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ITERATIVE ALGORITHM FOR FIXED POINTS OF MULTI-VALUED PSEUDO-CONTRACTIVE MAPPINGS IN BANACH SPACES

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Dedicated to Professor Simeon Reich on the occasion of his 65th birthday.

Abstract. Let $q > 1$ and let E be a q-uniformly smooth real Banach space and K be a nonempty, closed and convex subset of E. Let $CB(K)$ be the collection of all closed and bounded subsets of K. Suppose that $T : K \to CB(K)$ is a multi-valued bounded continuous pseudo-contractive mapping with a nonempty fixed point set. A new iteration scheme is constructed and the corresponding sequence $\{x_n\}$ is proved to converge strongly to a fixed point of T under appropriate conditions on the iteration parameters. Our theorems are significant improvements on several important recent results.

1. General introduction

For several years, the study of fixed point theory for *multi-valued nonlinear mappings* has attracted, and continues to attract, the interest of several well known mathematicians (see, for example, Brouwer [2], Downing and Kirk [12], Geanakoplos [14], Kakutani [16], Nadler [22], Nash [23, 24]).

Interest in the study of fixed point theory for multi-valued maps stems, perhaps, mainly from the fact that many problems in some areas of mathematics such as in *Game Theory and Market Economy* and in *Non-Smooth Differential Equations* can be written as fixed point problems for multi-valued maps. We describe briefly the connection of fixed point theory for multi-valued mappings with game theory.

Game Theory. In game theory and market economy, the existence of equilibrium was obtained by the application of a fixed point theorem. In fact, Nash [23, 24] showed the existence of equilibria for non-cooperative static games as a direct consequence of Brouwer [2] or Kakutani [16] fixed point theorem. More precisely, under some regularity conditions, given a game, there exists always a *multi-valued map* whose fixed points coincide with the equilibrium points of the game. A model example of such an application is the *Nash equilibrium theorem* (see, *e.g.,* [24]).

Consider a game $G = (u_n, K_n)$ with N players denoted by n, $n = 1, \cdots, N$, where $K_n \subset \mathbb{R}^{m_n}$ is the set of possible strategies of the n'th player and is assumed to be nonempty, compact and convex and $u_n : K := K_1 \times K_2 \cdots \times K_N \to \mathbb{R}$ is the payoff

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(or gain function) of the player n and is assumed to be continuous. The player n can take *individual actions*, represented by a vector $\sigma_n \in K_n$. All players together can take a *collective action*, which is a combined vector $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_N)$. For each n, $\sigma \in K$ and $z_n \in K_n$, we will use the following standard notations:

$$K_{-n} := K_1 \times \cdots \times K_{n-1} \times K_{n+1} \times \cdots \times K_N,$$

$$\sigma_{-n} := (\sigma_1, \cdots, \sigma_{n-1}, \sigma_{n+1}, \cdots, \sigma_N),$$

$$(z_n, \sigma_{-n}) := (\sigma_1, \cdots, \sigma_{n-1}, z_n, \sigma_{n+1}, \cdots, \sigma_N).$$

A strategy $\bar{\sigma}_n \in K_n$ permits the n'th player to maximize his gain *under the condition* that the *remaining players* have chosen their strategies σ_{-n} if and only if

$$u_n(\bar{\sigma}_n, \sigma_{-n}) = \max_{z_n \in K_n} u_n(z_n, \sigma_{-n}).$$

Now, let $T_n : K_{-n} \to 2^{K_n}$ be the multi-valued map defined by

$$T_n(\sigma_{-n}) := \operatorname*{Arg\,max}_{z_n \in K_n} u_n(z_n, \sigma_{-n}) \ \forall\, \sigma_{-n} \in K_{-n}.$$

Definition. A collective action $\bar{\sigma} = (\bar{\sigma}_1, \cdots, \bar{\sigma}_N) \in K$ is called a *Nash equilibrium point* if, for each n, $\bar{\sigma}_n$ is the best response for the n'th player to the action $\bar{\sigma}_{-n}$ made by the remaining players. That is, for each n,

$$u_n(\bar{\sigma}) = \max_{z_n \in K_n} u_n(z_n, \bar{\sigma}_{-n}) \tag{1.1}$$

or equivalently,

$$\bar{\sigma}_n \in T_n(\bar{\sigma}_{-n}). \tag{1.2}$$

This is equivalent to $\bar{\sigma}$ is a fixed point of the *multi-valued* map $T : K \to 2^K$ defined by

$$T(\sigma) := [T_1(\sigma_{-1}), T_2(\sigma_{-2}), \cdots, T_N(\sigma_{-N})].$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of *fixed point theory of multi-valued mappings.* However, it has been remarked that the applications of this fixed point theory to equilibrium problems are mostly static: they enhance understanding the conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. *This is part of the problem that is being addressed by iterative methods for fixed points of multi-valued mappings.*

Let E be a real normed space with dual E^* and let $S := \{x \in E : \|x\| = 1\}$. The space E is said to have *Gâteaux differentiable norm* if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S$; E is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S$, the limit is attained uniformly for $x \in S$.

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\}; \quad \tau > 0.$$

A normed linear space E is called *uniformly smooth* if $\lim\limits_{\tau\to 0}\dfrac{\rho_E(\tau)}{\tau} = 0$. It is well known (see, *e.g.* [10] p. 16, [20]) that ρ_E is nondecreasing. If there exist a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be q-*uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p \ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2 - \text{uniformly smooth} & \text{if} \quad 2 \leq p < \infty; \\ p - \text{uniformly smooth} & \text{if} \quad 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\}$$

where $\langle .,.\rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J. It is well known that if E is smooth, J_q is single-valued.

Every uniformly smooth real normed space has uniformly Gâteaux differentiable norm (see, *e.g.*, [10], p. 17).

Several iterative algorithms have been introduced and studied for approximating fixed points of *multi-valued* mappings.
Let K be a nonempty subset of a normed space E. The set K is called *proximinal* (see, *e.g.*, [26, 27, 31]) if for each $x \in E$, there exists $u \in K$ such that

$$d(x, u) = \inf\{\|x - y\| : y \in K\} = d(x, K),$$

where $d(x, y) = \|x - y\|$ for all $x, y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(K)$ and $P(K)$ denote the families of nonempty, closed and bounded subsets and nonempty, proximinal and bounded subsets of K, respectively. The *Hausdorff metric* on $CB(K)$ is defined by:

$$D(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}$$

for all $A, B \in CB(K)$. Let $T : D(T) \subseteq E \to CB(E)$ be a *multi-valued mapping* on E. A point $x \in D(T)$ is called a *fixed point of* T if $x \in Tx$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called L- *Lipschitzian* if there exists $L > 0$ such that

$$D(Tx, Ty) \leq L\|x - y\| \ \ \forall x, y \in D(T). \tag{1.3}$$

When $L \in (0, 1)$ in (1.3), we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

Several papers deal with the problem of *approximating* fixed points of *multi-valued nonexpansive* mappings (see, *e.g.*, [1, 18, 26, 27, 31], , and the references therein) and their generalizations (see, *e.g.*, [13]).

Sastry and Babu [27] introduced the following iterative schemes. Let $T : E \to P(E)$ be a multi-valued mapping and x^* be a fixed point of T. Define iteratively the sequence $\{x_n\}_{n\in\mathbb{N}}$ from $x_0 \in E$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n,\ y_n \in Tx_n,\ \|y_n - x^*\| = d(x^*, Tx_n), \tag{1.4}$$

where α_n is a real sequence in (0,1) satisfying the following conditions:
$(i)\ \sum_{n=1}^{\infty} \alpha_n = \infty$ and $(ii)\ \lim \alpha_n = 0$.

They also introduced the following sequence $\{x_n\}$:

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n z_n,\ z_n \in Tx_n,\ \|z_n - x^*\| = d(x^*, Tx_n), \\ \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n,\ u_n \in Ty_n,\ \|u_n - x^*\| = d(x^*, Ty_n), \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers satisfying the following conditions:

$$(i)\ \ 0 \le \alpha_n, \beta_n < 1,\ (ii)\ \lim_{n\to\infty} \beta_n = 0 \text{ and } (iii)\ \sum_{n=1}^{\infty} \alpha_n\beta_n = \infty.$$

They proved that if H is a real Hilbert space, K is a nonempty, compact and convex subset of H and $T : K \to P(K)$ is a multi-valued nonexpansive map with a fixed point p, then the sequence $\{x_n\}$ defined by (1.5) converges strongly to a fixed point of T.

Panyanak [26] extended this result to uniformly convex real Banach spaces.

Panyanak also modified the iteration schemes of Sastry and Babu [27]. Let K be a nonempty, closed and convex subset of a real Banach space and $T : K \to P(K)$ be a multi-valued map such that $F(T)$ is a nonempty proximinal subset of K.

The sequence of *Mann-type* (see, e.g., [21]) iterates is defined by $x_0 \in K$,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n y_n,\ \ \alpha_n \in [a, b],\ 0 < a < b < 1, \tag{1.6}$$

where $y_n \in Tx_n$ is such that $\|y_n - u_n\| = d(u_n, Tx_n)$ and $u_n \in F(T)$ is such that $\|x_n - u_n\| = d(x_n, F(T))$.

The sequence of *Ishikawa-type* (see, e.g., [15]) iterates is defined by $x_0 \in K$,

$$y_n = (1 - \beta_n)x_n + \beta_n z_n,\ \ \beta_n \in [a, b],\ 0 < a < b < 1, \tag{1.7}$$

where $z_n \in Tx_n$ is such that $\|z_n - u_n\| = d(u_n, Tx_n)$ and $u_n \in F(T)$ is such that $\|x_n - u_n\| = d(x_n, F(T))$.

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n z'_n, \ \ \alpha_n \in [a,b], \, 0 < a < b < 1, \tag{1.8}$$

where $z'_n \in Ty_n$ is such that $\|z'_n - v_n\| = d(v_n, Ty_n)$ and $v_n \in F(T)$ is such that $\|y_n - v_n\| = d(y_n, F(T))$.

A mapping $T : K \to CB(K)$ is said to satisfy *Condition (I)* if there exists a strictly increasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $d(x, T(x)) \geq f(d(x, F(T)) \ \forall x \in D$.

Panyanak [26] proved that if E is a uniformly convex real Banach space, K is a nonempty, closed, bounded and convex subset of E and $T : K \to P(K)$ is a multi-valued nonexpansive map that satisfies condition (I) such that $F(T)$ is a nonempty proximinal subset of K, then, the sequence $\{x_n\}$ defined by (1.6) converges strongly to a fixed point of T.

Remark 1.1. In defining the recursion formula (1.4), the authors take $y_n \in T(x_n)$ such that $\|y_n - x^*\| = d(x^*, Tx_n)$. The *existence* of y_n satisfying this condition is guaranteed by the assumption that the set Tx_n is proximinal. In general such a y_n is extremely difficult to pick. Furthermore, this condition involves a fixed point x^* of T that is being approximated. So, the recursion formula (1.4) is not convenient to use in application. Also, the recursion formulas defined in (1.7) and (1.8) are not convenient to use in application. The sequences $\{z_n\}$ and $\{z'_n\}$ are not known precisely. The restrictions $z_n \in Tx_n$, $\|z_n - u_n\| = d(u_n, Tx_n)$, $u_n \in F(T)$ and $z'_n \in Ty_n$, $\|z'_n - v_n\| = d(v_n, Ty_n)$, $v_n \in F(T)$ make them difficult to use. These restrictions on z_n and z'_n depend on $F(T)$, the fixed points set. So, the recursions formulas (1.7) and (1.8) are not easily useable.

As remarked by Nadler [22], the definition of the Hausdorff metric on $CB(E)$ gives the following useful result.

Lemma 1.2. *Let $A, B \in CB(E)$ and $a \in A$. For every $\gamma > 0$, there exists $b \in B$ such that $d(a,b) \leq D(A,B) + \gamma$.*

Song and Wang [31] used the idea of Lemma 1.2 to define the following iteration scheme.

Let K be a nonempty, closed and convex subset of a real Banach space and $T : K \to CB(K)$ be a multi-valued map. Let $\alpha_n, \beta_n \in [0,1]$ and $\gamma_n \in (0,\infty)$ be such that $\lim_{n\to\infty} \gamma_n = 0$. Choose $x_0 \in K$,

$$y_n = (1-\beta_n)x_n + \beta_n z_n, \quad x_{n+1} = (1-\alpha_n)x_n + \alpha_n u_n, \tag{1.9}$$

where $z_n \in Tx_n$, $u_n \in Ty_n$ are such that

$$\|z_n - u_n\| \leq D(Tx_n, Ty_n) + \gamma_n, \quad \|z_{n+1} - u_n\| \leq D(Tx_{n+1}, Ty_n) + \gamma_n, \, n \geq 0.$$

They then proved that if K is a nonempty, compact and convex subset of a uniformly convex real Banach space E and $T : K \to CB(K)$ is a multi-valued nonexpansive mapping with $F(T) \neq \emptyset$ satisfying $T(p) = \{p\}$ for all $p \in F(T)$, then the Ishikawa sequence defined by (1.9) converges strongly to a fixed point of T.

Shahzad and Zegeye [29] extended the results of Sastry and Babu [27], Panyanak [26] and Son and Wang [31] to multi-valued *quasi-nonexpansive maps* (i.e., mappings $T : K \to CB(K)$ such that $F(T) \neq \emptyset$ and $D(Tx, Tp) \leq \|x - p\| \quad \forall x \in K,\ p \in F(T)$). Also, in an attempt to remove the condition $T(p) = \{p\}$ for all $p \in F(T)$ in the result of Song and Wang, Shahzad and Zegeye introduced a new iteration scheme as follows:

Let K be a nonempty, closed and convex subset of a real Banach space, $T : K \to P(K)$ be a multi-valued map and let $P_T : K \to P(K)$ be defined as follows: $P_T x := \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Let $\alpha_n, \beta_n \in [0, 1]$. Choose $x_0 \in K$ and define $\{x_n\}$ as follows:

$$y_n = (1 - \beta_n)x_n + \beta_n z_n, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n u_n, \tag{1.10}$$

where $z_n \in P_T x_n$, $u_n \in P_T y_n$.

They then proved that if E is a uniformly convex real Banach space, K is a nonempty and convex subset of E and $T : K \to P(K)$ is a multi-valued map with $F(T) \neq \emptyset$ such that P_T is nonexpansive, and T satisfies Condition (I), then, $\{x_n\}$ defined by (1.10) converges strongly to a fixed point of T.

A class of single-valued *k-strictly pseudo-contractive maps* on Hilbert spaces was introduced by Browder and Petryshyn [4] as an important generalization of the class of nonexpansive mappings. This class of mappings is a superclass of the class of nonexpansive mappings.

Definition 1.3. Let K be a nonempty subset of a real Hilbert space H. A map $T : K \to H$ is called *k-strictly pseudo-contractive* if there exists $k \in (0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|x - y - (Tx - Ty)\|^2 \quad \forall x, y \in K.$$

Motivated by Definition 1.3, the present authors [11] introduced the following important class of *multi-valued strictly pseudo-contractive mappings in real Hilbert spaces* which is more general than the class of *multi-valued nonexpansive mappings.*

Definition 1.4. A multi-valued mapping $T : D(T) \subseteq H \to CB(H)$ is said to be *k-strictly pseudo-contractive* if there exists $k \in (0, 1)$ such that for all $x, y \in D(T)$ we have:

$$\Big(D(Tx, Ty)\Big)^2 \leq \|x - y\|^2 + k\|(x - u) - (y - v)\|^2 \ \forall\, u \in Tx,\ v \in Ty. \tag{1.11}$$

Then, they proved strong convergence theorems for this class of mappings. The recursion formula used in [11] is of the *Krasnoselskii-type* [19] which is known to be superior (see, *e.g.*, Remark 4 in [11]) to the recursion formula of Mann [21] or Ishikawa [15]. In fact, they proved the following theorems.

Theorem CA1 (Chidume *et al.* [11]). *Let K be a nonempty, closed and convex subset of a real Hilbert space H. Suppose that $T : K \to CB(K)$ is a multi-valued k-strictly pseudo-contractive mapping such that $F(T) \neq \emptyset$. Assume that $Tp = \{p\}$ for all $p \in F(T)$. Let $\{x_n\}$ be a sequence defined by*

$$x_0 \in K,\ x_{n+1} = (1-\lambda)x_n + \lambda y_n,\ n \geq 0,$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Theorem CA2 (Chidume *et al.* [11]). *Let K be a nonempty, compact and convex subset of a real Hilbert space H and $T : K \to CB(K)$ be a multi-valued k-strictly pseudo-contractive mapping with $F(T) \neq \emptyset$ such that $Tp = \{p\}$ for all $p \in F(T)$. Suppose that T is continuous. Let $\{x_n\}$ be a sequence defined by*

$$x_0 \in K,\ x_{n+1} = (1-\lambda)x_n + \lambda y_n, n \geq 0,$$

where $y_n \in Tx_n$ and $\lambda \in (0, 1-k)$. Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

In arbitrary real normed spaces, the following definition is known (see, *e.g.*, [10]).

Definition 1.5. Let E be real normed linear space. A map $T : D(T) \subset E \to E$ is called *pseudo-contractive* (see, *e.g.*, [5]) if the inequality

$$\|x - y\| \leq \|x - y + t((x - Tx)) - (y - Ty))\| \tag{1.12}$$

holds for each $x, y \in D(T)$ and for all $t > 0$. As a result of Kato [17], it follows from inequality (1.12) that T is *pseudo-contractive* if and only if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2, \tag{1.13}$$

where $J : E \to 2^{E^*}$ is the normalized duality mapping.

For *multi-valued pseudocontractions*, we have (see, *e.g.*, [25]) the following definition in arbitrary normed linear spaces.

Definition 1.6. Let E be a normed space. A multi-valued mapping $T : D(T) \to 2^E$ is called *pseudo-contractive* if for all $x, y \in D(T)$, we have

$$\langle u - v, j(x - y)\rangle \leq \|x - y\|^2 \quad \forall\, u \in Tx,\, v \in Ty. \tag{1.14}$$

The class of pseudo-contractive mappings is deeply connected with the class of accretive operators, where an operator A with domain $D(A)$ in E is called *accretive* if the inequality $\|x-y\| \leq \|x-y+s(u-v)\|$ holds for each $x, y \in D(A)$, $u \in Ax$, $v \in Ay$ and for all $s > 0$. We observe that A is accretive if and only if $T := I - A$ is pseudo-contractive and thus, the set of zeros of A, $N(A) := \{x \in D(A) : x \in A^{-1}(0)\}$, coincides with the fixed point set of T. The importance of these operators is well known (see, *e.g.*, [7, 8, 10]).

Remark 1.7. We note that for approximating a fixed point of a *multi-valued Lipschitz pseudo-contractive map* in a real Hilbert space, an example of Chidume and Mutangadura [7] shows that, even in the single-valued case, the Mann iteration method does not always converge strongly even in the setting of Theorem CA2.

Chidume and Zegeye [9] later introduced an iteration algorithm which converges in this setting. Precisely, they introduced the following algorithm. Let $x_1 \in K$ and define

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n T x_n - \lambda_n\theta_n(x_n - x_1),\ \forall\ n \geq 1, \tag{1.15}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(1) $\lambda_n(1+\theta_n) < 1$;
(2) $\lim \theta_n = 0$;
(3) $\sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \quad \lambda_n = o(\theta_n)$;
(4) $\lim_{n\to\infty} \frac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} = 0,\ \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$

Motivated by this algorithm, Ofoedu and Zegeye [25] introduced an iteration scheme for approximating a fixed point of a *multi-valued* Lipschitz pseudo-contractive mapping. They proved the following theorem.

Theorem OZ (Ofoedu and Zegeye [25]). *Let E be a reflexive real Banach space having uniformly Gâteaux differentiable norm, D be a nonempty open convex subset of E, such that every closed convex bounded nonempty subset of $\overline{D}$ has the fixed point property for nonexpansive self-mappings. Let $T : \overline{D} \to K(\overline{D})$ be a pseudo-contractive Lipschitzian mapping with constant $L > 0$ and let $u \in \overline{D}$ be fixed. Let $\{x_n\}$ be generated from arbitrary $x_0 \in \overline{D}$, $w_0 \in Tx_0$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n w_n - \lambda_n\theta_n(x_n - u),\ w_n \in Tx_n. \tag{1.16}$$

Suppose that $\|w_n - w_{n-1}\| = d(w_{n-1}, Tx_n)$, $n \geq 1$. If $F(T) \neq \emptyset$, then $\{x_n\}$ converges strongly to some fixed point of T.

Remark 1.8. To establish Theorem OZ, the authors assumed that $\|w_n - w_{n-1}\| = d(w_{n-1}, Tx_n)$ for all $n \geq 1$. A sufficient condition to guarantee this is to assume that for each x, the set Tx is proximinal. In this case, Tx is closed. If, in addition Tx is convex and E is, for example, a real Hilbert space, such w_n is characterized as follows:

$$\langle w_{n-1} - w_n, w_n - u_n\rangle \geq 0 \quad \forall u_n \in Tx_n.$$

Consequently, this condition requires that a sub-programme be constructed to first compute w_n at each step of the iteration process.

Remark 1.9. Nadler [22] remarked that requiring a *multi-valued mapping* to be *Lipschitz* is placing a *strong continuity condition* on the mapping. We shall therefore weaken this condition in our theorems. In fact, the Lipschitz condition of T in Theorem OZ will be weakened to *continuity* and *boundedness* for T.

Moreover, in many applications, the real Banach space E is either an L_p-space, a $W^{m,p}$-space, $1 < p < \infty$, $m \geq 1$, or a real Hilbert space. As has been remarked before, all these spaces are q-uniformly smooth and reflexive.

With Remarks 1.8 and 1.9 in mind, it is our purpose in this paper to prove strong convergence theorems for fixed points of *multi-valued bounded continuous pseudo-contractive maps* defined on q-uniformly smooth real Banach spaces. We use the recursion formula (1.16), dispensing with the restriction that $\|w_n - w_{n-1}\| = d(w_{n-1}, Tx_n)$ $\forall n \geq 1$. Furthermore, our iteration process, in the setting of q-uniformly smooth real Banach spaces, is direct, much more applicable than the process in Theorem OZ since it does not require the creation of a sub-programme to first compute w_n at each step of the iteration process. In particular, in q-uniformly smooth real Banach spaces, our theorems extend Theorem OZ *from multi-valued Lipschitz* pseudo-contractive mappings to the much more general class of *multi-valued continuous, bounded* and pseudo-contractive mappings.

2. Preliminaries

In the sequel we shall need the following results.

Lemma 2.1 (Xu, [34]). *Let $\{\rho_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$\rho_{n+1} \leq (1-\alpha_n)\rho_n + \alpha_n\sigma_n + \gamma_n, \;\; n \geq 0,$$

where,
(i) $\{\alpha_n\} \subset (0,1)$, $\sum \alpha_n = \infty$; *(ii)* $\limsup_{n\to\infty} \sigma_n \leq 0$;
(ii) $\gamma_n \geq 0$, $\sum \gamma_n < \infty$. *Then,* $\rho_n \to 0$ *as* $n \to \infty$.

Theorem 2.2 (Xu, [32]). *Let $q > 1$ and E be a smooth real Banach space. Then the following are equivalent.*
(i) *E is q-uniformly smooth.*
(ii) *There exists a constant $d_q > 0$ such that for all $x, y \in E$,*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + d_q||y||^q. \tag{2.1}$$

For the remainder of this paper, d_q will denote the constant appearing in Theorem 2.2.

Lemma 2.3. *Let E be a real normed linear space and $q > 1$. Then, the following inequality holds:*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x+y)\rangle \;\; \forall\, j_q(x+y) \in J_q(x+y), \;\; \forall\, x, y \in E. \tag{2.2}$$

Proof. Let $q > 1$ and $x, y \in E$. By the definition of J_q, the following holds

$$\begin{aligned} ||x+y||^q &= \langle x+y, j_q(x+y)\rangle \\ &= \langle x, j_q(x+y)\rangle + \langle y, j_q(x+y)\rangle \end{aligned}$$

for every $j_q(x+y) \in J_q(x+y)$. Using Schwartz and Young inequalities and the fact that $\|j_q(z)\| = \|z\|^{q-1}$ for all $z \in E$, we have

$$||x+y||^q \leq \|x\|\|x+y\|^{q-1} + \langle y, j_q(x+y)\rangle$$

$$\leq \quad \frac{1}{q}\|x\|^q + \frac{1}{p}\|x+y\|^q + \langle y, j_q(x+y)\rangle,$$

with $\frac{1}{p} + \frac{1}{q} = 1$. Observing that $1 - \frac{1}{p} = \frac{1}{q}$, then the results follows. □

3. Approximation of fixed points of multi-valued bounded continuous pseudo-contractive mappings

We shall make use of the following result.

Lemma 3.1 (Ofoedu and Zegeye [25])**.** *Let D be a nonempty open convex subset of a real Banach space E and $T : \overline{D} \to CB(E)$ be continuous (with respect to the Hausdorff metric) pseudo-contractive mapping satisfying weakly inward condition and $u \in \overline{D}$ be fixed. Then, for $t \in (0,1)$ there exists $y_t \in \overline{D}$ satisfying $y_t \in tTy_t + (1-t)u$. If, in addition, E is reflexive and has uniformly Gâteaux differentiable norm, and is such that every closed convex bounded subset of $\overline{D}$ has the fixed point property for nonexpansive self-mappings, then T has a fixed point if and only if $\{y_t\}$ remains bounded as $t \to 1$; moreover, in this case, $\{y_t\}$ converges strongly to a fixed point of T as $t \to 1$.*

Remark 3.2. If E is q-uniformly smooth, then E is reflexive and every nonempty, closed, convex and bounded subset of E has the fixed point property for nonexpansive self-mappings and has uniformly Gâteaux differentiable norm.

Remark 3.3. We note that in Lemma 3.1, in the case that $F(T) \neq \emptyset$, the path $\{y_t\}$ is bounded. Furthermore, if E is assumed to have a uniformly Gâteaux differentiable norm and is such that every closed convex and bounded subset of $\overline{D}$ has the fixed point property for nonexpansive self-mappings, then as $t \to 1$, the path $\{y_t\}$ converges strongly to a fixed point of T.

For the rest of this paper, $q > 1$ is a real number and $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:

(*i*) $\lim \theta_n = 0$;

(*ii*) $\lambda_n(1+\theta_n) < 1, \sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \quad \lambda_n^{q-1} = o(\theta_n)$;

(*iii*) $\limsup_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} \leq 0, \quad \sum_{n=1}^{\infty} \lambda_n^q < \infty.$

In what follows, $\{y_n\}$ denotes the sequence defined by $y_n := y_{t_n} = t_n z_n + (1-t_n)x_1$, where $x_1 \in \overline{D}$ and $t_n = (1+\theta_n)^{-1}\ \forall\ n \geq 1$, guaranteed by Lemma 3.1 and Remark 3.3. We observe that with this t_n, the sequence $\{y_n\}$ satisfies the following conditions:

$$\theta_n(y_n - x_1) + (y_n - z_n) = 0, \ n \geq 1; \tag{3.1}$$

$$y_n \to y^* \ \text{ with } \ y^* \in F(T). \tag{3.2}$$

for some $z_n \in Ty_n$.

We now prove the following theorem.

Theorem 3.4. *Let E be a q-uniformly smooth real Banach space and D be a nonempty, open and convex subset of E. Assume that $T : \overline{D} \to CB(\overline{D})$ is a multi-valued continuous (with respect to the Hausdorff metric), bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_1 \in \overline{D}$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1),\ u_n \in Tx_n. \tag{3.3}$$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n^{q-1} < \gamma_0\theta_n$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Let $x^* \in F(T)$. There exists $r > 0$ sufficiently large such that $x_1 \in B(x^*, r/2)$. Define $B := \overline{B(x^*, r)} \cap \overline{D}$. Since T is bounded, it follows that $(I-T)(B)$ is bounded. So,

$$M_0 := \sup\Big\{\|x - u + \theta(x - x_1)\|^q :\ x \in B,\, u \in Tx,\, 0 < \theta \leq 1\Big\} + 1 < \infty.$$

Set

$$M := d_q M_0;\ \ \gamma_0 := \Big(\frac{2^{q-1}-1}{2^q M}\Big) r^q.$$

Step1. *We prove that $\{x_n\}$ is bounded.* Indeed, it suffices to show that x_n is in B for all $n \geq 1$. The proof is by induction. By construction, $x_1 \in B$. Suppose that $x_n \in B$ for some $n \geq 1$. We prove that $x_{n+1} \in B$.

Using the recursion formula (3.3) and inequality (2.1) of Theorem 2.2, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &= \|x_n - x^* - \lambda_n(x_n - u_n + \theta_n(x_n - x_1))\|^q \\
&\leq \|x_n - x^*\|^q - q\lambda_n\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - x^*)\rangle \\
&\quad + d_q\lambda_n^q\|x_n - u_n + \theta_n(x_n - x_1)\|^q \\
&\leq \|x_n - x^*\|^q - q\lambda_n\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - x^*)\rangle + M\lambda_n^q. \qquad (3.4)
\end{aligned}$$

Using the fact that T is pseudo-contractive, we obtain

$$\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - x^*)\rangle \geq \theta_n\|x_n - x^*\|^q + \theta_n\langle x^* - x_1, j_q(x_n - x^*)\rangle.$$

Therefore, using inequality (3.4) and schwartz ineqality, we have the following estimates:

$$\begin{aligned}
\|x_{n+1} - x^*\|^q &\leq (1 - q\lambda_n\theta_n)\|x_n - x^*\|^q - q\lambda_n\theta_n\langle x^* - x_1, j_q(x_n - x^*)\rangle + M\lambda_n^q \\
&\leq (1 - q\lambda_n\theta_n)\|x_n - x^*\|^q + q\lambda_n\theta_n\|x^* - x_1\|\|x_n - x^*\|^{q-1} + M\lambda_n^q \\
&\leq (1 - q\lambda_n\theta_n)\|x_n - x^*\|^q + q\lambda_n\theta_n\Big(\frac{1}{q}\|x^* - x_1\|^q + \frac{1}{p}\|x_n - x^*\|^q\Big) \\
&\quad + M\lambda_n^q,
\end{aligned}$$

with $1/p + 1/q = 1$. Thus,

$$\|x_{n+1} - x^*\|^q \leq (1 - \lambda_n\theta_n)\|x_n - x^*\|^q + \lambda_n\theta_n\|x^* - x_1\|^q + M\lambda_n^q.$$

So, using the induction assumption, the fact that $x_1 \in B(x^*, r/2)$ and the condition $\lambda_n^{q-1} \leq \gamma_0\theta_n$, we obtain:

$$\|x_{n+1} - x^*\|^q \leq \Big(1 - \frac{1}{2}\lambda_n\theta_n\Big) r^q \leq r^q.$$

Therefore $x_{n+1} \in B$. Thus by induction, $\{x_n\}$ is bounded.

Step 2. *We prove that* $\{x_n\}$ *converges to a fixed point of* T. Let $\{y_n\}$ be the sequence obtained from Lemma 3.1 and satisfying equation (3.1) and condition (3.2).

Claim: $\|x_{n+1} - y_n\| \to 0$ as $n \to 0$. Since $\{x_n\}$ and $\{y_n\}$ are bounded and T is bounded, there exists some positive constant M such that:

$$\begin{aligned}
\|x_{n+1} - y_n\|^q &= \|x_n - y_n - \lambda_n(x_n - u_n + \theta_n(x_n - x_1))\|^q \\
&\leq \|x_n - y_n\|^q - q\lambda_n\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - y_n)\rangle \\
&\quad + d_q\lambda_n^q\|x_n - u_n + \theta_n(x_n - x_1)\|^q \\
&\leq \|x_n - y_n\|^q - q\lambda_n\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - y_n)\rangle + M\lambda_n^q.
\end{aligned}$$

Using equation (3.1) and the fact that T is pseudo-contractive, we have

$$\begin{aligned}
\langle x_n - u_n + \theta_n(x_n - x_1), j_q(x_n - y_n)\rangle &= \langle (x_n - u_n) - (y_n - z_n), j_q(x_n - y_n)\rangle \\
&\quad +\theta_n\|x_n - y_n\|^q \\
&\quad +\langle y_n - z_n + \theta_n(y_n - x_1), j_q(x_n - y_n)\rangle \\
&\geq \frac{\theta_n}{q}\Big\|x_n - y_n\Big\|^q.
\end{aligned}$$

Therefore,

$$\|x_{n+1} - y_n\|^q \leq (1 - \lambda_n\theta_n)\|x_n - y_n\|^q + M\lambda_n^q. \tag{3.5}$$

Using again the fact that T is pseudo-contractive, we obtain:

$$\|y_{n-1} - y_n\| \leq \Big\|y_{n-1} - y_n + \frac{1}{\theta_n}\Big[(y_{n-1} - z_{n-1}) - (y_n - z_n)\Big]\Big\|.$$

Observing from equation (3.1) that

$$y_{n-1} - y_n + \frac{1}{\theta_n}\Big[(y_{n-1} - z_{n-1}) - (y_n - z_n)\Big] = \frac{\theta_n - \theta_{n-1}}{\theta_n}(y_{n-1} - x_1),$$

it follows that

$$\|y_{n-1} - y_n\| \leq \left(\frac{\theta_{n-1} - \theta_n}{\theta_n}\right)\|y_{n-1} - x_1\|. \tag{3.6}$$

By Lemma 2.3, we have

$$\begin{aligned}
\|x_n - y_n\|^q &= \|(x_n - y_{n-1}) + (y_{n-1} - y_n)\|^q \\
&\leq \|x_n - y_{n-1}\|^q + q\langle y_{n-1} - y_n, j_q(x_n - y_n)\rangle.
\end{aligned}$$

Using Schwartz's inequality, we obtain:

$$\|x_n - y_n\|^q \leq \|x_n - y_{n-1}\|^q + q\|y_{n-1} - y_n\|\|x_n - y_n\|^{q-1}. \tag{3.7}$$

Using inequalities (3.5), (3.6), (3.7) and the fact that $\{x_n\}$ and $\{y_n\}$ are bounded, we have,

$$\begin{aligned}
\|x_{n+1} - y_n\|^q &\leq (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^q + C\Big(\frac{\theta_{n-1} - \theta_n}{\theta_n}\Big) + M\lambda_n^q \\
&= (1 - \lambda_n\theta_n)\|x_n - y_{n-1}\|^q + (\lambda_n\theta_n)\sigma_n + \gamma_n
\end{aligned}$$

for some positive constant C where

$$\sigma_n := \frac{C\Big(\frac{\theta_{n-1}-\theta_n}{\theta_n}\Big)}{\lambda_n\theta_n} = C\Big(\frac{\frac{\theta_{n-1}}{\theta_n}-1}{\lambda_n\theta_n}\Big),\ \ \gamma_n := M\lambda_n^q.$$

Thus, using Lemma 2.1, the conditions $\limsup\limits_{n\to\infty} \frac{\Big(\frac{\theta_{n-1}}{\theta_n}-1\Big)}{\lambda_n\theta_n} \leq 0$ and $\sum\limits_{n=1}^{\infty}\lambda_n^q < \infty$, it follows that $x_{n+1} - y_n \to 0$. From condition (3.2), we have that $x_n \to y^*$ and $y^* \in F(T)$. This completes the proof. □

Corollary 3.5. *Let E be a q-uniformly smooth real Banach space, $q > 1$ and D be a nonempty, open and convex subset of E. Assume that $T : \overline{D} \to CB(\overline{D})$ is a multi-valued Lipschitz pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_1 \in \overline{D}$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1),\ u_n \in Tx_n. \tag{3.8}$$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n^{q-1} < \gamma_0\theta_n$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. We need only to show that T is bounded. For this, let B be a nonempty bounded subset of K. We show that $T(B) := \cup_{x\in B}Tx$ is bounded. Let $y_0 \in T(B)$ be fixed. Then, there exists $x_0 \in B$ such that $y_0 \in Tx_0$. Set $r_1 :=$ diameter(B) and $r_2 :=$ diameter(Tx_0). We note that r_1 and r_2 are finite. Let $y_1, y_2 \in T(B)$. Then there exist $x_i \in B$ such that $y_i \in Tx_i$, $i = 1, 2$. Using Lemma 1.2 and the fact that T is Lipschitz, it follows that there exist $z_1, z_2 \in Tx_0$ such that

$$\begin{aligned}\|y_1 - y_2\| &\leq \|y_1 - z_1\| + \|z_1 - z_2\| + \|z_2 - y_2\| \\ &\leq 2(Lr_1 + 1) + r_2,\end{aligned}$$

which implies that diameter$(T(B)) < \infty$. Therefore, $T(B)$ is bounded. □

Corollary 3.6. *Let $E = L_p$, $1 < p < \infty$ and $q := \min\{2, p\}$. Let D be a nonempty, open and convex subset of E and $T : \overline{D} \to CB(\overline{D})$ be a multi-valued continuous (with respect to the Hausdorff metric), bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_1 \in \overline{D}$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1),\ u_n \in Tx_n. \tag{3.9}$$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n^{q-1} < \gamma_0\theta_n$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since L_p spaces, $1 < p < \infty$ are q-uniformly smooth spaces with $q := \min\{2, p\}$, the proof follows from Theorem 3.4. □

Corollary 3.7. *Let H be a real Hilbert space and D be a nonempty open convex subset of H. Assume that $T : \overline{D} \to CB(\overline{D})$ is a multi-valued continuous (with respect to the Hausdorff metric), bounded and pseudo-contractive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from arbitrary $x_1 \in \overline{D}$ by*

$$x_{n+1} := (1-\lambda_n)x_n + \lambda_n u_n - \lambda_n\theta_n(x_n - x_1),\ u_n \in Tx_n, \tag{3.10}$$

where $\{\lambda_n\}$ and $\{\theta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:

(*i*) $\lim \theta_n = 0$;

(*ii*) $\lambda_n(1+\theta_n) < 1, \sum_{n=1}^{\infty} \lambda_n\theta_n = \infty, \quad \lambda_n = o(\theta_n)$;

(*iii*) $\limsup_{n\to\infty} \dfrac{\left(\frac{\theta_{n-1}}{\theta_n} - 1\right)}{\lambda_n\theta_n} \leq 0, \quad \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$

Then, there exists a real constant $\gamma_0 > 0$ such that if $\lambda_n < \gamma_0\theta_n$, for all $n \geq 1$, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since Hilbert spaces are 2-uniformly smooth spaces, the proof follows from Theorem 3.4. □

Remark 3.8. Our theorems are significant improvements on recent results of Ofoedu and Zegeye [25] in the sense that the class of maps in our theorems is a superclass of the class of *Lipschitz pseudocontractions* studied by Ofoedu and Zegeye [25].

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