Journal of Nonlinear and Convex Analysis
Volume 15, Number 1, 2014, 257–267

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A NEW ITERATIVE ALGORITHM FOR APPROXIMATING ZEROS OF ACCRETIVE OPERATORS IN UNIFORMLY SMOOTH BANACH SPACES

C. E. CHIDUME*, C. O. CHIDUME, Y. SHEHU, AND J. N. EZEORA

Dedicated to Professor Simeon Reich on the occasion of his 65th birthday

Abstract. Let E be a uniformly smooth real Banach space and let $A : E \to E$ be a bounded accretive map which satisfies the range condition. A new iterative algorithm is constructed which converges strongly to a zero of A. This result is achieved by means of two celebrated theorems of Simeon Reich. An application of our theorem to convex minimization problem is also given.

1. General introduction

Many problems in applications can be modeled in the form $0 \in Ax$, where for example, $A : H \to 2^H$ is a *monotone* operator, *i.e.,* A satisfies the following inequality: $\langle u - v, x - y\rangle \geq 0 \; \forall u \in Ax, v \in Ay, x, y \in H$. Typical examples where monotone operators occur and satisfy the inclusion $0 \in Ax$ include the equilibrium state of *evolution equations* and critical points of some functionals defined on Hilbert spaces H. Let $f : H \to (-\infty, +\infty]$ be a proper, lower-semicontinuous convex function, then, it is known (see, *e.g.*, Minty [7] or Rockafellar [11]) that the multi-valued map $T := \partial f$, the *subdifferential* of f, is *maximal monotone*, where for $w \in H$,

$$\begin{aligned} w \in \partial f(x) &\Leftrightarrow f(y) - f(x) \geq \langle y - x, w\rangle \;\; \forall \, y \in H \\ &\Leftrightarrow x \in \operatorname{Argmin}(f - \langle \cdot, w\rangle). \end{aligned}$$

In this case, the solutions of the inclusion $0 \in \partial f(x)$, if any, correspond to the critical points of f, which are exactly its minimizers.

In general, consider the following problem:

$$\text{Find } u \in H \text{ such that } 0 \in Au \tag{1.1}$$

where H is a real Hilbert space and A is an *m-monotone operator* (defined below) on H. One of the classical algorithms for approximating a solution of (1.1), assuming existence, is the so-called *proximal point algorithm* introduced by Martinet [6] and studied further by Rockafellar [10] and a host of other authors. More precisely, given $x_k \in H$, an approximation of a solution of (1.1), the proximal point algorithm generates the next iterate x_{k+1} by solving the following equation

$$x_{k+1} = \Big(I + \frac{1}{\lambda_k} A\Big)^{-1}(x_k) + e_k, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 47H04, 47H06, 47H15, 47H17, 47J25.
Key words and phrases. Accretive mappings, range condition, uniformly smooth spaces.
*Corresponding author.

where $\lambda_k > 0$ is a regularizing parameter. If the sequence $\{\lambda_k\}_{k=1}^{\infty}$ is bounded from above, then the resulting sequence $\{x_k\}_{k=1}^{\infty}$ of proximal point iterates converges *weakly* to a solution of (1.1), provided that a solution exists (Rockafellar [10]).

Rockafellar then posed the following question:

Q1. Does the proximal point algorithm always converge *strongly*?

This question was resolved in the negative by Güler [2] who produced a proper closed convex function g in the infinite dimensional Hilbert space l_2 for which the proximal point algorithm converges *weakly* but *not strongly*. This naturally raised the following question:

Q2. Can the proximal point algorithm be modified to guarantee *strong convergence*?

It is clear that the proximal point algorithm (1.2), even if it converges strongly, is not at all convenient to use. This is because *at each step of the iteration process*, one has to compute $\left(I + \frac{1}{\lambda_k}A\right)^{-1}(x_k)$ and this is certainly not convenient. Consequently, Chidume and Djitte [1] posed the following question, which perhaps, is more important than **Q2**.

Q3. Can an iteration process be developed which will not involve the computation of $\left(I + \frac{1}{\lambda_k}A\right)^{-1}(x_k)$ at each step of the iteration process and which will guarantee *strong convergence* to a solution of (1.1)?

With respect to **Q2**, Solodov and Svaiter [12] were the first to propose a modification of the proximal point algorithm which guarantees strong convergence in a real Hilbert space. Their algorithm is as follows:

Algorithm. Choose any $x^0 \in H$ and $\sigma \in [0, 1)$. At iteration k, having x_k, choose $\mu_k > 0$, and find (y_k, v_k) an inexact solution of $0 \in Tx + \mu_k(x - x_k)$, with tolerance σ. Define

$$C_k := \{z \in H \mid \langle z - y^k, v^k\rangle \leq 0\},$$

and

$$Q_k := \{z \in H \mid \langle z - x^k, x^0 - x^k\rangle \leq 0\}.$$

Take

$$x_{k+1} = P_{C_k \cap Q_k}(x^0).$$

The authors themselves noted ([12], p.195) that``... at each iteration, there are two subproblems to be solved... '': (i) find an inexact solution of the proximal point algorithm, and (ii) find the projection of x^0 onto $C_k \cap Q_k$, the intersection of the two halfspaces.

Kamimura and Takahashi [3], extended the work of Solodov and Svaiter [12] to the framwork of Banach spaces that are both uniformly convex and uniformly smooth.

Xu [14] noted that`` ... Solodov and Svaiter's algorithm, though strongly convergent, does need more computing time due to the projection in the second subproblem... ''

Xu [13] then proposed and studied the following algorithm:

$$x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\Big(I + c_n T\Big)^{-1}(x_n) + e_n,\ n \geq 0. \tag{1.3}$$

He proved that (1.3) converges strongly provided that the sequences $\{\alpha_n\}$ and $\{c_n\}$ of real numbers and the sequence $\{e_n\}$ of errors are chosen appropriately. He argued that once $u_n := \Big(I + c_n T\Big)^{-1}(x_n) + e_n$ has been calculated, the calculation of the mean $\alpha_n x_0 + (1-\alpha_n)u_n$ is much easier than the projection of x_0 onto $C_n \cap Q_n$ mentioned earlier, and so his algorithm seems simpler than that of Solodov and Svaiter [12].

Lehdili and Moudafi [4] considered the technique of the proximal map and the Tikhonov regularization to introduce the so-called *Prox-Tikhonov* method which generates the sequence $\{x_n\}$ by the algorithm:

$$x_{n+1} = J^{T_n}_{\lambda_n} x_n,\ n \geq 0, \tag{1.4}$$

where $T_n := \mu_n T + T$, $\mu_n > 0$ is viewed as a Tikhonov regularization of T and $J^{T_n}_{\lambda_n} := (I + \lambda_n T_n)^{-1}$. Using the notation of variational distance, Lehdili and Moudafi [4] proved strong convergence theorems for the algorithm (1.4) and its perturbed version, under appropriate conditions on the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

Xu [14] studied the algorithm (1.4). He used the technique of nonexpansive mappings to get convergence theorems for the perturbed version of the algorithm (1.4), under much relaxed conditions on the sequences $\{\lambda_n\}$ and $\{\mu_n\}$.

Another modification of the proximal point algorithm, perhaps the most significant, which yields strong convergence, is implicitly contained in the following theorem of Reich.

Theorem 1.1 (Reich, [9]). *Let E be a real uniformly smooth Banach space and $A : D(A) \subseteq E \to E$ be an accretive mapping with $cl(D(A))$ convex. Suppose A satisfies the range condition $D \subseteq R(I + sA)$, $\forall\ s > 0$. Suppose that $0 \in R(A)$, then for each $x \in D$, the strong limit $\lim J^A_s x$ exists and belongs to $N(A)$. If we denote $\lim J^A_s x$ by Qx, then $Q : D \to N(A)$ is the unique sunny nonexpansive retraction of D onto $N(A)$.*

We have seen that, in response to **Q2**, *all* modifications of the classical proximal point algorithm to obtain strong convergence so far studied still involve the computation of $\Big(I + c_n T\Big)^{-1}(x_n)$ at each step of the process.

In the case that A is maximal monotone and bounded, Chidume and Djitte [1] gave an affirmative answer to **Q3** by proving the following important theorem:

Theorem CD (Chidume and Djitte [1]). *Let E be a 2-uniformly smooth real Banach space and let $A : E \to E$ be a bounded m-accretive map. For arbitrary $x_1 \in E$, define the sequence $\{x_n\}$ iteratively by*

$$x_{n+1} := x_n - \lambda_n A x_n - \lambda_n \theta_n (x_n - x_1), \; n \geq 1, \tag{1.5}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lim \theta_n = 0$*; and $\{\theta_n\}$ is decreasing;*

(2) $\sum_{n=1}^{\infty} \lambda_n \theta_n = \infty, \quad \lambda_n = o(\theta_n)$*;*

(3) $\lim_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n \theta_n} = 0, \; \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$

Suppose that the equation $Ax = 0$ has a solution. Then, there exists a constant $\gamma_0 > 0$ such that if $\lambda_n \leq \gamma_0 \theta_n \;\forall\, n \geq 1$, $\{x_n\}$ converges strongly to a solution of the equation $Ax = 0$.

Remark 1.2. We note that 2-uniformly smooth Banach spaces include L_p spaces, $2 \leq p < \infty$ but do not include L_p spaces, $1 < p < 2$.

It is our purpose in this paper to prove a significant improvement of Theorem CD in the following sense. First, our recursion formula will be simpler than the one in Theorem CD, requiring only one iteration parameter instead of two required in Theorem CD. Secondly, our theorem will be proved in the much more general *uniformly smooth real Banach spaces*. As is well known, these spaces include L_p spaces, $1 < p < \infty$. These results are achieved by using two celebrated theorems of Simeon Reich ([8], [9]). An application of our theorem to convex minimization problem is also given.

2. PRELIMINARIES

Let $A : H \to H$ be a monotone map. A is called m-monotone if $R(I + \lambda A) = H$ for *some* $\lambda > 0$. It is well known that if A is m-monotone, it satisfies the range condition, that is, $R(I + \lambda A) = H$ for *all* $\lambda > 0$ (see, e.g., Chidume and Djitte [1] for a recent proof).

In the sequel, we shall use the following lemmas.

Lemma 2.1 (Reich, [8]). *Let E be a real uniformly smooth Banach space. Then, there exists a nondecreasing continuous function*

$$\beta : [0, \infty) \to [0, \infty),$$

satisfying the following conditions:
(i) $\beta(ct) \leq c\beta(t) \;\forall c \geq 1$;

(ii) $\underset{t\to 0^+}{lim}\,\beta(t) = 0$, and,

$$||x+y||^2 \leq ||x||^2 + 2\text{Re}\langle y, j(x)\rangle + \max\{||x||, 1\}||y||\beta(||y||)\forall x, y \in E.$$

Lemma 2.2 (See e.g., [13]). *Let $\{\lambda_n\}_{n\geq 1}$ be a sequence of non-negative real numbers satisfying the condition*

$$\lambda_{n+1} \leq (1-\omega_n)\lambda_n + \omega_n\sigma_n,\ n \geq 0,$$

where $\{\omega_n\}_{n\geq 0}$ and $\{\sigma_n\}_{n\geq 0}$ are sequences of real numbers such that $\{\omega_n\}_{n\geq 1} \subset [0,1]$, $\sum_{n=1}^{\infty}\omega_n = +\infty$ and $\limsup \sigma_n \leq 0$. Then $\lambda_n \to 0$ as $n \to \infty$.

Lemma 2.3 (Xu and Roach, [15]). *Let E be a real uniformly smooth Banach space. Then, there exist constants D and C such that for all $x, y \in E, j(x) \in J(x)$; the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x)\rangle + D\max\{||x|| + ||y||, \frac{1}{2}C\}\rho_E(||y||)$$

where ρ_E denotes the modulus of smoothness of E.

Lemma 2.4 (Lindenstrauss and Tzafriri, [5]). *In L_p(or ℓ_p) spaces, $1 < p < \infty$,*

$$\rho_{L_p}(\tau) = \begin{cases} (1+\tau^p)^{\frac{1}{p}} - 1 < \frac{1}{p}\tau^p;\ 1 < p < 2 \\ \frac{p-1}{2}\tau^2 + o(\tau^2) < \frac{p-1}{2}\tau^2;\ p \geq 2. \end{cases}$$

3. Main result

We prove the following theorem. In the theorem, β is the function defined in Lemma 2.1.

Theorem 3.1. *Let E be a uniformly smooth real Banach space and let $A : E \to E$ be a bounded accretive map which satisfies the range condition. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}$ be iteratively defined by*

$$x_{n+1} := x_n - \lambda_n A x_n - \lambda_n(x_n - x_1),\ n \geq 1, \tag{3.1}$$

where $\{\lambda_n\}$ is a sequence in $(0,1)$ satisfying the following conditions:

(1) $\lim_{n\to\infty} \lambda_n = 0$
(2) $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Suppose that the equation $Ax = 0$ has a solution. Then, there exists a constant $\gamma_0 > 0$ such that if $\beta(\lambda_n) < \gamma_0$, $\{x_n\}$ converges strongly to a solution of the equation $Ax = 0$.

Proof. We first prove that $\{x_n\}$ is bounded. Let $x^* \in A^{-1}(0)$, since x_1 is fixed in E there exists $r > 0$ sufficiently large such that $x_1 \in B(x^*, \frac{r}{2})$. Define $B := \overline{B(x^*, r)}$. Since A is bounded, $A(B)$ is bounded. Define

$$\begin{aligned} M_0 &:= \max\{||x - x^*|| : x \in B\} \\ M_1 &:= \sup\{||Ax + (x - x_1)|| : x \in B\} + 1 \\ M &:= M_0 M_1^2 \text{ and } \gamma_0 = \frac{r^2}{4M}. \end{aligned}$$

Observe that $\lambda_n < \gamma_0 \; \forall n \geq 1$ implies $\lambda_n \beta(\lambda_n) < \lambda_n \frac{r^2}{4M}$. We now prove by induction that $x_n \in B \; \forall \; n \geq 1$. By construction, $x_1 \in B$. Assume that $x_n \in B$ for some $n \geq 1$. We show that $x_{n+1} \in B$. From the recursion formula (3.1) and Lemma 2.1, we have that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||x_n - x^* - \lambda_n(Ax_n + (x_n - x_1)||^2 \\ &\leq ||x_n - x^*||^2 - 2\lambda_n\langle Ax_n, j(x_n - x^*)\rangle - 2\lambda_n\langle x_n - x_1, j(x_n - x^*)\rangle \\ &\quad + \max\{||x_n - x^*||, 1\}||\lambda_n[Ax_n + (x_n - x_1)]|| \\ &\quad \times \beta\Big(||\lambda_n[Ax_n + (x_n - x_1)]||\Big) \\ &\leq ||x_n - x^*||^2 - 2\lambda_n\langle Ax_n, j(x_n - x^*)\rangle - 2\lambda_n\langle x_n - x_1, j(x_n - x^*)\rangle \\ &\quad + \max\{||x_n - x^*||, 1\} \times \lambda_n||Ax_n + (x_n - x_1)|| \\ &\quad \times \beta\Big(\lambda_n||Ax_n + (x_n - x_1)||\Big). \end{aligned} \tag{3.2}$$

Since A is accretive and $x^* \in A^{-1}(0)$, then $\langle Ax_n, j(x_n - x^*)\rangle \geq 0$. Hence, we obtain that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &\leq ||x_n - x^*||^2 - 2\lambda_n||x_n - x^*||^2 + 2\lambda_n\langle x_1 - x^*, j(x_n - x^*)\rangle \\ &\quad + M_0 M_1^2 \lambda_n \beta(\lambda_n) \\ &\leq (1 - 2\lambda_n)||x_n - x^*||^2 + \lambda_n(||x_1 - x^*||^2 + ||x_n - x^*||^2) \\ &\quad + \lambda_n M \beta(\lambda_n) \\ &\leq (1 - \lambda_n)r^2 + \lambda_n \frac{r^2}{4} + \lambda_n \frac{r^2}{4} \\ &= \Big(1 - \frac{\lambda_n}{2}\Big)r^2 \leq r^2. \end{aligned} \tag{3.3}$$

This implies that $x_{n+1} \in B$, so by induction, $x_n \in B \; \forall \; n \geq 1$. Therefore, $\{x_n\}$ is bounded.

We now prove $x_n \to x^*$ as $n \to \infty$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, we have that $\{Ax_n\}_{n=1}^{\infty}$ is bounded. Observe that, if for all $\gamma > 0$, we define $A_\gamma : E \to E$ by $A_\gamma x = \gamma Ax \; \forall \; x \in E$, then we easily see that A_γ is bounded and satisfies the range condition since A satisfies the range condition. Furthermore,

$$A^{-1}(0) = A_\gamma^{-1}(0) = F(J_s^{A_\gamma}),$$

where $J_s^{A_\gamma}$ is the resolvent of the operator A_γ, $\forall \gamma > 0$. Observe that

$$||A_\gamma x_n|| = \gamma||Ax_n|| \leq \gamma \sup_{x \in B'} ||Ax||, \; \forall n \geq 1$$

(where $B' = B \cup \{x_1, x_2, \ldots, x_{n_0 - 1}\}$). This implies that $\lim_{\gamma \to 0} ||A_\gamma x_n|| = 0$. From Theorem 1.1, we get that $\lim_{s \to \infty} J_s^{A_\gamma} x_1 = x^* \in A^{-1}(0)$.
Define

$$\zeta_n := \max\{\langle x_1 - x^*, j(x_n - x^*)\rangle, 0\}, \; \forall \; n \geq 1,$$

then $\lim_{n\to\infty} \zeta_n = 0$. We prove this. Since $J_s^{A_\gamma} = (I + sA_\gamma)^{-1}$, we obtain $(I + sA_\gamma)J_s^{A_\gamma} x_1 = x_1$. Therefore,

$$A_\gamma o J_s^{A_\gamma} x_1 = \frac{1}{s}\Big(x_1 - J_s^{A_\gamma} x_1\Big).$$

Since A is accretive, we have that A_γ is accretive and so

$$\Big\langle A_\gamma x_n - \frac{1}{s}\Big(x_1 - J_s^{A_\gamma} x_1\Big), j(x_n - J_s^{A_\gamma} x_1)\Big\rangle \geq 0 \;\forall\; s > 0,\; \gamma > 0.$$

This implies that there exists a constant $K > 0$, such that

$$\begin{aligned}\langle x_1 - J_s^{A_\gamma} x_1, j(x_n - J_s^{A_\gamma} x_1)\rangle &\leq s\langle A_\gamma x_n, j(x_n - J_s^{A_\gamma} x_1)\rangle \\ &\leq sK||A_\gamma x_n||.\end{aligned}$$

Hence, $\limsup_{\gamma\to 0}\langle x_1 - J_s^{A_\gamma} x_1, j(x_n - J_s^{A_\gamma} x_1)\rangle \leq 0 \;\forall\; n \geq 1$. Therefore, for any $\varepsilon > 0$, there exists $\delta := \delta(\varepsilon) > 0$ such that for all $\gamma \in (0, \delta]$,

$$\langle x_1 - J_s^{A_\gamma} x_1, j(x_n - J_s^{A_\gamma} x_1)\rangle < \varepsilon.$$

In particular, for $\gamma = \delta$, there exists $K_0 > 0$ such that

$$\begin{aligned}\langle x_1 - x^*, j(x_n - x^*)\rangle &= \langle x_1 - x^*, j(x_n - x^*) - j(x_n - J_s^{A_\delta} x_1)\rangle \\ &\quad + \langle x_1 - J_s^{A_\delta} x_1, j(x_n - J_s^{A_\delta} x_1)\rangle \\ &\quad + \langle J_s^{A_\delta} x_1 - x^*, j(x_n - J_s^{A_\delta} x_1)\rangle \\ &< \langle x_1 - x^*, j(x_n - x^*) - j(x_n - J_s^{A_\delta} x_1)\rangle \\ &\quad + K_0||J_s^{A_\delta} x_1 - x^*|| + \varepsilon \\ &\leq ||x_1 - x^*||||j(x_n - x^*) - j(x_n - J_s^{A_\delta} x_1)|| \\ &\quad + K_0||J_s^{A_\delta} x_1 - x^*|| + \varepsilon.\end{aligned}$$

This implies that

$$\begin{aligned}&\limsup_{n\to\infty}\Big(\limsup_{s\to\infty}\langle x_1 - x^*, j(x_n - x^*)\rangle\Big) \leq K_0 \limsup_{n\to\infty}\Big(\limsup_{s\to\infty}||J_s^{A_\delta} x_1 - x^*||\Big) \\ &+ \limsup_{n\to\infty}\Big(\limsup_{s\to\infty}||x_1 - x^*||||j(x_n - x^*) - j(x_n - J_s^{A_\delta} x_1)||\Big) + \varepsilon.\end{aligned}$$

Since E is uniformly smooth, J is norm-to-norm uniformly continuous on bounded subsets of E. Then, we have

$$\limsup_{n\to\infty}\langle x_1 - x^*, j(x_n - x^*)\rangle \leq \varepsilon.$$

This implies that

$$\limsup_{n\to\infty}\langle x_1 - x^*, j(x_n - x^*)\rangle \leq 0. \tag{3.4}$$

Using (3.4), we get that $\limsup_{n\to\infty}\zeta_n = 0$. From (3.1), we obtain

$$\begin{aligned}||x_{n+1} - x^*||^2 &\leq (1 - 2\lambda_n)||x_n - x^*||^2 + 2\lambda_n\langle x_1 - x^*, j(x_n - x^*)\rangle + \lambda_n\beta(\lambda_n)M \\ &\leq (1 - 2\lambda_n)||x_n - x^*||^2 + 2\lambda_n\zeta_n + \lambda_n\beta(\lambda_n)M \\ &= (1 - 2\lambda_n)||x_n - x^*||^2 + \lambda_n\sigma_n, \qquad (3.5)\end{aligned}$$

where $\sigma_n := 2\zeta_n + \beta(\lambda_n)M$. Clearly, $\limsup \sigma_n \leq 0$, so by conditions (i) and (ii) and applying Lemma 2.2 to (3.5), we conclude that $x_n \to x^*,\ n \to \infty$, completing the proof. □

4. Convergence theorems for the special case of $L_p,\ 1 < p < \infty$

In this section, using a result of Xu and Roach (Lemma 2.3), a result of Lindenstrauss and Tzafriri (Lemma 2.4) and following the method of proof of Theorem 3.1, the following theorems are easily proved.

Theorem 4.1. *Let $E = L_p, 1 < p < 2$ and let $A : E \to E$ be a bounded accretive map which satisfies the range condition. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}$ be iteratively defined by*

$$x_{n+1} := x_n - \lambda_n A x_n - \lambda_n(x_n - x_1),\ n \geq 1, \tag{4.1}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(1) $\lim\limits_{n\to\infty} \lambda_n = 0$
(2) $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Suppose that the equation $Ax = 0$ has a solution. Then, there exists a constant $\gamma_1 > 0$ such that if $\lambda_n < \gamma_1$, the sequence $\{x_n\}$ converges strongly to a solution of the equation $Ax = 0$.

Theorem 4.2. *Let $E = L_p, 2 \leq p < \infty$ and let $A : E \to E$ be a bounded accretive map which satisfies the range condition. For arbitrary $x_1 \in E$, let the sequence $\{x_n\}$ be iteratively defined by*

$$x_{n+1} := x_n - \lambda_n A x_n - \lambda_n(x_n - x_1),\ n \geq 1, \tag{4.2}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(1) $\lim\limits_{n\to\infty} \lambda_n = 0$
(2) $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Suppose that the equation $Ax = 0$ has a solution. Then, there exists a constant $\gamma_2 > 0$ such that if $\lambda_n < \gamma_2$, the sequence $\{x_n\}$ converges strongly to a solution of the equation $Ax = 0$.

Remark 4.3. Following the method of proof of Theorem 3.1 and using Lemma 2.3 and Lemma 2.4, the condition $\beta(\lambda_n) < \gamma_0$ is replaced with the condition $\lambda_n < \gamma_1$ in Theorem 4.1, where $\gamma_1 := \left(\frac{r^2}{4M*}\right)^{\frac{1}{p-1}}$ for some constant $M* > 0$ and with $\lambda_n < \gamma_2$ in Theorem 4.2, where $\gamma_2 := \frac{r^2}{4M^{**}}$ for some constant $M^{**} > 0$.

Remark 4.4. Condition 1 and continuity of β imply that $\beta(\lambda_n) \to 0$ as $n \to \infty$. Consequently, the condition $\beta(\lambda_n) < \gamma_0$ is always satisfied for sufficiently large n.

Remark 4.5. 1. As has been remarked in the Introduction, the recursion formula (3.1) is simpler than that of Theorem CD. We note that the desirable choice $\lambda = \frac{1}{n}$ is applicable in our theorems which is not the case in Theorem CD.
2. Theorem 3.1 is applicable in arbitrary uniformly smooth real Banach spaces. In particular, it is applicable in L_p spaces *for all* $1 < p < \infty$ which is not the case in Theorem CD.

5. Application to convex minimization problems

In this section, we investigate the problem of finding a minimizer of a continuously differentiable convex function in real Hilbert spaces. In fact, let $f : H \to (-\infty, +\infty]$ be a proper lower semicontinuous convex function. We have observed that the equation $0 \in \partial f(x)$ is equivalent to $f(x) = \min_{y \in H} f(y)$.

Note that if $f : H \to (-\infty, +\infty]$ is differentiable at a point x, then $\partial f(x) = \{\nabla f(x)\}$, where $\nabla f(x)$ is the gradient of f at x.

The following basic results are well known.

Lemma 5.1. *Let $f : H \to \mathbb{R}$ be a real-valued convex differentiable function and $a \in H$. Then, the following hold.*

(1) *The point a is a minimizer of f if and only if $\nabla f(a) = 0$.*
(2) *If f is bounded on bounded subsets of H, then for every $x_0 \in H$ and $r > 0$, there exists $\gamma > 0$ such that f is γ-Lipschitzian on $B(x_0, r)$, i.e.*

$$|f(x) - f(y)| \leq \gamma \|x - y\| \;\forall\, x, y \in B(x_0, r).$$

Lemma 5.2. *Let $f : H \to \mathbb{R}$ be a real-valued convex differentiable function and $a \in H$. Assume that f is bounded on bounded subsets of H. Then, the gradient map $\nabla f : H \to H$ is bounded on bounded subsets of H.*

Proof. Let $x_0 \in H$ and $r > 0$. Set $B := B(x_0, r)$. We show that $\nabla f(B)$ is bounded in H. From lemma 5.1, there exists $\gamma > 0$ such that

$$|f(x) - f(y)| \leq \gamma \|x - y\| \;\forall\, x, y \in B. \tag{5.1}$$

Let $z^* \in \nabla f(B)$ and $x^* \in B$ such that $z^* = \nabla f(x^*)$. Since B is open, for all $u \in H$, there exists $t > 0$ such that $x^* + tu \in B$. Using the fact that $z^* = \nabla f(x^*)$ and inequality (5.1), it follows that

$$\begin{aligned} \langle z^*, tu \rangle &\leq f(x^* + tu) - f(x^*) \\ &\leq t\gamma \|u\| \end{aligned}$$

so that

$$\langle z^*, u \rangle \leq \gamma \|u\| \;\forall\, u \in H.$$

Therefore $\|z^*\| \leq \gamma$. Hence $\nabla f(B)$ is bounded. □

We now prove the following theorem.

Theorem 5.3. *Let H be real Hilbert space. Assume that $f : H \to \mathbb{R}$ is a real valued bounded, convex and continuously differentiable function. Let $\{x_n\}$ be the sequence generated from arbitrary $x_1 \in H$ by*

$$x_{n+1} := x_n - \lambda_n \nabla f(x_n) - \lambda_n (x_n - x_1), \;\forall\, n \geq 1, \tag{5.2}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(1) $\lim_{n \to \infty} \lambda_n = 0$
(2) $\sum_{n=1}^{\infty} \lambda_n = \infty$.

If f has a minimizer on H, then there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a minimizer of f.

Proof. From [11] and Lemma 5.1, we have that the gradient map $\nabla f : H \to H$ is an m-monotone mapping hence satisfies the range condition (see, e.g., Chidume and Djitte, [1]), and $\nabla f(a) = 0$ if and only if a is a minimizer of f in H. Using the fact that f is continuously differentiable, bounded and Lemma 5.2, it follows that the gradient map $\nabla f : H \to H$ is bounded and satisfies the range condition. Therefore, the conclusion follows from Theorem 4.2. □

References

[1] C. E. Chidume and N. Djitte, *Strong convergence theorems for zeros of bounded maximal monotone nonlinear operators*, Abstract and Applied Analysis, **Volume 2012**, Article ID 681348, 19 pages, doi:10.1155/2012/681348.

[2] O. Güler, *On the convergence of the proximal point algorithm for convex minimization*, SIAM J. Control Optim. **29** (1991), 403–419.

[3] S. Kamimura and W. Takahashi; *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. of Optimization **13** (2003), 938–945.

[4] N. Lehdili and A. Moudafi; *Combining the proximal algorithm and Tikhonov regularization*, Optimization **37** (1996), 239–252.

[5] J. Lindenstrauss and L. Tzafriri; *Classical Banach spaces II: Function Spaces*, Ergebnisse Math. Grenzgebiete Bd. **97**, Springer-Verlag, Berlin, 1979.

[6] B. Martinet, *regularization d' inéquations variationelles par approximations successives*, Revue Francaise d'informatique et de Recherche operationelle,**4** (1970), 154–159.

[7] G. J. Minty, *Monotone (nonlinear) operator in Hilbert space*, Duke Math.**29** (1962), 341–346.

[8] S. Reich, *Constructive techniques for accretive and monotone operators* in Applied Nonlinear Analysis, Academic Press, New York, 1979, pp. 335–345.

[9] S. Reich, *Strong convergent theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 287–292.

[10] R. T. Rockafellar, *Monotone operator and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877–898.

[11] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

[12] M. V. Solodov and B. F. Svaiter, *Forcing strong convergence of proximal point iterations in a Hilber space*, Math. Program., Ser. A **87** (2000), 189–202.

[13] H. K. Xu, *Iterative algorithms for nonlinear operators,* J. London Math. Soc. **66** (2002), 240–256.

[14] H. K. Xu, *A regularization Method for the proximal point algorithm*, J. Global Opt. **36**(2006), 115–125.

[15] Z. B. Xu and G. F. Roach, *Characteristic inequalities for uniformly convex and uniformly smooth Banach spaces*, J. Math. Anal. Appl. **157** (1991), 189–210.

Manuscript received December 23, 2013
revised June 10, 2013

C. E. Chidume
African University of Science and Technology, Abuja, Nigeria
E-mail address: cchidume@aust.edu.ng

C. O. Chidume
Auburn University Department of Mathematics and Statistics Auburn, Alabama, U.S.A.
E-mail address: chidugc@auburn.edu

Y. Shehu
African University of Science and Technology, Abuja, Nigeria
E-mail address: deltanougt2006@yahoo.com

J. N. Ezeora
African University of Science and Technology, Abuja, Nigeria
E-mail address: jerryezeora@yahoo.com