family $f_t := g_t^{-1} : \mathbb{H} \to \Omega_t$, $t \geq 0$, satisfies the following initial value problem for the chordal Loewner PDE:*

$$\frac{\partial f_t(z)}{\partial t} = \frac{2}{\lambda(t) - z} \frac{\partial f_t(z)}{\partial z}, \qquad f_0 = \mathsf{id}_{\mathbb{H}}.$$

Nowadays the chordal Loewner evolution is more often considered in this "decreasing" framework rather than in the "increasing" framework used by Kufarev et al [25], see e.g. [6]. In particular, the geometry of the so-called Loewner hulls $K_t := \mathbb{H} \setminus f_t(\mathbb{H})$ was studied in a series of papers launched by [32, 28]. One of the basic questions addressed is under which analytical conditions on the control function λ the hulls K_t are formed by a growing slit along a Jordan arc and what are the relations between the regularity of λ and that of the slit. A similar question for the radial Loewner equation was considered, apparently for the first time, by Kufarev [23] in 1946, who proved that if the control function k in equation (1.3) has bounded derivative on a segment $[0, T]$, then for $0 \leq s \leq t \leq T$ the corresponding functions $\varphi_{s,t}$ map $\mathbb{D}$ onto $\mathbb{D}$ minus a slit along a C^1-smooth Jordan arc. Without attempting to give the complete list we mention some recent papers on the topic [29, 36, 30, 21, 39, 35].

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[4]For simplicity we formulate here only the special case of Theorem 4.6 from [27], when the family of measures μ_t, defining the vector field in the right-hand side of (a generalization of) the chordal Loewner equation, is a family of Dirac measures. In this case the equation in [27, Theorem 4.6] reduces to the above chordal Loewner equation (1.6).

[5]In this case "maximal", or "non-extendable", means that there are no solutions $[0, T) \ni t \mapsto \tilde{w}_z(t) \in \mathbb{H}$ to (1.6), with $T \in (0, +\infty]$, such that w_z is the restriction of $\tilde{w}_z$ to a proper subset of $[0, T)$.

In a similar way the radial Loewner evolution (both associated with the original "slit" Loewner equation and the more general Loewner – Kufarev equation) can be considered in the "decreasing" way when the functions f_t are self-maps of $\mathbb{D}$ with the image domains $f_t(\mathbb{D})$ forming a decreasing family. Another variant, appeared in the recent publications, is the so-called *whole-plane* radial Loewner evolution defined for all $t \in \mathbb{R}$, see e.g. [27, §4.3], which however can be reduced to increasing and decreasing Loewner evolutions.

In addition to the frameworks described above, we would like to mention several studies [19, 16, 17, 13, 20, 18] devoted to the infinitesimal representation of various semigroups in $\mathsf{Hol}(\mathbb{D},\mathbb{D})$, which fits very well in a general heuristic scheme containing the increasing radial and chordal Loewner evolutions. Although independent proofs have been given for different concrete examples, there are common ideas which it is pertinent to sketch here without going much into details. Let $\mathcal{U} \subset \mathsf{Hol}(\mathbb{D},\mathbb{D})$ be a semigroup w.r.t. the operation of composition containing the neutral element $\mathsf{id}_{\mathbb{D}}$. Consider a continuous one-parameter subsemigroup $(\phi_t) \subset \mathcal{U}$, i.e. a family $(\phi_t)_{t\ge 0}$ in $\mathcal{U}$ such that $\phi_0 = \mathsf{id}_{\mathbb{D}}$, $\phi_t \circ \phi_s = \phi_{t+s}$ for any $s, t \ge 0$, and $\phi_t(z) \to z$ as $t \to +0$ for all $z \in \mathbb{D}$. It is known that for each $z \in \mathbb{D}$ the function $t \mapsto \phi_t(z)$ is the solution to the initial value problem

$$\frac{dw}{dt} = G(w), \quad w|_{t=0} = z, \tag{1.7}$$

with some holomorphic function $G : \mathbb{D} \to \mathbb{C}$ called the *infinitesimal generator* of (ϕ_t). Denote the set of all infinitesimal generators of one-parameter semigroups in $\mathcal{U}$ by $\mathcal{G}_{\mathcal{U}}$.

In contrast to the theory of finite-dimensional Lie groups, the union of all one-parameter semigroups in $\mathcal{U}$ does not coincide with $\mathcal{U}$ in general. That is why one has to consider a non-autonomous version of equation (1.7) of the form

$$dw/dt = G(w, t), \tag{1.8}$$

where the non-autonomous vector field G is subject to the condition $G(\cdot, t) \in \mathcal{G}_{\mathcal{U}}$ for a.e. $t \ge 0$.

Under some reasonable conditions on G, for any $s \ge 0$ and $z \in \mathbb{D}$ there exists a unique solution $[s, +\infty) \ni t \mapsto w = w_{z,s}(t)$ to (1.8) with the initial condition $w(s) = z \in \mathbb{D}$, which is defined for all $t \ge s$ and depends on z holomorphically. Equation (1.8) is said to *give the infinitesimal representation* of the semigroup $\mathcal{U}$ if the union of all *evolution families* $(\varphi_{s,t}) = (z \mapsto w_{z,s}(t))$, $t \ge s \ge 0$, generated by (1.8) coincides with $\mathcal{U}$. The simplest and most classical example is the radial Loewner – Kufarev ODE giving the infinitesimal representation of the semigroup $\mathcal{U}_0 := \{\varphi \in \mathsf{Hol}(\mathbb{D},\mathbb{D}) : \varphi(0) = 0, \varphi'(0) > 0\}$, for which $\mathcal{G}_{\mathcal{U}_0}$ is known to be equal to

$$\Big\{z \mapsto -zp(z) : p \in \mathsf{Hol}\big(\mathbb{D}, \{\zeta : \mathsf{Re}\,\zeta > 0\}\big),\ \mathsf{Im}\,p(0) = 0\Big\} \cup \Big\{G(z) \equiv 0\Big\}.$$

Such infinitesimal representations turn out to be very useful, because typically, from the analytic point of view, the structure of $\mathcal{G}_{\mathcal{U}}$ is much simpler than that of $\mathcal{U}$. Note that the infinitesimal representation of $\mathcal{U}_0$ given by the Loewner – Kufarev ODE is the main tool that "captures" the univalence condition in de Branges' proof of the Bieberbach conjecture.

1.2. **General approach in Loewner Theory.** Recently a general version of Loewner evolution, which includes the radial and chordal variants as very special cases, has been introduced by Bracci and the two first authors [7, 8], and studied further in [5, 9, 10]. Relying partially on the theory of one-parameter semigroups, which can be regarded as the autonomous version of Loewner theory, they have given an intrinsic definition of an evolution family in *the whole semigroup* $\mathsf{Hol}(\mathbb{D},\mathbb{D})$ as follows.

Definition 1.1 ([7])**.** A family $(\varphi_{s,t})_{t\ge s\ge 0}$ of holomorphic self-maps of the unit disc is an *evolution family of order* d with $d\in[1,+\infty]$ (in short, an L^d*-evolution family*) if it satisfies the following conditions:

EF1. $\varphi_{s,s}=\mathsf{id}_{\mathbb{D}}$ for all $s\ge 0$,
EF2. $\varphi_{s,t}=\varphi_{u,t}\circ\varphi_{s,u}$ whenever $0\le s\le u\le t<+\infty$,
EF3. for every $z\in\mathbb{D}$ and every $T>0$ there exists a non-negative function $k_{z,T}\in L^d([0,T],\mathbb{R})$ such that

$$|\varphi_{s,u}(z)-\varphi_{s,t}(z)|\le\int_u^t k_{z,T}(\xi)d\xi$$

whenever $0\le s\le u\le t\le T$.

Condition EF3 is to guarantee that any evolution family can be obtained via solutions of an ODE which resembles both the radial and chordal Loewner – Kufarev equations, as well as other versions of the Loewner equation established in [19, 16, 17, 13, 20, 18]. The vector fields that drive this generalized Loewner – Kufarev ODE are referred to as *Herglotz vector fields.*

Definition 1.2 ([7])**.** Let $d\in[1,+\infty]$. A *weak holomorphic vector field of order* d in the unit disc $\mathbb{D}$ is a function $G:\mathbb{D}\times[0,+\infty)\to\mathbb{C}$ with the following properties:

WHVF1. for all $z\in\mathbb{D}$, the function $[0,+\infty)\ni t\mapsto G(z,t)$ is measurable,
WHVF2. for all $t\in[0,+\infty)$, the function $\mathbb{D}\ni z\mapsto G(z,t)$ is holomorphic,
WHVF3. for any compact set $K\subset\mathbb{D}$ and any $T>0$ there exists a non-negative function $k_{K,T}\in L^d([0,T],\mathbb{R})$ such that

$$|G(z,t)|\le k_{K,T}(t)$$

for all $z\in K$ and for almost every $t\in[0,T]$.

We say that G is a *(generalized) Herglotz vector field* of order d if, in addition to conditions WHVF1 – WHVF3 above, for almost every $t\in[0,+\infty)$ the holomorphic function $G(\cdot,t)$ is an infinitesimal generator of a one-parameter semigroup in $\mathsf{Hol}(\mathbb{D},\mathbb{D})$.

Remark 1.3. By [7, Theorem 4.8], the class of all Herglotz vector fields of order d coincides with that of all functions $G:\mathbb{D}\times[0,+\infty)\to\mathbb{C}$ which can represented in the form

$$G(z,t):=(\tau(t)-z)(1-\overline{\tau(t)}z)p(z,t)\quad\text{for all } z\in\mathbb{D}\text{ and a.e. } t\ge 0, \tag{1.9}$$

where τ is any measurable function from $[0,+\infty)$ to $\overline{\mathbb{D}}$ and $p:\mathbb{D}\times[0,+\infty)\to\mathbb{C}$ satisfies the following conditions: (1) $p(\cdot,t)$ is holomorphic in $\mathbb{D}$ for every $t\ge 0$

with $\mathsf{Re}\, p(z,t) \geq 0$ for all $z \in \mathbb{D}$ and $t \geq 0$; (2) $p(z,\cdot)$ is in $L^d_{\mathrm{loc}}([0,+\infty),\mathbb{C})$ for every $z \in \mathbb{D}$.

The generalized Loewner – Kufarev equation

$$\frac{dw}{dt} = G(w,t),\ t \geq s, \quad w(s) = z, \tag{1.10}$$

resembles the radial Loewner – Kufarev ODE when $\tau \equiv 0$ and $p(0,t) \equiv 1$. Furthermore, with the help of the Cayley map between $\mathbb{D}$ and $\mathbb{H}$, the chordal Loewner equation appears to be the special case of (1.10) with $\tau \equiv 1$.

In a similar way, different notions of evolution families considered previously in the literature can be reduced to special cases of L^d-evolution families defined above.

Equation (1.10) establishes a 1-to-1 correspondence between evolution families of order d and Herglotz vector fields of the same order. Namely, the following theorem takes place.

Theorem B ([7, Theorem 1.1]). *For any evolution family $(\varphi_{s,t})$ of order $d \in [1,+\infty]$ there exists an (essentially) unique Herglotz vector field $G(z,t)$ of order d such that for every $z \in \mathbb{D}$ and every $s \geq 0$ the function $[s,+\infty) \ni t \mapsto w_{z,s}(t) := \varphi_{s,t}(z)$ solves the initial value problem* (1.10).

Conversely, given any Herglotz vector field $G(z,t)$ of order $d \in [1,+\infty]$, for every $z \in \mathbb{D}$ and every $s \geq 0$ there exists a unique solution $[s,+\infty) \ni t \mapsto w_{z,s}(t)$ to the initial value problem (1.10). *The formula $\varphi_{s,t}(z) := w_{z,s}(t)$ for all $s \geq 0$, all $t \geq s$, and all $z \in \mathbb{D}$, defines an evolution family $(\varphi_{s,t})$ of order d.*

Here by *essential uniqueness* we mean that two Herglotz vector fields $G_1(z,t)$ and $G_2(z,t)$ corresponding to the same evolution family must coincide for a.e. $t \geq 0$.

The general notion of a Loewner chain has been given[6] in [9].

Definition 1.4 ([9]). A family $(f_t)_{t\geq 0}$ of holomorphic maps of $\mathbb{D}$ is called a *Loewner chain of order d* with $d \in [1,+\infty]$ (in short, an *L^d-Loewner chain*) if it satisfies the following conditions:

LC1. each function $f_t : \mathbb{D} \to \mathbb{C}$ is univalent,

LC2. $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s < t < +\infty$,

LC3. for any compact set $K \subset \mathbb{D}$ and any $T > 0$ there exists a non-negative function $k_{K,T} \in L^d([0,T],\mathbb{R})$ such that

$$|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi)d\xi$$

whenever $z \in K$ and $0 \leq s \leq t \leq T$.

This definition of (generalized) Loewner chains matches the abstract notion of evolution family introduced in [7]. In particular the following statement holds.

[6]See also [5] for a straightforward extension of this notion to complex manifolds. It is worth mentioning that, in that paper, the construction of an associated Loewner chain for a given evolution family is based on arguments borrowed from Category theory and so it differs notably from the one we used in [9, Theorems 1.3 and 1.6].

Theorem C ([9, Theorem 1.3]). *For any Loewner chain (f_t) of order $d \in [1, +\infty]$, if we define*

$$\varphi_{s,t} := f_t^{-1} \circ f_s \quad \textit{whenever } 0 \le s \le t,$$

then $(\varphi_{s,t})$ is an evolution family of the same order d. Conversely, for any evolution family $(\varphi_{s,t})$ of order $d \in [1, +\infty]$, there exists a Loewner chain (f_t) of the same order d such that

$$f_t \circ \varphi_{s,t} = f_s \quad \textit{whenever } 0 \le s \le t.$$

In the situation of this theorem we say that the Loewner chain (f_t) and the evolution family $(\varphi_{s,t})$ are *associated with* each other. It was proved in [9] that given an evolution family $(\varphi_{s,t})$, an associated Loewner chain (f_t) is unique up to conformal maps of $\cup_{t\ge 0} f_t(\mathbb{D})$.

Thus in the framework of the approach described above the essence of Loewner theory is represented by the essentially 1-to-1 correspondence among Loewner chains, evolution families, and Herglotz vector fields.

Remark 1.5. Definition 1.1 does not require elements of an evolution family to be univalent. However, this condition is automatically satisfied. Indeed, by Theorem B, any evolution family $(\varphi_{s,t})$ can be obtained via solutions to the generalized Loewner – Kufarev ODE. Hence the univalence of $\varphi_{s,t}$'s follows from the uniqueness of solutions to this ODE. For an essentially different direct proof see [8, Proposition 3].

To conclude the section, we mention that an analogous approach has been suggested for the Loewner theory in the annulus [11, 12]. However, in this paper we restrict ourselves to the Loewner theory for simply connected domains.

1.3. **Aim of the paper and main results.** As one can see from what is stated in Subsect. 1.1, there are essentially two different ways to deal with Loewner evolution, regardless of which specific variant of the theory we consider. One of them involves conformal maps f_t onto an increasing family of simply connected domains described by (one or another version of) the Loewner equations. in this case, the Loewner ODE being equipped with an initial condition at the left end-point generate, a non-autonomous semi-flow consisting of holomorphic self-maps $\varphi_{s,t}$ of the reference domain ($\mathbb{D}$ or $\mathbb{H}$). The other way to study Loewner evolution, which can be colloquially called "decreasing", is formally obtained by reversing the direction of the time parameter t and changing correspondingly the sign in the Loewner equations. The initial condition for the Loewner ODE is again given *at the left end-point* and hence in contrast to the "increasing" case, the solutions do not extend unrestrictedly in time. The functions generated by such initial value problem map a decreasing family of simply connected domains conformally onto the reference domain.

It is clear that locally in time there is no big difference between these two ways of reasoning: the connection between them is easily established by considering, for an arbitrary $T > 0$, the parameter change $t \mapsto T - t$. At the same time the diversity of settings within which the Loewner evolution has been considered in literature may cause certain difficulties. The primary aim of this paper is to describe rigorously how the general approach discussed in the previous subsection can be "translated" to the "decreasing" setting. First of all we introduce "decreasing" analogues of Loewner chains and evolution families.

For a set $E \subset \mathbb{R}$ we denote by $\Delta(E)$ the "upper triangle" $\{(s,t) : s,t \in E,\ s \le t\}$.

Definition 1.6. Let $d \in [1,+\infty]$. A family $(f_t)_{t\ge 0}$ of holomorphic self-maps of the unit disk $\mathbb{D}$ will be called a *decreasing Loewner chain of order* d (or, in short, *decreasing* L^d*-chain*) if it satisfies the following conditions:

DC1. each function $f_t : \mathbb{D} \to \mathbb{D}$ is univalent,

DC2. $f_0 = \mathsf{id}_{\mathbb{D}}$ and $f_t(\mathbb{D}) \subset f_s(\mathbb{D})$ for all $(s,t) \in \Delta\big([0,+\infty)\big)$,

DC3. for any compact set $K \subset \mathbb{D}$ and all $T > 0$ there exists a non-negative function $k_{K,T} \in L^d\big([0,T],\mathbb{R}\big)$ such that

$$|f_s(z) - f_t(z)| \le \int_s^t k_{K,T}(\xi)d\xi$$

for all $z \in K$ and all $(s,t) \in \Delta\big([0,T]\big)$.

Remark 1.7. Note that the difference of this definition from that of an increasing Loewner chain resides not only in the opposite inclusion sign in condition DC2. We also assume that $f_0 = \mathsf{id}_{\mathbb{D}}$, which we always can do by composing with a conformal mapping, while in the "increasing" case no such restriction is imposed, because in general, it may be impossible to employ any conformal mapping except for linear transformations of $\mathbb{C}$.

Remark 1.8. As we will see in Sect. 3, by means of the Cayley map between $\mathbb{D}$ and $\mathbb{H}$, the notion of *chordal Loewner families* introduced by Bauer [6] reduces to a special case of decreasing Loewner chains of order $d = +\infty$.

Now we introduce reverse evolution families, the "decreasing" counterparts of L^d-evolution families.

Definition 1.9. Let $d \in [1,+\infty]$. A family $(\varphi_{s,t})_{0\le s\le t}$ of holomorphic self-maps of the unit disk $\mathbb{D}$ is a *reverse evolution family of order* d if the following conditions are fulfilled:

REF1. $\varphi_{s,s} = \mathsf{id}_{\mathbb{D}}$,

REF2. $\varphi_{s,t} = \varphi_{s,u} \circ \varphi_{u,t}$ for all $0 \le s \le u \le t < +\infty$,

REF3. for any $z \in \mathbb{D}$ and any $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0,T],\mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \le \int_u^t k_{z,T}(\xi)d\xi$$

for all $s,t,u \in [0,T]$ satisfying inequality $s \le u \le t$.

Remark 1.10. Speaking informally, comparing with Definition 1.1, the parameters s and t switch their roles. That is why condition EF3 is not converted to condition REF3 under the "time reversing trick".

Using the results of [7, 9] we deduce relations among decreasing Loewner chains, reverse evolution families and Herglotz vector fields. In particular, we obtain the following extension of Theorem A (and its more abstract form in [27, Chapter 4]).

Theorem 1.11. *Let G be a Herglotz vector field of order $d \in [1, +\infty]$. Then:*

(i) *For every $z \in \mathbb{D}$, there exists a unique maximal*[7] *solution $w(t) = w_z(t) \in \mathbb{D}$ to the following initial value problem*

$$\frac{dw}{dt} = -G(w,t), \qquad w(0) = z\,. \tag{1.11}$$

(ii) *For every $t \geq 0$, the set Ω_t of all $z \in \mathbb{D}$ for which w_z is defined at the point t is a simply connected domain and the function $g_t : \Omega_t \to \mathbb{D}$ defined by $g_t(z) := w_z(t)$ for all $z \in \Omega_t$, maps Ω_t conformally onto $\mathbb{D}$.*

(iii) *The functions $f_t := g_t^{-1} : \mathbb{D} \to \Omega_t$ form a decreasing Loewner chain of order d, which is the unique solution to the following initial value problem for the Loewner – Kufarev PDE:*

$$\frac{\partial f_t(z)}{\partial t} = \frac{\partial f_t(z)}{\partial z} G(z,t), \qquad f_0 = \mathsf{id}_{\mathbb{D}}. \tag{1.12}$$

Remark 1.12. The right-hand side of (1.11) and that of (1.10) in Subsect. 1.2 are not continuous in t. Hence these equations should be understood as Carathéodory ODEs, see e.g. [15, §I.1], [26, Ch. 18] for the general theory of such equations, or [11, §2] for the basic results in the case of the right-hand side holomorphic w.r.t. w. As for PDE (1.12) there does not seem to be any abstract theory that suits completely well our purposes. We discuss the notion of a solution to (1.12) in Subsect. 2.1.

Some of the statements we get are obtained by careful "translation" of known results under the change of the time parameter, while other statements, in particular those involving time regularity of evolution families, require deeper ideas. As an interesting 'byproduct" we prove the following assertion.

Denote by $AC^d(X,Y)$, where $X \subset \mathbb{R}$ and $Y \subset \mathbb{C}$, the class of all locally absolutely continuous functions $f : X \to Y$ whose derivative is of class L^d_{loc}.

Theorem 1.13. *Let $(f_t)_{t\geq 0}$ be a family of holomorphic functions in the unit disc $\mathbb{D}$ satisfying conditions LC1 and LC2. Then (f_t) is an L^d-Loewner chain if and only if*

LC3w. *For any $T > 0$ there exist two distinct points $\zeta_1, \zeta_2 \in \mathbb{D}$ such that the mappings $[0,T] \ni t \mapsto f_t(\zeta_j)$, $j = 1, 2$, both belong to $AC^d([0,T], \mathbb{D})$.*

Remark 1.14. It follows immediately from Definitions 1.4 and 1.6 that the above theorem is equivalent to the following statement: *a family $(f_t)_{t\geq 0}$ of holomorphic functions $f_t : \mathbb{D} \to \mathbb{C}$ satisfying conditions DC1 and DC2 is a decreasing Loewner chain of order d if and only if the above condition LC3w holds.*

The rest of the paper is organized as follows. In Section 2 we make precise definition of what is meant by solutions to the generalized Loewner – Kufarev PDE (1.12) and prove some auxiliary propositions and lemmas.

In Section 3 we discuss relationship between decreasing Loewner chains and Herglotz vector fields. In particular, we prove Theorem 1.11. We also establish a kind of inverse theorem (Theorem 3.2).

[7] in this case "maximal", or "non-extendable", means that there are no solutions $[0,T) \ni t \mapsto \tilde{w}_z(t) \in \mathbb{D}$ to (1.11), with $T \in (0, +\infty]$, such that w_z is the restriction of $\tilde{w}_z$ to a proper subset of $[0,T)$.

Section 4 is devoted to the study of the relationship of reversed evolution families on the one side, and decreasing Loewner chains together with the corresponding Herglotz vector fields on the other side.

Finally in Section 5 we prove Theorem 1.13.

2. Auxiliary statements

2.1. **Solutions to the generalized Loewner – Kufarev PDE.** Consider the (generalized) Loewner – Kufarev PDE

$$\frac{\partial F(z,t)}{\partial t} = \epsilon \frac{\partial F(z,t)}{\partial z} G(z,t), \tag{2.1}$$

where G is a Herglotz vector field and ϵ is a constant in $\{-1,+1\}$ whose value depends on whether we deal with the increasing ($\epsilon \equiv -1$) or decreasing ($\epsilon \equiv 1$) variant of the theory.

The formulation of Theorems 1.11, 3.2, and 4.2 contains the notion of a solution to the Loewner – Kufarev PDE, which should be defined more precisely. Similar to the classical text [34, Chapter 6], we give the definition as follows.

Definition 2.1. By a solution to the Loewner – Kufarev PDE equation (2.1) we mean a function $F : \mathbb{D} \times E \to \mathbb{C}$, where $E \subset [0,+\infty)$ is an interval, such that

S1. F is continuous in $\mathbb{D} \times E$;
S2. for every $t \in E$ the function $F(\cdot,t)$ is holomorphic in $\mathbb{D}$;
S3. for every $z \in \mathbb{D}$ the function $F(z,\cdot)$ is locally absolutely continuous in E;
S4. for every $z \in \mathbb{D}$ equality (2.1) holds a.e. in E.

Remark 2.2. Condition S4 means exactly the following: for each $z \in \mathbb{D}$ there exists a null-set $M_z \subset E$ such that (2.1) holds for all $z \in \mathbb{D}$ and all $t \in E \setminus M_z$. Note that *a priori* the sets M_z can depend on z. We show below that under conditions S1 – S3 this set can be chosen *independently* of z.

We complete this subsection with the proof of the following technical statement, which might appear implicitly, in one or another context, in some works on Loewner theory.

Proposition 2.3. *Let $\epsilon \in \{-1,+1\}$ and let G be a Herglotz vector field of some order $d \in [1,+\infty]$. Suppose that a function $F : \mathbb{D} \times E \to \mathbb{C}$ satisfies conditions S1 – S4. Then the following assertions hold:*

(i) *The function $F(z,t)$ is locally absolutely continuous in t uniformly w.r.t. z on every compact subset of $\mathbb{D}$, i.e. for any compact set $K \subset \mathbb{D}$ and any compact interval $I \subset E$ the mapping $t \mapsto F(\cdot,t) \in \mathsf{Hol}(\mathbb{D},\mathbb{C})$ is absolutely continuous on I w.r.t. the metric $d_K(f,g) := \max_{z\in K} |f(z) - g(z)|$.*

(ii) *There is a null-set $N \subset [0,+\infty)$ such that for all $z \in \mathbb{D}$ and all $t \in E \setminus N$ the partial derivative $\partial F(z,t)/\partial t$ exists and equality (2.1) holds. Moreover, for each $t \in E \setminus N$, $\big(F(z,t+h) - F(z,t)\big)/h \to \partial F(z,t)/\partial t$ locally uniformly in $\mathbb{D}$ as $h \to 0$.*

(iii) *For any solution $t \mapsto w(t) \in \mathbb{D}$ to the generalized Loewner – Kufarev ODE*

$$\dot{w} = -\epsilon\, G(w,t) \tag{2.2}$$

the function $t \mapsto F\big(w(t),t\big)$ is constant in the domain of w intersected with E.

Proof. By S3 for any $z \in \mathbb{D}$ the map $F(z,\cdot)$ is locally absolutely continuous on E. Hence by S4, for any $z \in \mathbb{D}$ and any $(s,t) \in \Delta(E)$,

$$F(z,t) - F(z,s) = \int_s^t \frac{\partial F(z,\xi)}{\partial t}\,d\xi = \epsilon \int_s^t \frac{\partial F(z,\xi)}{\partial z} G(z,\xi)\,d\xi. \tag{2.3}$$

Fix now any compact interval $I \subset E$ and any closed disk $K := \{z : |z| \le r\}$, with some $r \in (0,1)$. From S1 it follows that $\big(F(\cdot,t)\big)_{t\in I}$ forms a compact subset in $\mathsf{Hol}(\mathbb{D},\mathbb{C})$. Therefore there exists a constant $C>0$ such that $|\partial F(z,t)/\partial z| \le C$ for all $z \in K$ and all $t \in I$. Combined with equality (2.3) and condition WHVF3 from Definition 1.2 of a Herglotz vector field, this fact implies that there exists a non-negative function $k_{I,K} \in L^d\big(I,\mathbb{R}\big)$ such that

$$|F(z,t) - F(z,s)| \le C \int_s^t |G(z,\xi)|\,d\xi \le C \int_s^t k_{I,K}(\xi)\,d\xi \tag{2.4}$$

for any $z \in K$ and any $(s,t) \in \Delta(I)$. Assertion (i) follows now immediately.

Applying the above argument to a sequence of closed disks ($K_n \subset \mathbb{D}$) and a sequence of compact intervals ($I_k \subset E$) whose unions cover $\mathbb{D}$ and E, respectively, one can easily construct a sequence of non-negative functions $k_n \in L^d_{\mathrm{loc}}(E,\mathbb{R})$ such that for each $n \in \mathbb{N}$,

$$|F(z,t) - F(z,s)| \le \int_s^t k_n(\xi)\,d\xi \tag{2.5}$$

for all $(s,t) \in \Delta(E)$ and all $z \in K_n$. Choose any $t_0 \in E$. Since the functions

$$Q_n(t) := \int_{t_0}^t k_n(\xi)\,d\xi, \quad t \in E,\ n \in \mathbb{N},$$

are absolutely continuous, there exists a null-set $N_0 \subset E$ such that $Q_n'(t)$ exists finitely for all $t \in E \setminus N_0$ and all $n \in \mathbb{N}$. Now from (2.5) it follows that for each $t \in E \setminus N_0$ and some $\varepsilon > 0$ small enough the family

$$\mathcal{F}_t := \left\{ \frac{F(\cdot,t') - F(\cdot,t)}{t'-t} : t' \in E,\ 0 < |t'-t| < \varepsilon \right\}$$

is bounded on each of K_n's. Hence $\mathcal{F}_t$ is relatively compact in $\mathsf{Hol}(\mathbb{D},\mathbb{C})$ provided $t \in E \setminus N_0$. For each $z \in \mathbb{D}$ by M_z we denote the null set in E aside which F satisfies (2.1). Following a standard technique we apply now Vitali's principle to the family $\mathcal{F}_t$ and the set $\{z_j := 1 - 1/(j+1) : j \in \mathbb{N}\}$ in order to conclude that if $t \in E \setminus N_0$ and $t \notin \cup_{j\in\mathbb{N}} M_{z_j}$, then

$$\frac{F(z,t') - F(z,t)}{t'-t} \to \epsilon \frac{\partial F(z,t)}{\partial z} G(z,t)$$

locally uniformly in $\mathbb{D}$ as $t' \to t$, $t' \in E$. This proves (ii) with $N := N_0 \cup \big(\cup_{j\in\mathbb{N}} M_{z_j}\big)$.

To prove (iii), we note that due to compactness of $\{F(\cdot,t)\}_{t\in I}$ for any compact interval $I \subset E$, the limit

$$\lim_{\zeta \to z} \frac{F(\zeta,t) - F(z,t)}{\zeta - z} = \frac{\partial F(z,t)}{\partial z} \tag{2.6}$$

is attained uniformly w.r.t. $t \in I$ for any fixed $z \in \mathbb{D}$. This justifies the formal computation

$$\frac{dF(w(t),t)}{dt} = \dot{w}(t)\, \frac{\partial F(z,t)}{\partial z}\bigg|_{z:=w(t)} + \frac{\partial F(z,t)}{\partial t}\bigg|_{z:=w(t)}$$
$$= -\epsilon\, G(w(t),t)\, \frac{\partial F(z,t)}{\partial z}\bigg|_{z:=w(t)} + \left(\epsilon\, \frac{\partial F(z,t)}{\partial z} G(z,t)\right)\bigg|_{z:=w(t)} = 0$$

for any $z \in \mathbb{D}$ and a.e. $t \in E$. From (i) and compactness of $\{F(\cdot,t)\}_{t\in I}$ for compact intervals $I \subset E$ it follows that $t \mapsto F(w(t),t)$ is locally absolutely continuous in its domain. Thus we may conclude that this function is constant. The proof is now complete. □

2.2. **Some lemmas.** In what follows we will take advantage of several lemmas proved in this subsection. We start with a kind of "rigidity" lemma, going back to the Schwarz lemma and the classical growth estimate for holomorphic functions with positive real part.

Denote by $\rho_{\mathbb{D}}$ the pseudohyperbolic distance in $\mathbb{D}$, i.e. let

$$\rho_{\mathbb{D}}(z,w) := \left|\frac{w-z}{1-\bar{w}\,z}\right|, \quad \text{for all } z, w \in \mathbb{D}.$$

Lemma 2.4. *There exists a universal constant $C > 0$ such that for any $\psi \in \mathsf{Hol}(\mathbb{D},\mathbb{D})$ with $\psi(0) = 0$, any $r \in (0,1)$, and any $\zeta_0 \in \mathbb{D}\setminus\{0\}$,*

$$\rho_{\mathbb{D}}\big(\psi(\zeta),\zeta\big) \le \frac{C|\psi(\zeta_0)-\zeta_0|}{|\zeta_0|(1-|\zeta_0|^2)(1-r)^2} \tag{2.7}$$

whenever $|\zeta| \le r$.

Proof. The proof is based on [11, Lemma 3.8], according to which

$$\big|\psi(\zeta)-\zeta\big| \le \frac{C|\psi(\zeta_0)-\zeta_0|}{|\zeta_0|(1-|\zeta_0|^2)(1-r^2)} \quad \text{whenever } |\zeta| \le r,$$

where $C > 0$ is a universal constant.

From the Schwarz lemma it follows that $\big|1-\bar{\zeta}\,\psi(\zeta)\big| \ge 1-|\zeta|^2$. Hence to deduce (2.7) it remains to notice that $1/(1-r^2) \le 1/(1-r)$. □

Corollary 2.5. *For any $\varphi \in \mathsf{Hol}(\mathbb{D},\mathbb{D})$, any $\zeta, \zeta_0 \in \mathbb{D}\setminus\{0\}$, and any $r \in (0,1)$,*

$$\rho_{\mathbb{D}}\big(\varphi(\zeta),\zeta\big) \le C\frac{|\varphi(\zeta_0)-\zeta_0| + 4|\varphi(0)|}{|\zeta_0|(1-|\zeta_0|)^2(1-r)^2} \quad \textit{whenever } |\zeta| \le r, \tag{2.8}$$

where C is the universal constant from Lemma 2.4.

Proof. Apply Lemma 2.4 for the function $\psi := \ell \circ \varphi$, where

$$\ell(z) := \frac{z-w_0}{1-\overline{w}_0 z}, \quad w_0 := \varphi(0).$$

Let us estimate first $|\ell(z)-z|$. For any $z \in \mathbb{D}$ we have

$$\big|\ell(z)-z\big| = \left|\frac{\overline{w}_0 z^2 - w_0}{1-\overline{w}_0 z}\right| \le \frac{2|w_0|}{1-|z|} = \frac{2|\varphi(0)|}{1-|z|}. \tag{2.9}$$

Similarly, $|w-\ell^{-1}(w)| \le 2|\varphi(0)|/(1-|w|)$ for all $w \in \mathbb{D}$. Setting $w := \psi(\zeta_0)$ and bearing in mind that in this case $|w| \le |\zeta_0|$ by the Schwarz lemma, we therefore conclude that

$$(2.10)\quad \big|\psi(\zeta_0)-\zeta_0\big| \le \big|\varphi(\zeta_0)-\zeta_0\big| + \big|\psi(\zeta_0)-\ell^{-1}\big(\psi(\zeta_0)\big)\big| \le \big|\varphi(\zeta_0)-\zeta_0\big| + \frac{2|\varphi(0)|}{1-|\zeta_0|}.$$

By the invariance of the pseudohyperbolic distance under automorphisms of $\mathbb{D}$ and by the triangle inequality, for each $\zeta \in \mathbb{D}$ we have

$$\rho_{\mathbb{D}}\big(\varphi(\zeta),\zeta\big) = \rho_{\mathbb{D}}\big(\psi(\zeta),\ell(\zeta)\big) \le \rho_{\mathbb{D}}(\ell(\zeta),\zeta) + \rho_{\mathbb{D}}\big(\psi(\zeta),\zeta\big).$$

Inequality (2.9) implies that $\rho_{\mathbb{D}}\big(\ell(\zeta),\zeta\big) \le \big|\ell(\zeta)-\zeta\big|/\big(1-|\zeta|\big) \le 2|\varphi(0)|/\big(1-|\zeta|\big)^2$, while the estimate for $\rho_{\mathbb{D}}\big(\psi(\zeta),\zeta\big)$ is obtained from (2.10) and (2.7).

Now (2.8) follows easily. □

Corollary 2.6. *For every $r, R, \rho \in (0,1)$ there exists a constant $\tilde{C} = \tilde{C}(r,R,\rho)$ such that for any $\varphi \in \mathsf{Hol}(\mathbb{D},\mathbb{D})$,*

$$(2.11)\qquad \rho_{\mathbb{D}}\big(\varphi(\zeta),\zeta\big) \le \tilde{C}(r,R,\rho)\big(|\varphi(\zeta_1)-\zeta_1| + |\varphi(\zeta_2)-\zeta_2|\big)$$

whenever $\zeta,\zeta_1,\zeta_2 \in \mathbb{D}$ satisfy the conditions $|\zeta| \le r$, $|\zeta_j| \le R$, $j=1,2$, and $\rho_{\mathbb{D}}(\zeta_2,\zeta_1) \ge \rho$.

Proof. The statement of the corollary can be obtained in the following way. Applying Corollary 2.5 for $\ell^{-1}\circ\varphi\circ\ell$ and $\zeta_0 := \ell^{-1}(\zeta_2)$, where ℓ is an automorphism of $\mathbb{D}$ sending 0 to ζ_1, one obtains an estimate for $\rho_{\mathbb{D}}\big(\varphi(\ell(\zeta)),\ell(\zeta)\big) = \rho_{\mathbb{D}}\big((\ell^{-1}\circ\varphi\circ\ell)(\zeta),\zeta\big)$.

Now substitute $\ell^{-1}(\zeta)$ for ζ in order to deduce (2.11). To carry out these estimates, we also use the fact that $|\ell^{-1}(z)| \le (|z|+|\zeta_1|)/(1+|z\zeta_1|)$ and $|(\ell^{-1})'(z)| \le 2/(1-|\zeta_1|)$ for all $z \in \mathbb{D}$. Since the concrete expression for the constant $\tilde{C}(r,R,\rho)$ is not important for our purposes, we omit the details. □

Lemma 2.7. *Let $(f_t)_{t\in[0,T]}$ be a family of holomorphic univalent functions $f_t : \mathbb{D} \to \mathbb{C}$, satisfying the following conditions:*

(i) *$\{f_t : t \in [0,T]\}$ is a normal family in $\mathbb{D}$;*
(ii) *there exist two points $\zeta_1,\zeta_2 \in \mathbb{D}$, $\zeta_1 \ne \zeta_2$, such that the functions $[0,T] \ni t \mapsto f_t(\zeta_j) \in \mathbb{C}$ are continuous for $j=1,2$.*

Then for any compact set $K \subset \mathbb{D}$ there exists a constant $M_K > 0$ such that

$$(2.12)\qquad |z_1 - z_2| \le M_K |f_t(z_1) - f_t(z_2)|$$

for any $t \in [0,T]$, any $z_1 \in K$ and all $z_2 \in \mathbb{D}$.

Proof. Assume the conclusion is false. Then there exist sequences $(z_n^{(1)})$, $(z_n^{(2)})$, (t_n) and a compact set $K \subset \mathbb{D}$ such that for all $n \in \mathbb{N}$ we have:

(a) $z_n^{(1)} \in K$, $z_n^{(2)} \in \mathbb{D}$ and $t_n \in [0,T]$;
(b) $|z_n^{(1)} - z_n^{(2)}| > n|f_{t_n}(z_n^{(1)}) - f_{t_n}(z_n^{(2)})|$.

Since $|z_n^{(1)} - z_n^{(2)}| < 2$, from (b) it follows that

(c) $|f_{t_n}(z_n^{(1)}) - f_{t_n}(z_n^{(2)})| \to 0$ as $n \to +\infty$.

Recall that by (i), (f_t) constitutes a normal family in $\mathbb{D}$. Hence using (a) and passing if necessary to subsequences, we can assume that

(d) $z_n^{(1)} \to z_0$ and $f_{t_n} \to f$ locally uniformly in $\mathbb{D}$ as $n \to +\infty$

for some $z_0 \in K$ and some $f \in \mathsf{Hol}(\mathbb{D}, \overline{\mathbb{C}})$.

By condition (ii) the functions $[0,T] \ni t \mapsto f_t(\zeta_j)$, $j = 1, 2$, are continuous. Moreover, $f_t(\zeta_1) \neq f_t(\zeta_2)$ for all $t \in [0,T]$. Therefore, $|f_{t_n}(\zeta_1) - f_{t_n}(\zeta_2)| > m$ and $|f_{t_n}(\zeta_1)| < M$ for all $n \in \mathbb{N}$ and some constants $m > 0$ and $M > 0$ not depending on n. Hence f is a holomorphic function in $\mathbb{D}$ different from a constant. With the help of the Hurwitz theorem it follows that f is univalent in $\mathbb{D}$.

Now fix any closed disk U centered at $w_0 := f(z_0)$ and lying in $f(\mathbb{D})$ and let U' be a closed disk of smaller radius centered also at w_0. Assertions (c) and (d) imply that there exists n_0 such that for all $n > n_0$, $n \in \mathbb{N}$ we have

(e) $f_{t_n}(z_n^{(j)}) \in U'$, $j = 1, 2$, and $U \subset f_{t_n}(\mathbb{D})$.

In particular, it follows that $(f_{t_n}^{-1})' \to (f^{-1})'$ uniformly on U' as $n \to +\infty$, $n > n_0$. Therefore there exists a constant $C > 0$ such that $|(f_{t_n}^{-1})'(w)| < C$ for all $n > n_0$, $n \in \mathbb{N}$ and all $w \in U'$. Finally, taking into account (e), we get

$$|z_n^{(1)} - z_n^{(2)}| < C|f_{t_n}(z_n^{(1)}) - f_{t_n}(z_n^{(2)})|$$

for all $n > n_0$, $n \in \mathbb{N}$. This contradicts assumption (b) and hence completes the proof of (2.12). □

3. Decreasing Loewner chains and Herglotz vector Fields

As we mentioned in the Introduction, while the classical Loewner theory deals with increasing Loewner chains over $[0, +\infty)$, the SLE theory, having recently caused a burst of interest to Loewner theory, is based on the "decreasing" counterpart of the classical constructions of [31, 22, 33, 34, 25, 2, 3, 19]. The variant of the Loewner ODE underlying the chordal SLE is the following equation

$$\dot{w} = \frac{2}{w - \lambda(t)}, \quad t \geq 0, \ w(0) = \zeta, \tag{3.1}$$

where the initial condition ζ is chosen in the upper half-plane $\mathbb{H} := \{\zeta : \mathsf{Im}\,\zeta > 0\}$ and $\lambda : [0, +\infty) \to \mathbb{R}$ is a continuous control function. The stochastic ODE describing SLE is obtained by substituting the Brownian motion (B_t) times a positive factor for $\lambda(t)$.

Remark 3.1. A more general (deterministic) form of (3.1) was studied by Bauer [6].

By means of the Cayley map $H(z) := i(1+z)/(1-z)$ from $\mathbb{D}$ onto $\mathbb{H}$ equation (3.1) can be rewritten as $\dot{w} = -G_{\lambda(t)}(w)$, where $G_\lambda(z) := (1/2)(1-z^3)\big(1-H^{-1}(\lambda)\big)/\big(z-H^{-1}(\lambda)\big)$ for all $z \in \mathbb{D}$ and all $\lambda \in \mathbb{R}$. With $\lambda(t)$ being continuous, it is easy to see that $\mathbb{D} \times [0, +\infty) \ni (z, t) \mapsto G_{\lambda(t)}(z)$ is a Herglotz vector field of order $d = +\infty$ (see Definition 1.2). More generally, although the definitions given in [2, 3, 4, 19, 6] differ, the authors of these works considered essentially the same class $\mathfrak{R}$ of non-autonomous vector fields in $\mathbb{H}$ consisting of functions $p : \mathbb{H} \times [0, +\infty) \to \mathbb{H}$ measurable in t for each $z \in \mathbb{H}$ and representable, for a.e. $t \geq 0$ fixed, in the following form

$$p(\zeta, t) = \int_{\mathbb{R}} \frac{d\mu_t(x)}{x - \zeta},$$

where μ_t, for each $t \geq 0$, is a probability measure on $\mathbb{R}$. Via the Cayley map, this class corresponds to the vector fields in $\mathbb{D}$ given by $G_p(z,t) = p(H(z),t)/H'(z)$. Using the estimate $|p(\zeta,t)| \leq 1/\mathsf{Im}\,\zeta$ for all $\zeta \in \mathbb{H}$, a.e. $t \geq 0$ and all $p \in \mathfrak{R}$ (see, e.g., [4, Lemma 1]) it is easy to see that the vector fields G_p represent again a particular case of Herglotz vector fields of order $d = +\infty$.

Thus Theorem 1.11, which we are going to prove in this section, can be regarded as an extension of Theorem A as well as of its more abstract form in [27, §4.1].

We also will establish a kind of inverse statement for Theorem 1.11.

Theorem 3.2. *Suppose (f_t) is a decreasing Loewner chain of order $d \in [1,+\infty]$. Denote $\Omega_t := f_t(\mathbb{D})$, $g_t := f_t^{-1} : \Omega_t \to \mathbb{D}$ for all $t \geq 0$, and $t(z) := \sup\{t \geq 0 : z \in \Omega_t\}$ for all $z \in \mathbb{D}$. Then there exists a Herglotz vector field G of order d and a null-set $N \subset [0,+\infty)$ such that the following three statements hold.*

(i) *For every $t \in [0,+\infty) \setminus N$ the function*

$$z \mapsto \frac{\partial g_t(z)}{\partial t} := \lim_{h\to 0} \frac{g_{t+h}(z) - g_t(z)}{h}$$

is well defined and holomorphic in Ω_t. Moreover, for every $z \in \mathbb{D}$ the function $[0,t(z)) \ni t \mapsto w_z(t) := g_t(z)$ is the maximal solution to the following initial value problem for the generalized Loewner–Kufarev ODE

$$\frac{dw}{dt} = -G(w,t),\ t \geq 0, \quad w(0) = z. \tag{3.2}$$

Substituting $w_z(t)$ for $w(t)$ turns this equation into equality that holds for all $t \in [0,t(z)) \setminus N$.

(ii) *The function $F(z,t) := f_t(z)$, $t \geq 0$, $z \in \mathbb{D}$, is a solution, in the sense of Definition 2.1, to the following generalized Loewner–Kufarev PDE*

$$\frac{\partial F(z,t)}{\partial t} = \frac{\partial F(z,t)}{\partial z} G(z,t) \tag{3.3}$$

with the initial condition $F(\cdot,0) = \mathsf{id}_{\mathbb{D}}$. Substituting $f_t(z)$ for $F(z,t)$ turns this equation into equality that holds for all $t \in [0,+\infty) \setminus N$ and all $z \in \mathbb{D}$.

(iii) *Given any holomorphic function $F_0 : \mathbb{D} \to \mathbb{C}$, the initial value problem $F(\cdot,0) = F_0$ for PDE (3.3) has a unique solution $(z,t) \mapsto F(z,t) \in \mathbb{C}$, which is defined for all $(z,t) \in \mathbb{D} \times [0,+\infty)$ and is given by the formula $F(\cdot,t) = F_0 \circ f_t$, $t \in [0,+\infty)$.*

The Herglotz vector field G for which at least one of these statements holds[8] *is essentially unique, i.e. any two such Herglotz vector fields should agree for all $t \in [0,+\infty) \setminus M$ and all $z \in \mathbb{D}$, where $M \subset [0,+\infty)$ is a null-set.*

Remark 3.3. In terminology of [6], a *Loewner chordal family* $(F_t)_{t\geq 0}$ is a family of holomorphic self-maps of $\mathbb{H}$ satisfying $F_t(\mathbb{H}) \subset F_s(\mathbb{H})$ whenever $0 \leq s \leq t$ and such that for each $t \geq 0$,

$$F_t(\zeta) = \zeta - \frac{t}{\zeta} + \gamma_t(\zeta),$$

[8]The null-set N in these statements may of course depend on G.

where $\gamma_t : \mathbb{H} \to \mathbb{C}$ is a holomorphic function with[9] $\angle\lim_{\zeta\to\infty} \zeta\gamma_t(\zeta) = 0$. It is easy to see that C^1-curves in $\mathbb{H} \cup \{\infty\}$ going to ∞ within a Stolz angle are mapped by each F_t onto C^1-curves, with the angles between them at ∞ being preserved. Hence for any $s \geq 0$, any $k > 1$ and any $k' > k$ there exist $R, R' > 0$ such that $D(R,k) \subset F_s\big(D(R',k')\big)$, where by $D(R,k)$ we denote the angular domain $\{\zeta : k\,\mathsf{Im}\,\zeta > |\zeta| > R\}$. Using this fact it is not difficult to show that for any $s \geq 0$ and any $t \geq 0$ the holomorphic function $\Phi_{s,t} := F_s^{-1} \circ F_t : \mathbb{H} \to \mathbb{H}$ has a similar expansion $\Phi_{s,t}(\zeta) = \zeta - (t-s)/\zeta + \gamma_{s,t}(\zeta)$ with $\angle\lim_{\zeta\to\infty} \zeta\gamma_{s,t}(\zeta) = 0$. Then $|\Phi_{s,t}(\zeta) - \zeta| \leq (t-s)/\mathsf{Im}\,\zeta$, see, e.g., [2, p. 7–12] or [10, p. 567–568]. Since for each $T > 0$ the family $\{F_s : s \in [0,T]\}$ is locally uniformly bounded (again use $|F_s(\zeta) - \zeta| \leq s/\mathsf{Im}\,\zeta$), this inequality leads to an estimate for $|F_t(\zeta) - F_s(\zeta)| = |F_s(\Phi_{s,t}(\zeta)) - F_s(\zeta)|$ in terms of $t-s$, which in turn implies that up to the Cayley map, Loewner chordal families defined in [6] are a particular case of decreasing Loewner chains of order d, introduced in this paper. Thus Theorem 3.2(ii) can be regarded as an extension of [6, Theorem 5.3] to the general case, while Theorem 1.11(iii) represents an extension of [6, Theorem 5.6].

3.1. **Proof of Theorem 3.2.** Let us fix any $T > 0$ and define

$$h_t^T := \begin{cases} f_{T-t}, & \text{if } t \in [0,T], \\ \mathsf{id}_{\mathbb{D}}, & \text{if } t \in (T,+\infty). \end{cases}$$

It is easy to see that $(h_t^T)_{t\geq 0}$ is an (increasing) Loewner chains of order d. By Theorem C there exists an evolution family $(\varphi_{s,t}^T)$ of order d such that $h_s^T = h_t^T \circ \varphi_{s,t}^T$ whenever $0 \leq s \leq t$. In particular,

$$f_{T-s} = \varphi_{s,T}^T \quad \text{for each } s \in [0,T]. \tag{3.4}$$

Denote by G_T the Herglotz vector field of order d that corresponds to the evolution family $(\varphi_{s,t}^T)$ in the sense of Theorem B. Then by [7, Theorem 6.6], $\partial\varphi_{s,T}^T(z)/\partial s = -G_T(z,s)(\varphi_{s,T}^T)'(z)$ for all $z \in \mathbb{D}$ and a.e. $s \in [0,T]$. Note that from the very definition of a decreasing Loewner chain it follows easily that $F(z,t) := f_t(z)$ satisfies conditions S1 – S3. Hence using (3.4), we may conclude, in accordance with Subsect. 2.1, that $F|_{\mathbb{D}\times[0,T]}$ is a solution to (3.3) with $G(z,t) := G_T(z,T-t)$ for all $z \in \mathbb{D}$ and all $t \in [0,T]$. The vector field G is defined by (f_t) via (3.3) uniquely up to a null-set in $[0,T]$. That is why there exists a null-set N_0 and a function $G : \mathbb{D} \times [0,+\infty) \to \mathbb{C}$ holomorphic in the first (complex) variable and measurable in the second (real) variable such that for each $n \in \mathbb{N}$, we have $G(z,t) = G_n(z,n-t)$ for all $t \in [0,n] \setminus N_0$ and all $z \in \mathbb{D}$. Clearly, such a function G is a Herglotz vector field of order d. In this way $F(z,t) = f_t(z)$ becomes a solution to (3.3) on the whole semiaxis $[0,+\infty)$. Bearing in mind Proposition 2.3(ii) and the fact that by definition $f_0 = \mathsf{id}_{\mathbb{D}}$, we see that we have proved assertion (ii).

Let us now prove (i). Fix again $T > 0$. By construction,

$$g_s|_{\Omega_t} = \varphi_{T-t,T-s}^T \circ g_t \quad \text{for all } (s,t) \in \Delta([0,T]). \tag{3.5}$$

Fix any $z \in \mathbb{D}$. Choose any $t_0 \leq t(z)$ and let $T := [t_0] + 1$, where $[x]$ stands for the integer part of x. Then by (3.5), $g_s(z) = \varphi_{T-t_0,T-s}^T\big(g_{t_0}\big)$ for all $s \in [0,t_0]$. It is

[9] Here $\angle\lim$ stands for the angular limit.

known, see, e.g., [11, Theorem 3.6(iii)], that for each evolution family (i.e., for each $T > 0$ in our case) there exists a null-set $N(T) \subset [0, +\infty)$ such that for every $z \in \mathbb{D}$ and every $s \geq 0$, $(\partial/\partial t)\varphi^T_{s,t}(z)$ exists and equals $G_T\big(\varphi^T_{s,t}(z), t\big)$ whenever $t \geq s$ and $t \notin N(T)$. Note that $N(T)$ depends neither on z, nor on t. Bearing in mind that $(\varphi^T_{s,t})$ is locally absolutely continuous in t, we therefore conclude that $s \mapsto g_s(z)$ is absolutely continuous on $[0, t_0]$ and that $(d/ds)g_s(z)$ exists and equals $-G(g_s(z), s)$ for all $s \in [0, t_0] \setminus (N_0 \cup N(T))$.

Since t_0 can be chosen arbitrarily close to $t(z)$, the above argument proves assertion (i) with $N := N_0 \cup \big(\cup_{n\in\mathbb{N}} N(n)\big)$, except for the fact that the solution $w_z(t) := g_t(z)$ to (3.2) has no extension beyond $t = t(z)$. Assume on the contrary that $t(z) < +\infty$ and that such an extension w^*_z exists. Denote by E_z its domain of definition. Then by Proposition 2.3(iii) and assertion (ii) of the theorem we are proving now, $f_t(w^*_z(t)) = z$ for all $t \in E_z$. In particular, $z \in f_t(\mathbb{D}) = \Omega_t$ for all $t \in E_z$. Hence $\sup E_z \leq t(z)$. This contradiction completes the proof of assertion (i).

It remains to show that assertion (iii) holds. First of all, the fact that $F(z, t) := F_0\big(f_t(z)\big)$ solves the initial value problem $F(\cdot, 0) = F_0$ for (3.3) follows immediately from (ii). To prove the uniqueness of the solution, we recall that by assertion (i) we proved above, for any $z \in \mathbb{D}$ the function $[0, t(z)) \ni z \mapsto g_t(z)$ solves (3.2). Therefore if F is any solution to (3.3) with $F(\cdots, 0) = F_0$, then by Proposition 2.3(iii), $F(g_t(z), t) = F(g_0(z), 0) = F_0(z)$ for any $t \geq 0$ and any $z \in \Omega_t$. The proof of (iii) is now finished, because $g_t = f_t^{-1}$ for all $t \geq 0$.

Finally the essential uniqueness of G holds because each of the equations (3.3) and (3.2) defines G uniquely up to a null-set in $[0, +\infty)$. □

3.2. **Proof of Theorem 1.11.** Statement (i) of this theorem is a standard fact in the theory of Carathéofory ODEs, see, e.g., [11, Sect. 2], which in particular contains the proof of a more general statement [11, Theorem 2.3(i)].

In order to prove (ii) we fix an arbitrary $T > 0$ and consider the following Herglotz vector field of order d,

$$G_T(z, t) := \begin{cases} G(z, T - t), & \text{if } t \in [0, T], \\ 0, & \text{if } t > T. \end{cases}$$

By Theorem (B), there exists an evolution family $(\varphi^T_{s,t})$ of order d such that for each $\zeta \in \mathbb{D}$ and $s \geq 0$, the function $[s, +\infty) \ni t \mapsto w^T_{\zeta,s}(t) := \varphi^T_{s,t}(\zeta) \in \mathbb{D}$ is the unique solution to the initial value problem

$$\frac{dw}{dt} = G_T(w, t),\ t \geq s, \quad w(s) = \zeta. \tag{3.6}$$

It follows that for any $z \in \tilde{\Omega}_T := \varphi^T_{0,T}(\mathbb{D})$ the unique solution $t \mapsto w_z(t) =: g_t(z)$ to the initial value problem (1.11) is defined at least for all $t \in [0, T]$ and given for these t by the formula $w_z(t) = \varphi^T_{0,T-t}(\zeta)$, where $\zeta := \big(\varphi^T_{0,T}\big)^{-1}(z)$. Therefore, $\tilde{\Omega}_T \subset \Omega_T$. On the other hand, the uniqueness of the solution to (3.6) implies that for any $z \in \Omega_T$, the restriction $w_z|_{[0,T]}$ coincides with $[0, T] \ni t \mapsto \varphi^T_{0,T-t}(\zeta)$, where $\zeta := w_z(T)$. In particular, for any $z \in \Omega_T$, we have $z = w_z(0) = \varphi^T_{0,T}(\zeta) \in \tilde{\Omega}_T$. Thus $\tilde{\Omega}_T = \Omega_T$ and $g_T = \big(\varphi^T_{0,T}\big)^{-1} : \tilde{\Omega}_t \to \mathbb{D}$. Since $T > 0$ is arbitrary, this proves (ii).

To prove (iii), fix again an arbitrary $T > 0$. By the above argument, for each $t \in [0, T]$,

$$g_t|_{\Omega_T} = \varphi^T_{0,T-t} \circ \left(\varphi^T_{0,T}\right)^{-1} = \varphi^T_{0,T-t} \circ \left(\varphi^T_{T-t,T} \circ \varphi^T_{0,T-t}\right)^{-1} = \varphi^{-1}_{T-t,T}|_{\Omega_T},$$

where we used condition EF2 from Definition 1.1 of an evolution family. By the uniqueness principle for holomorphic functions this means that $f_t = g_t^{-1} = \varphi_{T-t,T}$ for all $t \in [0, T]$. Consider the family

$$h^T_t := \begin{cases} \varphi_{t,T}, & \text{if } t \in [0, T], \\ \mathsf{id}_{\mathbb{D}}, & \text{if } t \in [0, T]. \end{cases}$$

Clearly,

$$f_t = h^T_{T-t} \quad \text{for all } t \in [0, T]. \tag{3.7}$$

Note that by Remark 1.5, each function $\varphi^T_{s,t}$ is univalent in $\mathbb{D}$. Hence for each $t \ge 0$, h^T_t is univalent in $\mathbb{D}$. Furthermore, $\varphi^T_{s,t} = \mathsf{id}_{\mathbb{D}}$ whenever $t \ge s \ge T$ because $G_T(\cdot, t) \equiv 0$ for all $t \ge T$. Taking into account EF2, we conclude also that $\varphi^T_{s,t} = \varphi^T_{s,T}$ if $0 \le s \le T \le t$. Now using again EF2, it is easy to see that $h^T_t \circ \varphi^T_{s,t} = h_s$ whenever $0 \le s \le t$. By [9, Lemma 3.2], (h^T_t) is a Loewner chain of order d. Since $T > 0$ can be chosen arbitrarily, in view of (3.7), this implies that (f_t) is a decreasing Loewner chain of order d. As we have already mentioned in the proof of Theorem 3.2, any (decreasing or increasing) Loewner chain satisfies conditions S1 – S3 in the definition of a solution to the generalized Loewner – Kufarev PDE, formulated in Subsect. 2.1.

Let us show that S4 is also satisfied. Indeed, according to [7, Theorem 6.6] for each $z \in \mathbb{D}$ and for a.e. $t \in [0, T]$,

$$\frac{\partial f_t(z)}{\partial t} = \frac{\partial \varphi^T_{T-t,T}(z)}{\partial t} = \frac{\partial \varphi^T_{T-t,T}(z)}{\partial z} G_T(z, T-t) = \frac{\partial f_t(z)}{\partial z} G(z, t).$$

Again, since one can choose $T > 0$ arbitrarily large, S4 holds for the whole semiaxis $E = [0, +\infty)$. Thus $\mathbb{D} \times [0, +\infty) \ni (z, t) \mapsto f_t(z)$ is a solution to (1.12). The uniqueness of the solution is proved in the same way as in Theorem 3.2. This completes the proof of (iii). □

4. Reverse evolution families versus decreasing Loewner chains and Herglotz vector fields

In this section we would like to discuss in more detail the notion of a reverse evolution family, introduced in Subsect. 1.3, see Definition 1.9, and its relationship with decreasing Loewner chains and Herglotz vector fields.

4.1. Statements of results.

4.1. Statements of results. The theorem below is an analogue of Theorem C.

Theorem 4.1. *For each $d \in [1, +\infty]$ The formula*

$$\varphi_{s,t} := f_s^{-1} \circ f_t, \quad (s, t) \in \Delta\big([0, +\infty)\big), \tag{4.1}$$

establishes a 1-to-1 correspondence between decreasing Loewner chains (f_t) of order d and reverse evolution families $(\varphi_{s,t})$ of the same order d. Namely, for every decreasing Loewner chain (f_t) of order d the family $(\varphi_{s,t})_{(s,t)\in\Delta([0,+\infty))}$ defined *by* (4.1) *is a reverse evolution family of order* d. *Conversely, for any reverse evolution family* $(\varphi_{s,t})$ *of order* d *the family* $\big(f_t\big)_{t\ge0} = \big(\varphi_{0,t}\big)_{t\ge0}$ *is a decreasing Loewner chain of order* d *satisfying equality* (4.1).

In the situation described in the above theorem we will say that the decreasing Loewner chain (f_t) and the reverse evolution family $(\varphi_{s,t})$ are *associated with* each other.

In Theorem 1.11 we described the solutions to the generalized Loewner – Kufarev ODE $dw/ds = -G(w,s)$ with the initial condition at $s = 0$. It appeared that this initial value problem generate, for a fixed Herglotz vector field G and variable initial data, the family of the inverse mappings $g_s := f_s^{-1}$ of some decreasing Loewner chain (f_t). The following theorem shows that if instead we consider the initial condition *at the right end-point* $w(t) = z \in \mathbb{D}$, where $t > 0$ and the solutions are looked for on the interval $s \in [0,t]$, then we obtain the reverse evolution family $(\varphi_{s,t})$ associated with (f_t). The converse statement is true and it is also included in this theorem, which can be regarded as a "decreasing" analogue of Theorem B.

Theorem 4.2. *Let* $d \in [1,+\infty]$. *The following statements hold:*

(i) *The generalized Loewner – Kufarev ODE*

$$\frac{dw}{ds} = -G(w,s),\ \ s \in [0,t], \quad w(t) = z, \tag{4.2}$$

establishes essentially a 1-to-1 correspondence between reverse evolution families $(\varphi_{s,t})$ *of order* d *and Herglotz vector fields* G *of the same order. Namely, given a reverse evolution family* $(\varphi_{s,t})$ *of order* d, *there exists an essentially unique*[10] *Herglotz vector field* G *of order* d *such that for each* $t \ge 0$ *and* $z \in \mathbb{D}$ *the function* $[0,t] \ni s \mapsto w(s) := \varphi_{s,t}(z)$ *solves the initial value problem* (4.2). *Conversely, given a Herglotz vector field* G *of order* d, *for every* $t > 0$ *and every* $z \in \mathbb{D}$ *the initial value problem* (4.2) *has a unique solution* $s \mapsto w = w_{z,t}(s)$ *defined for all* $s \in [0,t]$ *and the formula* $\varphi_{s,t}(z) := w_{z,t}(s)$ *for all* $z \in \mathbb{D}$ *and all* $(s,t) \in \Delta\big([0,+\infty)\big)$ *defines a reverse evolution family* $(\varphi_{s,t})$ *of order* d.

(ii) *Let* $(\varphi_{s,t})$ *and* G *be as in statement* (*i*) *above. Then a family* $(f_t)_{t\ge0}$ *of holomorphic functions in* $\mathbb{D}$ *is the decreasing Loewner chain associated with* $(\varphi_{s,t})$ *if and only if the function* $F : \mathbb{D}\times[0,+\infty) \to \mathbb{C}$, *defined by* $F(z,t) := f_t(z)$ *for all* $t \ge 0$ *and all* $z \in \mathbb{D}$, *is a solution to the following initial value problem for the Loewner – Kufarev PDE*

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial z} G(z,t), \quad t \ge 0, \quad F(\cdot,0) = \mathsf{id}_{\mathbb{D}}. \tag{4.3}$$

In the proofs of the above theorems we will use the following statement of independent interest, which provides several alternatives for condition REF3 in Definition 1.9 equivalent to REF3 under the assumption that REF1 and REF2 are satisfied.

Proposition 4.3. *Let* $d \in [1,+\infty]$. *Suppose that a family* $(\varphi_{s,t})_{0\le s\le t}$ *of holomorphic self-maps of* $\mathbb{D}$ *satisfies conditions REF1 and REF2 from Definition 1.9. Then the following conditions are equivalent:*

[10] This means "unique up a null-set on the t-axis".

(i) *for each $T > 0$ there exist two distinct points $\zeta_j \in \mathbb{D}$, $j = 1, 2$, and a non-negative function $k_T \in L^d([0, T], \mathbb{R})$ such that for $j = 1, 2$,*

$$|\varphi_{s,t}(\zeta_j) - \varphi_{s,u}(\zeta_j)| \le \int_u^t k_T(\xi) d\xi \tag{4.4}$$

whenever $0 \le s \le u \le t \le T$.

(ii) *$(\varphi_{s,t})$ satisfies condition REF3, i.e. $(\varphi_{s,t})$ is a reverse evolution family of order d.*

(iii) *for any $T > 0$ and any compact set $K \subset \mathbb{D}$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that for all $z \in K$,*

$$|\varphi_{s,t}(z) - \varphi_{s,u}(z)| \le \int_u^t k_{K,T}(\xi) d\xi \tag{4.5}$$

whenever $0 \le s \le u \le t \le T$.

(iv) *for any $T > 0$ there exists an evolution family $(\varphi^T_{s,t})$ of order d such that $\varphi_{s,t} = \varphi^T_{T-t,T-s}$ for all $(s, t) \in \Delta\big([0, T]\big)$.*

(v) *for any $T > 0$ and any compact set $K \subset \mathbb{D}$ there exists a non-negative function $k_{K,T} \in L^d([0, T], \mathbb{R})$ such that for all $z \in K$,*

$$|\varphi_{s,t}(z) - \varphi_{u,t}(z)| \le \int_s^u k_{K,T}(\xi) d\xi \tag{4.6}$$

whenever $0 \le s \le u \le t \le T$.

In particular, every element of any reverse evolution family is a univalent function in $\mathbb{D}$.

Remark 4.4. Before stating the proofs we would like to mention that the families $\{B(a, b; \cdot)\}_{0 \le a \le b}$ introduced in [6] and referred there to as *semigroups associated with chordal Loewner families* are a special case of reverse evolution families of order $d = +\infty$. Accordingly, the results we have stated in this section extend Theorems 5.4 and 5.5 from [6].

The equivalence of (i), (iii), and (iv) in Proposition 4.3, being formulated in the "increasing" context, immediately leads to the following statement of some independent interest.

Corollary 4.5. *Let $d \in [1, +\infty]$. Suppose that a family $(\varphi_{s,t})_{0 \le s \le t}$ of holomorphic self-maps of $\mathbb{D}$ satisfies conditions EF1 and EF2 from Definition 1.1. Then the following conditions are equivalent:*

(i) *$(\varphi_{s,t})$ satisfies condition EF3, i.e. $(\varphi_{s,t})$ is a evolution family of order d.*

(ii) *for any $T > 0$ there exist two distinct points $\zeta_j \in \mathbb{D}$, $j = 1, 2$, and a non-negative function $k_T \in L^d([0, T], \mathbb{R})$ such that for $j = 1, 2$,*

$$|\varphi_{s,t}(\zeta_j) - \varphi_{u,t}(\zeta_j)| \le \int_s^u k_T(\xi) d\xi$$

whenever $0 \le s \le u \le t \le T$.

(iii) *for any* $T > 0$ *and any compact set* $K \subset \mathbb{D}$ *there exists a non-negative function* $k_{K,T} \in L^d([0,T],\mathbb{R})$ *such that for all* $z \in K$,

$$|\varphi_{s,t}(z) - \varphi_{u,t}(z)| \le \int_s^u k_{K,T}(\xi)d\xi$$

whenever $0 \le s \le u \le t \le T$.

4.2. **Proof of Proposition 4.3.** Recall that by $\rho_{\mathbb{D}}(\cdot,\cdot)$ we denote the pseudohyperbolic distance in $\mathbb{D}$. Note that

$$|w - z| \le \rho_{\mathbb{D}}(z,w)$$

for any $z, w \in \mathbb{D}$. Hence combining conditions REF1 and REF2, the Schwarz−Pick lemma, and Corollary 2.6, we deduce that for any compact set $K \subset \mathbb{D}$ and any two distinct points $\zeta_1, \zeta_2 \in \mathbb{D}$ there exists $M = M(K, \zeta_1, \zeta_2) > 0$ such that

$$\begin{aligned}(4.7)\quad \big|\varphi_{s,t}(z) - \varphi_{s,u}(z)\big| &\le \rho_{\mathbb{D}}\big(\varphi_{s,t}(z), \varphi_{s,u}(z)\big) = \rho_{\mathbb{D}}\big(\varphi_{s,u}(\varphi_{u,t}(z)), \varphi_{s,u}(z)\big)\\ &\le \rho_{\mathbb{D}}\big(\varphi_{u,t}(z), z\big) \le M\big(|\varphi_{u,t}(\zeta_1) - \zeta_1| + |\varphi_{u,t}(\zeta_2) - \zeta_2|\big)\end{aligned}$$

for all $z \in K$ and all $s, u, t \ge 0$ such that $s \le u \le t$.

Obviously, (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). Moreover, substituting (u, u, t) for (s, u, t) in REF3 and bearing in mind (4.7), one can easily see that (i) implies (iii).

Let us prove that (iv) $\Longrightarrow$ (iii). Fix any $T > 0$ and a compact set $K \subset \mathbb{D}$. Choose any two distinct points $z_1, z_2 \in \mathbb{D}$. Denote $w_j(s) := \varphi_{s,T}(\zeta_j)$ for all $s \in [0,T]$ and $j = 1, 2$. From REF2 it follows that for any $(u,t) \in \Delta\big([0,T]\big)$ and $j = 1, 2$,

$$(4.8)\qquad \big|\varphi_{u,t}\big(w_j(t)\big) - w_j(t)\big| = \big|w_j(u) - w_j(t)\big|.$$

By (iv) there exists an evolution family $(\varphi^T_{s,t})$ of order d such that $\varphi_{s,t} = \varphi^T_{T-t,T-s}$ for all $(s,t) \in \Delta\big([0,T]\big)$. In particular, $w_j(s) = \varphi^T_{0,T-s}(z_j)$ for all $s \in [0,T]$ and $j = 1, 2$. Hence the functions w_1 and w_2 are of class AC^d on $[0,T]$. Moreover, with all $\varphi^T_{s,t}$'s being univalent by Remark 1.5, these two functions do not share common values. In particular, it follows that there exist $R \in (0,1)$ and $\rho > 0$ such that $|w_j(t)| \le R$, $j = 1, 2$, and $\rho_{\mathbb{D}}\big(w_1(t), w_2(t)\big) \ge \rho$ for all $t \in [0,T]$. Apply now Corollary 2.6 with $w_j(t)$ substituted for ζ_j to deduce from (4.8) that there is a constant $M_1 = M_1(K, T, z_1, z_2) > 0$ such that

$$(4.9)\qquad \rho_{\mathbb{D}}\big(\varphi_{u,t}(z), z\big) \le M_1\big(|w_1(t) - w_1(u)| + |w_2(t) - w_2(u)|\big)$$

for all $(u,t) \in \Delta\big([0,T]\big)$ and all $z \in K$. Inequalities (4.7) and (4.9) imply together that

$$\big|\varphi_{s,t}(z) - \varphi_{s,u}(z)\big| \le M_1\big(|w_1(t) - w_1(u)| + |w_2(t) - w_2(u)|\big)$$

for all $(u,t) \in \Delta\big([0,T]\big)$ and all $z \in K$. Recalling that $w_1, w_2 \in AC^d\big([0,T],\mathbb{D}\big)$, we see that assertion (iii) holds with

$$k_{K,T}(\xi) := M_1\big(|w_1'(\xi)| + |w_2'(\xi)|\big).$$

Now assume (i) and let us prove (v). To this end we again fix any $T > 0$ and any compact set $K \subset \mathbb{D}$. Let us show first that from (i) it follows that there exists $r \in (0,1)$ such that

$$(4.10)\qquad \varphi_{u,t}(K) \subset r\mathbb{D} \quad \text{for all } (u,t) \in \Delta\big([0,T]\big).$$

Assume on the contrary that there exists a sequence $\big((u_n,t_n)\big)\subset\Delta\big([0,T]\big)$ such that $\sup_K|\varphi_{u_n,t_n}|\to 1$ as $n\to+\infty$. Without loss of generality we may also assume that $(u_n,t_n)\to(u_0,t_0)$ as $n\to+\infty$ for some $(u_0,t_0)\in\Delta\big([0,T]\big)$.

Note that by (i),

$$|\varphi_{u,t}(\zeta_j)-\zeta_j|\le\omega(t-u)\quad\text{for } j=1,2 \text{ and for any } (u,t)\in\Delta\big([0,T]\big), \tag{4.11}$$

where $\omega(\cdot)$ stands for the modulus of continuity of $[0,T]\ni t\mapsto\int_0^t k_T(\xi)d\xi$. Hence from (4.7) it follows that $t_0\neq u_0$ and $\varphi_{u_n,t_n}-\varphi_{u_n,t_0}\to 0$ uniformly on K as $n\to+\infty$. Therefore

$$\sup_K|\varphi_{u_n,t_0}|\to 1\qquad\text{as } n\to+\infty. \tag{4.12}$$

Now choose $\delta\in(0,t_0-u_0)$ in such a way that $\omega(2\delta)<\varepsilon_0:=\big(1-|\zeta_1|\big)/2$, and let $n_0\in\mathbb{N}$ be such that $|u_n-u_0|<\delta$ for all natural $n\ge n_0$. Then $|\varphi_{u_n,u_0+\delta}(\zeta_1)|<1-\varepsilon_0$ for all $n>n_0$. Bearing in mind that $\varphi_{u_n,u_0+\delta}(\mathbb{D})\subset\mathbb{D}$ for all $n\in\mathbb{N}$, we conclude that $\{\varphi_{u_n,u_0+\delta}:n>n_0\}$ is relatively compact in $\mathsf{Hol}(\mathbb{D},\mathbb{D})$. In view of the fact that by REF2, $\varphi_{u_n,t_0}=\varphi_{u_n,u_0+\delta}\circ\varphi_{u_0+\delta,t_0}$ for all $n>n_0$, this contradicts (4.12) and thus proves (4.10) for some $r\in(0,1)$ depending on the compact set K and $T>0$.

Now taking advantage of REF2, we get

$$\big|\varphi_{s,t}(z)-\varphi_{u,t}(z)\big|=|\varphi_{s,u}(\zeta)-\zeta|,\quad \zeta:=\varphi_{u,t}(z)$$

for any $z\in\mathbb{D}$ and any $s,u,t\ge 0$ such that $s\le u\le t$. Using (4.10) we apply Corollary 2.6 to conclude that there exists $M_2=M_2(T,K,\zeta_1,\zeta_2)>0$ such that

$$\big|\varphi_{s,t}(z)-\varphi_{u,t}(z)\big|\le M_2\big(|\varphi_{u,t}(\zeta_1)-\zeta_1|+|\varphi_{u,t}(\zeta_2)-\zeta_2|\big)$$

whenever $z\in K$ and $0\le s\le u\le t\le T$. Now it follows easily from (4.4) with (u,u,t) substituted for (s,u,t) that there exists a non-negative function $\hat{k}_{K,T}:=M_2k_T\in L^d\big([0,T],\mathbb{R}\big)$ such that for all $z\in K$,

$$|\varphi_{s,t}(z)-\varphi_{u,t}(z)|\le\int_s^u\hat{k}_{K,T}(\xi)d\xi \tag{4.13}$$

whenever $0\le s\le u\le t\le T$.

To complete the proof assume (v) and let us prove (iv). Write

$$\varphi^T_{s,t}:=\begin{cases}\varphi_{T-t,T-s}, & \text{if } 0\le s\le t\le T,\\ \varphi_{0,T-s}, & \text{if } 0\le s\le T\le t,\\ \mathsf{id}_{\mathbb{D}}, & \text{if } T\le s\le t.\end{cases} \tag{4.14}$$

In other words, for all $s\ge 0$ and all $t\ge s$, we have $\varphi^T_{s,t}=\varphi_{\tau(t),\tau(s)}$, where $\tau(t)$ stands for $\max\{0,T-t\}$.

It is easy to see that $(\varphi^T_{s,t})$ satisfies conditions EF1 and EF2 in Definition 1.1. Moreover, combining (4.13) and (4.14), for all $z\in\mathbb{D}$ and all s,u,t such that $0\le s\le u\le t$ we have

$$\big|\varphi^T_{s,t}(z)-\varphi^T_{s,u}(z)\big|=\big|\varphi_{\tau(t),\tau(s)}(z)-\varphi_{\tau(u),\tau(s)}(z)\big|\le\int_u^t\tilde{k}_{K,T}(\xi),$$

where $\tilde{k}_{K,T}(\xi):=k_{K,T}(T-\xi)$ for all $\xi\in[0,T]$ and $\tilde{k}_{K,T}(\xi):=0$ for all $\xi>T$. It follows that $(\varphi^T_{s,t})$ satisfies EF3. Thus (iv) is true.

Finally, using the implication (ii) $\Longrightarrow$ (iv) we can easily conclude that all the elements of any reverse evolution family is a univalent function in $\mathbb{D}$, since by Remark 1.5 this is the case for elements of evolution families. Now the proof of Proposition 4.3 is complete. □

4.3. **Proof of Theorem 4.1.** Assume first that we are given a decreasing Loewner chain (f_t) of order d. Fix any $T>0$. Then the family $(h_t^T)_{t\ge 0}$ defined by

$$h_t^T := \begin{cases} f_{T-t}, & \text{if } t\in[0,T], \\ \mathsf{id}_{\mathbb{D}}, & \text{if } t\in(T,+\infty), \end{cases} \tag{4.15}$$

is an (increasing) Loewner chain. Hence by Theorem C, the formula $\varphi_{s,t}^T := h_t^{-1}\circ h_s$ for all $s\ge 0$ and all $t\ge s$ defines an evolution family of order d. Note that for all $(s,t)\in\Delta\big([0,T]\big)$ we have $\varphi_{s,t}=\varphi_{T-t,T-s}^T$, where $(\varphi_{s,t})$ is defined by (4.1). Thus, bearing in mind that $T>0$ can be chosen arbitrarily, we conclude that by Proposition 4.3, $(\varphi_{s,t})$ is a reverse evolution family of order d.

To prove the converse statement we assume now that $(\varphi_{s,t})$ is a reverse evolution family of order d. We have to show that $f_t:=\varphi_{0,t}$ is a decreasing Loewner chain of order d satisfying (4.1). Recall that by Proposition 4.3, the functions $\varphi_{s,t}$ are univalent in $\mathbb{D}$. It follows that (f_t) satisfies LC1. By the same reason, REF2 implies (4.1). Moreover, REF2 implies also that (f_t) satisfies LC2. Finally, LC3 follows immediately from assertion (iii) of Proposition 4.3. The proof is complete. □

4.4. **Proof of Theorem 4.2.** We start with the proof of (i). Assume at first that we are given a reverse evolution family $(\varphi_{s,t})$ of order d and let us prove that there exists a Herglotz vector field of order d that generate $(\varphi_{s,t})$ via (4.2). To this end we fix $T>0$ and apply Proposition 4.3, according to which there exists an evolution family $(\varphi_{s,t}^T)$ of order d such that $\varphi_{s,t}=\varphi_{T-t,T-s}^T$ whenever $0\le s\le t\le T$. In turn, according to Theorem B there exists a Herglotz vector field G_T of order d such that for any $s\ge 0$ and any $z\in\mathbb{D}$ the function $[s,+\infty)\ni t\mapsto w(t):=\varphi_{s,t}^T(z)$ is the unique solution to the equation $dw/dt=G_T(w,t)$, $t\ge s$, with the initial condition $w(s)=z$. It follows that for each $t\in(0,T]$ and each $z\in\mathbb{D}$ the function $[0,t]\ni s\mapsto w_{z,t}(s):=\varphi_{s,t}(z)$ is the unique solution to the initial value problem

$$\frac{dw}{ds}=-G_T(w,T-s),\ s\in[0,t],\quad w(t)=z. \tag{4.16}$$

Since the functions $(z,s)\mapsto w_{z,t}(s)$ define G_T via (4.16) uniquely up to a null-set in $[0,T]$, there exists a function $G:\mathbb{D}\times[0,+\infty)\to\mathbb{C}$ such that for each $n\in\mathbb{N}$, $G(\cdot,t)=G_n(\cdot,n-t)$ for a.e. $t\in[0,n]$. Clearly, G is the desired Herglotz vector field of order d.

Now we pass to the converse statement. So assume that we are given a Herglotz vector field G of order d and let us prove that it generates a reverse evolution family $(\varphi_{s,t})$ of order d. Again fix any $T>0$. Arguing as in the proof of Theorem 1.11(ii), one can construct an evolution family $(\varphi_{s,t}^T)$ of order d such that for each $z\in\mathbb{D}$ and each $t\in(0,T]$ the function $[0,t]\ni s\mapsto w_{z,t}(s):=\varphi_{T-t,T-s}^T(z)$ is the unique solution to the initial value problem (4.2). Note that by uniqueness of the solution, $w_{z,t}$ does not depend on T. Hence there exists a unique family $(\varphi_{s,t})_{0\le s\le t}$ of holomorphic functions $\varphi_{s,t} : \mathbb{D} \to \mathbb{D}$ such that

$$\varphi_{s,t}(z) = w_{z,t}(s) = \varphi^T_{T-t,T-s}(z) \tag{4.17}$$

for all $z \in \mathbb{D}$, all $(s,t) \in \Delta\big([0,+\infty)\big)$, and all $T \geq t$.

Take now any $u, s, t \geq 0$ such that $u \leq s \leq t$. Choose any $T \geq t$. Then combining (4.17) with conditions EF1 and EF2 for $(\varphi^T_{s,t})$ we easily obtain REF1 and REF2. Furthermore, applying now Proposition 4.3 we conclude that REF3 holds as well. Hence $(\varphi_{s,t})$ is a reverse evolution family of order d. To complete the proof of (i) it remains to recall that by (4.17), for each $t \geq 0$ and each $z \in \mathbb{D}$, $[0,t] \ni s \mapsto \varphi_{s,t}(z)$ is the unique solution to (4.2).

Now let us prove (ii). First let us assume that $F(z,t) := f_t(z)$ solves (4.3). Then by Theorem 1.11, (f_t) is a decreasing Loewner chain of order d, with the functions $w_\zeta(s) := f_s^{-1}(\zeta)$, where $\zeta \in \mathbb{D}$, being solutions, on their domains of definition, to the equation $dw/ds = G(w,s)$. For each $z \in \mathbb{D}$ and each $t > 0$, set $\zeta := \zeta(z,t) = f_t(z)$. Then by condition DC2 in Definition 1.6, $w_{\zeta(z,t)}$ is defined for all $s \in [0,t]$. Moreover, it satisfies the initial condition $w_{\zeta(z,t)}(t) = z$. By the uniqueness of the solution to (4.2), it follows that $w_{\zeta(z,t)}(s) = \varphi_{s,t}(z)$ for all $z \in \mathbb{D}$ and all $(s,t) \in \Delta\big([0,+\infty)\big)$. On the other hand, by construction $w_{\zeta(z,t)(s)} = f_s^{-1} \circ f_t(z)$ for all such z, s and t. Thus the decreasing Loewner chain (f_t) is associated with the reverse evolution family $(\varphi_{s,t})$.

It remains to prove the converse statement. So we assume that (f_t) is the decreasing Loewner chain of order d associated with $(\varphi_{s,t})$. We have to show that $F(z,t) := f_t(z)$ solves (4.3). By Theorem 3.2, the function F is a solution to the generalized Loewner – Kufarev PDE

$$\frac{\partial F(z,t)}{\partial t} = \frac{\partial F(z,t)}{\partial z} G^*(z,t), \quad t \geq 0, \ z \in \mathbb{D},$$

for some Herglotz vector field G^*, which *a priori* can be different from G. We have to prove that actually G^* and G coincide aside a null set on the t-axis. To this end consider the reverse evolution family $(\varphi^*_{s,t})$ generated in the sense of statement (i) of the theorem we are proving by the Herglotz vector field G^*. By what we have already showed, (f_t) must be the Loewner chain associated with $(\varphi^*_{s,t})$. By the very definition, this means that $(\varphi^*_{s,t}) = (\varphi_{s,t})$. Now the uniqueness of the Herglotz vector field in statement (i) implies that G and G^* essentially coincide. This completes the proof. □

5. Two-point characterization of Loewner chains

The regularity of the L^d-Loewner chains, both increasing and decreasing, w.r.t. the time parameter t is described by (literally identical) conditions LC3 and DC3, requiring that the Loewner chain, considered as a mapping $[0,+\infty) \ni t \mapsto f_t \in \mathsf{Hol}(\mathbb{D},\mathbb{C})$, must be locally absolutely continuous. At the same time in the classical theory, both in chordal and radial variants, the regularity w.r.t. t is achieved by controlling a unique increasing parameter, such as $f_t'(0)$ in the radial case. Theorem 1.13, which we are going to prove in this section, states that condition LC3 can be replaced by an *a priori* weaker condition in the spirit of the classical theory. This theorem has an immediate consequence for decreasing Loewner chains.

Corollary 5.1. *Let $(f_t)_{t\geq 0}$ be a family of functions satisfying conditions DC1 and DC2 from Definition 1.6 and let $d \in [1, +\infty]$. Then (f_t) is a decreasing Loewner chain of order d if and only if for every $T > 0$ there exist two distinct points $\zeta_1, \zeta_2 \in \mathbb{D}$ such that the functions $t \mapsto w_j(t) := f_t(\zeta_j)$ belong to the class $AC^d\big([0,T],\mathbb{C}\big)$ for $j = 1, 2$.*

Since Corollary 5.1 follows directly from Theorem 1.13 by means of the change of variable $t \mapsto T - t$, we omit the proof.

Proof of Theorem 1.13. The implication LC3 $\Longrightarrow$ LC3w is obvious. So assume that (f_t) satisfies LC1, LC2, and LC3w and let us prove that (f_t) is a Loewner chain of order d.

Fix any $T > 0$. For $(s,t) \in \Delta([0,T])$ consider $\varphi_{s,t} := f_t^{-1} \circ f_s : \mathbb{D} \to \mathbb{D}$. Since $f_t(\mathbb{D}) \subset f_T(\mathbb{D})$ for all $t \in [0,T]$ and the map $[0,T] \ni t \mapsto f_t(\zeta_1)$ is continuous, the set $\{f_t : t \in [0,T]\}$ is relatively compact in $\mathsf{Hol}(\mathbb{D},\mathbb{C})$. Therefore applying Lemma 2.7 for $K := \{\zeta_1, \zeta_2\}$, $z_2 = z_2(s,t) := \varphi_{s,t}(z_1)$, $z_1 := \zeta_j$, we get

$$|\varphi_{s,t}(\zeta_j) - \zeta_j| \leq M|f_t(\zeta_j) - f_s(\zeta_j)|, \quad j = 1, 2, \tag{5.1}$$

for all $(s,t) \in \Delta([0,T])$, where $M := M_{\{\zeta_1,\zeta_2\}} > 0$ does not depend on s and t.

Combining (5.1) with Corollary 2.6 for $\varphi := \varphi_{s,t}$ and taking into account that

$$|w - z| \leq \rho_{\mathbb{D}}(w, z)$$

for all $z, w \in \mathbb{D}$, we conclude that for each $r \in (0,1)$,

$$\begin{aligned} \big|\varphi_{s,t}(\zeta) - \zeta\big| &\leq \tilde{C}(r, R_0, \rho_0)\big(|\varphi_{s,t}(\zeta_1) - \zeta_1| + |\varphi_{s,t}(\zeta_2) - \zeta_2|\big) \\ &\leq C_0\big(|f_t(\zeta_1) - f_s(\zeta_1)| + |f_t(\zeta_2) - f_s(\zeta_2)|\big), \end{aligned} \tag{5.2}$$

whenever $|\zeta| \leq r$ and $(s,t) \in \Delta([0,T])$, where $C_0 := M\tilde{C}(r, R_0, \rho_0)$, $R_0 := \max\{|\zeta_1|, |\zeta_2|\}$, and $\rho_0 := \rho_{\mathbb{D}}(\zeta_2, \zeta_1)$.

Since the mappings $[0,T] \ni t \mapsto f_t(\zeta_j)$, $j = 1, 2$, are continuous, inequality (5.2) implies that for each $r \in (0,1)$ there exists $\delta > 0$ such that $|\varphi_{s,t}(\zeta)| \leq r' := (1+r)/2$ for all ζ with $|\zeta| \leq r$ and all $(s,t) \in \Delta([0,T])$ with $t < s + \delta$. Recall that $\{f_t : t \in [0,T]\}$ is relatively compact in $\mathsf{Hol}(\mathbb{D},\mathbb{C})$. It follows that $|f_t'(z)|$ is uniformly bounded on the disk $\{z : |z| \leq r'\}$. As a result, from (5.2) we get

$$|f_t(\zeta) - f_s(\zeta)| \leq M_1|\varphi_{s,t}(\zeta) - \zeta| \leq C(r)\big(|f_t(\zeta_1) - f_s(\zeta_1)| + |f_t(\zeta_2) - f_s(\zeta_2)|\big), \tag{5.3}$$

for any ζ with $|\zeta| \leq r$ and any $(s,t) \in \Delta([0,T])$ with $t < s+\delta$, where $C(r) := M_1 C_0$ and $M_1 = M_1(r') := \sup\big\{|f_t'(z)| : |z| \leq r',\ t \in [0,T]\big\}$.

Let K be any compact set in $\mathbb{D}$. Choose $r \in (0,1)$ in the above argument in such a way that $|\zeta| \leq r$ for all $\zeta \in K$. Set $k_{K,T}(\xi) := C(r)\big(k_1(\xi) + k_2(\xi)\big)$ for all $\xi \in [0,T]$, where $k_j(t) := |df_t(\zeta_j)/dt|$ for $j = 1, 2$ and all $t \in [0,T]$. Obviously, $k_{K,T} \geq 0$ and belongs to $L^d_{\mathrm{loc}}\big([0,T],\mathbb{R}\big)$. From (5.3) it follows that whenever $0 \leq s \leq t \leq T$, $t < s + \delta$, and $\zeta \in K$, we will have

$$\big|f_t(\zeta) - f_s(\zeta)\big| \leq \int_s^t k_{K,T}(\xi)\, d\xi. \tag{5.4}$$

Using the triangle inequality in the right-hand side of (5.4) and the additivity of the integral in its left-hand side, it is easy to remove the restriction $t < s + \delta$. Hence (f_t) satisfies LC3 and thus it is Loewner chain of order d. $\square$

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