Journal of Nonlinear and Convex Analysis
Volume 15, Number 1, 2014, 219–240

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

COMPOSITE VISCOSITY APPROXIMATION METHODS FOR EQUILIBRIUM PROBLEM, VARIATIONAL INEQUALITY AND COMMON FIXED POINTS

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Dedicated to Professor Simeon Reich on the occasion of his 65th birthday

Abstract. In this paper, we present a new composite viscosity approximation method, and prove the strong convergence of the method to a common fixed point of a finite number of nonexpansive mappings that also solves a suitable equilibrium problem and an appropriate variational inequality.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, C be a nonempty closed convex subset of H and P_C be the metric projection from H onto C. Let $T : C \to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T and by $\mathbf{R}$ the set of all real numbers. A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

A mapping $A : C \to H$ is called α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \ge \alpha \|x - y\|^2, \quad \forall x, y \in C.$$

For a given mapping $A : C \to H$, we consider the following variational inequality (VI) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \ge 0, \quad \forall x \in C. \tag{1.1}$$

The solution set of the VI (1.1) is denoted by $\mathrm{VI}(C, A)$. We remark that the variational inequality was first discussed by Lions [12] and now is well known. In 2003, for finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ when $C \subset H$ is nonempty, closed and convex, $S : C \to C$ is nonexpansive and $A : C \to H$ is α-inverse strongly monotone, Takahashi and Toyoda [23] introduced the following Mann's type iterative

2010 *Mathematics Subject Classification.* 49J40, 47J20, 47H10, 65K05, 47H09.

Key words and phrases. Hierarchical fixed points, equilibrium problem, nonexpansive mapping, contraction, variational inequality, projection.

*This research was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133) and Ph.D. Program Foundation of Ministry of Education of China (20123127110002).

†This work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PNII-ID-PCE-2011-3-0094.

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algorithm:

$$(1.2) \qquad \begin{cases} x_0 = x \in C \text{ chosen arbitrarily,} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C (x_n - \lambda_n A x_n), \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0, 2\alpha)$. It was shown in [23] that, if $\mathrm{Fix}(S) \cap \mathrm{VI}(C,A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to some $z \in \mathrm{Fix}(S) \cap \mathrm{VI}(C,A)$. Further, given a contractive mapping $f : C \to C$, an α-inverse-strongly monotone mapping $A : C \to H$ and a nonexpansive mapping $T : C \to C$, Jung [10] introduced the following two-step iterative scheme by the viscosity approximation method

$$(1.3) \qquad \begin{cases} x_0 = x \in C \text{ chosen arbitrarily,} \\ y_n = \alpha_n f(x_n) + (1-\alpha_n) T P_C (x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\beta_n) y_n + \beta_n T P_C (y_n - \lambda_n A y_n), \quad \forall n \geq 0, \end{cases}$$

where $\{\lambda_n\} \subset (0, 2\alpha)$ and $\{\alpha_n\}, \{\beta_n\} \subset [0,1)$. It was proven in [10] that, if $\mathrm{Fix}(T) \cap \mathrm{VI}(C,A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $q = P_{\mathrm{Fix}(T) \cap \mathrm{VI}(C,A)} f(q)$.

On the other hand, if C is the fixed point set $\mathrm{Fix}(T)$ of a nonexpansive mapping T and S is another nonexpansive mapping (not necessarily with fixed points), the VI (1.1) becomes the variational inequality of finding $x^* \in \mathrm{Fix}(T)$ such that

$$(1.4) \qquad \langle (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

This problem, introduced by Mainge and Moudafi [17,18], is called hierarchical fixed point problem. It is clear that if S has fixed points, then they are solutions of the VI (1.4).

If S is a ρ-contraction (i.e., $\|Sx - Sy\| \leq \rho \|x - y\|$ for some $0 < \rho < 1$) the set of solutions of the VI (1.4) is a singleton and it is well-known as viscosity problem. This was last introduced by Moudafi [15] and also developed by Xu [26]. In this case, it is easy to see that solving the VI (1.4) is equivalent to finding a fixed point of the nonexpansive mapping $P_{\mathrm{Fix}(T)} S$, where $P_{\mathrm{Fix}(T)}$ is the metric projection on the closed and convex set $\mathrm{Fix}(T)$.

In the literature, the recent research work shows that variational inequalities like the VI (1.1) cover several topics, for example, monotone inclusions, convex optimization and quadratic minimization over fixed point sets; see [13,15,24,26] for more details.

At present, there are generally two main approaches to the variational inequality. The first, known as a hierarchical fixed point approach, was introduced by Mainge and Moudafi [17]. This approach, in the implicit frame, generates a double-index net $\{x_{s,t} : (s,t) \in (0,1) \times (0,1)\}$ satisfying the fixed point equation

$$x_{s,t} = t f(x_{s,t}) + (1-t)(sSx_{s,t} + (1-s)Tx_{s,t})$$

where f is a ρ-contraction on C. In [17], the authors gave the following theorem.

Theorem 1.1. *The net $x_{s,t}$ strongly converges, as $t \to 0$, to x_s, where x_s satisfies $x_s = P_{\mathrm{Fix}(sS+(1-s)T)} f(x_s)$. Moreover, the net x_s, in turn, weakly converges, as $s \to 0$, to a solution x_∞ of the VI* (1.4).

Here, it is worth pointing out that Mainge and Moudafi [17] stated the problem of the strong convergence of the net $x_{s,t}$ when $(t,s) \to (0,0)$ jointly, to a solution of the VI (1.4). A negative answer to this question is given in [5].

In [18], Moudafi and Mainge studied the explicit scheme introducing the iterative algorithm

$$x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)(\alpha_n S x_n + (1-\alpha_n)T x_n), \tag{1.5}$$

where $\{\alpha_n\}, \{\lambda_n\}$ are sequences in $(0,1)$ and proving the strong convergence to a solution-point of the VI (1.4).

Theorem 1.2. *Assume that the following hold*

(P0) $\mathrm{Fix}(T) \cap \mathrm{int}(C) \neq \emptyset$;
(P1) $\alpha_n = o(\lambda_n)$ *and* $\sum_n \alpha_n = \infty$;
(P2) $\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n \lambda_n} = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_n \lambda_{n-1} \alpha_n} = 0$;
(P3) *there exist two constants* θ *and* k *such that*

$$\|x - Tx\| \geq k \cdot \mathrm{dist}(x, \mathrm{Fix}(T))^{\theta}, \quad \forall x \in C;$$

(P4) $\lambda_n^{1+\frac{1}{\theta}} = o(\alpha_n)$.

Suppose that $\{x_n\}$ *is bounded. Then* $\{x_n\}$ *strongly converges to a solution of the VI* (1.4).

A different approach was introduced by Yao, Liou and Marino [28]. That is, their two-step iterative algorithm generates a sequence $\{x_n\}$ by the explicit scheme

$$\begin{cases} y_n = \beta_n S x_n + (1-\beta_n) x_n, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T y_n, \quad \forall n \geq 1. \end{cases} \tag{1.6}$$

Theorem 1.3. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* S *and* T *be two nonexpansive mappings on* C *into itself. Let* $f : C \to C$ *be a* ρ*-contraction and* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *two real sequences in* $(0,1)$. *Assume that the sequence* $\{x_n\}$ *generated by scheme* (1.6) *is bounded and*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \frac{1}{\alpha_n} |\frac{1}{\beta_n} - \frac{1}{\beta_{n-1}}| = 0$, $\lim_{n\to\infty} \frac{1}{\beta_n} |1 - \frac{\alpha_{n-1}}{\alpha_n}| = 0$;
(iii) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$, $\lim_{n\to\infty} \frac{\beta_n^2}{\alpha_n} = 0$;
(iv) $\mathrm{Fix}(T) \cap \mathrm{int}(C) \neq \emptyset$;
(v) *there exists a constant* $k > 0$ *such that* $\|x - Tx\| \geq k \cdot \mathrm{dist}(x, \mathrm{Fix}(T))$ *for each* $x \in C$, *where* $\mathrm{dist}(x, \mathrm{Fix}(T)) = \inf_{y \in \mathrm{Fix}(T)} \|x - y\|$.

Then the sequence $\{x_n\}$ *strongly converges to* $\tilde{x} = P_{\Omega} f(\tilde{x})$ *which solves the VI* (1.4).

In addition, if $C = \mathrm{Fix}(T)$ and $F(x,y) := \langle (I - S)x, y - x \rangle$, the VI (1.4) can be reformulated as the problem of finding $x^* \in C$ such that

$$F(x^*, y) \geq 0, \quad \forall y \in C, \tag{1.7}$$

i.e., as an equilibrium problem. In [2, 19], it is shown that formulation (1.7) covers monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems and certain fixed point problems (see [9]).

It is worth to remark that, in the case of the VI (1.4), the induced bifunction $F(x,y) := \langle (I-S)x, y-x\rangle$ satisfies the following conditions:

(f1) $F(x,x) = 0$ for all $x \in C$;

(f2) $F(x,y) + F(y,x) \le 0$ for all $(x,y) \in C \times C$ (i.e., F is monotone);

(f3) for each $x, y, z \in C$

$$\limsup_{t\to 0} F(tz + (1-t)x, y) \le F(x,y);$$

i.e., F is hemicontinuous in the first variable.

(f4) the function $y \mapsto F(x,y)$ is convex and lower semicontinuous for each $x \in C$.

Recently, many authors have generalized the classical equilibrium problem introduced by Combettes and Hirstoaga [8] by introducing "perturbations" to the function F; for example, Moudafi [16] studied the equilibrium problem of finding $x^* \in C$ such that

$$F(x^*, y) + \langle Ax^*, y - x^*\rangle \ge 0, \quad \forall y \in C,$$

where A is an α-inverse strongly monotone operator. In [4, 20, 21], the authors studied the mixed problem of finding $x^* \in C$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \ge 0, \quad \forall y \in C$$

with φ being an opportune mapping.

In this paper, we study the equilibrium problem (EP) of finding $x^* \in C$ such that

$$F(x^*, y) + h(x^*, y) \ge 0, \quad \forall y \in C, \tag{1.8}$$

that includes all previous equilibrium problems as special cases.

On the other hand, for a long time, many authors were interested in the construction of iterative algorithms that weakly or strongly converge to a common fixed point of a family of nonexpansive mappings (see e.g., [1, 3, 11]). In [25], Xu proved that the sequence generated by

$$x_{n+1} = (I - \epsilon_{n+1}A)T_{n+1}x_n + \epsilon_{n+1}u$$

where $T_n = T_{n \bmod N}$, strongly converges to a solution of a quadratic minimization problem under the assumption

$$\mathrm{Fix}(T_1T_2\cdots T_N) = \mathrm{Fix}(T_NT_1\cdots T_{N-1}) = \mathrm{Fix}(T_2T_3\cdots T_1).$$

In [27], Yao studied the viscosity approximation of a common fixed point of the family of mappings under the lack of the last hypothesis. In [7], Colao, Marino and Xu used a different approach to obtain the convergence of a more general scheme that involves an equilibrium problem.

Very recently, Marino, Muglia and Yao [14] introduced a multi-step iterative scheme

$$\begin{cases} F(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1}S_1u_n + (1-\beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_iu_n + (1-\beta_{n,i})y_{n,i-1}, \quad i = 2, \dots, N, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_{n,N}, \end{cases} \tag{1.9}$$

with $f : C \to C$ a ρ-contraction and $\{\alpha_n\}, \{\beta_{n,i}\}_{i=1}^N \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, that generalizes the two-step iterative scheme in [28] for two nonexpansive mappings to a finite family of nonexpansive mappings $T, S_i : C \to C,\ i = 1, \ldots, N$, and proved that the iterative scheme (1.9) converges strongly to a common fixed point of the mappings that is also an equilibrium point of the EP (1.8).

Combining the two-step iterative scheme in [10] and the multi-step iterative scheme in [14] by virtue of the viscosity approximation method and the Mann iterative method, we introduce and consider a composite viscosity iterative scheme for finding a common element of the solution set $\mathrm{VI}(C, A)$ of the variational inequality (1.1), the solution set $\mathrm{EP}(F, h)$ of the equilibrium problem (1.8) and the common fixed point set of a finite family of nonexpansive mappings $T, S_i : C \to C,\ i = 1, \ldots, N$, in the setting of infinite-dimensional Hilbert space.

In this paper, we study the composite viscosity iterative scheme that generalizes the two-step iterative scheme in [28] for two nonexpansive mappings, the two-step iterative scheme in [10] for the VI (1.1) and a nonexpansive mapping, and the multi-step iterative scheme in [14] for a finite family of nonexpansive mappings, to the VI (1.1) and a finite family of nonexpansive mappings. It is proved that this iterative scheme converges strongly to a common fixed point of the mappings $T, S_i : C \to C,\ i = 1, \ldots, N$, that is also an equilibrium point of the EP (1.8) and a solution of the VI (1.1).

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$ and $\omega_s(x_n)$ to denote the strong ω-limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}$$

and

$$\omega_s(x_n) := \{x \in H : x_{n_i} \to x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Recall that the metric (or nearest point) projection from H onto K is the mapping $P_K : H \to K$ which assigns to each point $x \in H$ the unique point $P_K x \in K$ satisfying the property

$$\|x - P_K x\| = \inf_{y \in K} \|x - y\| =: d(x, K).$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *For given $x \in H$ and $z \in K$:*

(i) $z = P_K x \iff \langle x - z, y - z \rangle \leq 0,\ \forall y \in K;$

(ii) $z = P_K x \iff \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in K;$

(iii) $\langle P_K x - P_K y, x - y \rangle \geq \|P_K x - P_K y\|^2,\ \forall y \in H$, *which hence implies that P_K is nonexpansive and monotone.*

The following lemma appears implicitly in the paper of Reineermann [22].

Lemma 2.2 ([22]). *Let H be a real Hilbert space. Then, for all $x, y \in H$ and $\lambda \in [0,1]$,*

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

In the sequel, we will indicate with $\mathrm{EP}(F,h)$ the set of solutions of (1.8).

Lemma 2.3 ([6]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbf{R}$ be a bi-function such that*

- (f1) *$F(x,x) = 0$ for all $x \in C$;*
- (f2) *F is monotone and upper hemicontinuous in the first variable;*
- (f3) *F is lower semicontinuous and convex in the second variable.*

Let $h : C \times C \to \mathbf{R}$ be a bi-function such that

- (h1) *$h(x,x) = 0$ for all $x \in C$;*
- (h2) *h is monotone and weakly upper semicontinuous in the first variable;*
- (h3) *h is convex in the second variable.*

Moreover, let us suppose that

(H) for fixed $r > 0$ and $x \in C$, there exists a bounded $K \subset C$ and $\hat{x} \in K$ such that for all $z \in C \setminus K$, $-F(\hat{x}, z) + h(z, \hat{x}) + \frac{1}{r}\langle \hat{x} - z, z - x\rangle < 0$.
For $r > 0$ and $x \in H$, let $T_r : H \to 2^C$ be a mapping defined by

$$T_r x = \{z \in C : F(z,y) + h(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\} \tag{2.1}$$

called the resolvent of F and h. Then

1. *$T_r x \neq \emptyset$;*
2. *$T_r x$ is a singleton;*
3. *T_r is firmly nonexpansive;*
4. *$\mathrm{EP}(F,h) = \mathrm{Fix}(T_r)$ and it is closed and convex.*

Lemma 2.4 ([6]). *Let us suppose that (f1)-(f3), (h1)-(h3) and (H) hold. Let $x, y \in H$, $r_1, r_2 > 0$. Then*

$$\|T_{r_2}y - T_{r_1}x\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}y - y\|.$$

Lemma 2.5 ([14]). *Suppose that the hypotheses of Lemma 2.3 are satisfied. Let $\{r_n\}$ be a sequence in $(0,\infty)$ with $\liminf_{n\to\infty} r_n > 0$. Suppose that $\{x_n\}$ is a bounded sequence. Then the following statements are equivalent and true:*

- (a) *if $\|x_n - T_{r_n}x_n\| \to 0$ as $n \to \infty$, the weak cluster points of $\{x_n\}$ satisfies the problem*
$$F(x,y) + h(x,y) \geq 0, \quad \forall y \in C,$$
i.e., $\omega_w(x_n) \subseteq \mathrm{EP}(F,h)$.
- (b) *the demiclosedness principle holds in the sense that, if $x_n \rightharpoonup x^*$ and $\|x_n - T_{r_n}x_n\| \to 0$ as $n \to \infty$, then $(I - T_{r_k})x^* = 0$ for all $k \geq 1$.*

Lemma 2.6 ([24]). *Assume that $\{a_n\}$ is a sequence of nonnegative numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbf{R}$ such that

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) *either* $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

The following result is an immediate consequence of inner product.

Lemma 2.7. *In a real Hilbert space H, there holds the following inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

3. Main results

Let us consider the following composite viscosity iterative scheme

$$\text{(3.1)} \qquad \begin{cases} F(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1}) u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, \quad i = 2, \dots, N, \\ y_n = \alpha_n f(y_{n,N}) + (1 - \alpha_n) T P_C(y_{n,N} - \lambda_n A y_{n,N}), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n T P_C(y_n - \lambda_n A y_n), \quad \forall n \geq 1, \end{cases}$$

where

the mapping $f : C \to C$ is a ρ-contraction;
$A : C \to H$ is an α-inverse-strongly monotone mapping;
$S_i, T : C \to C$ are nonexpansive mappings for each $i = 1, \dots, N$;
$F, h : C \times C \to \mathbf{R}$ are two bi-functions satisfying the hypotheses of Lemma 2.3;
$\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ with $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$;
$\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
$\{\beta_{n,i}\}$ is a sequence in $(0, 1)$ for each $i = 1, \dots, N$;
$\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$.

Lemma 3.1. *Let us suppose that $\Omega = \mathrm{Fix}(T) \cap (\cap_i \mathrm{Fix}(S_i)) \cap \mathrm{EP}(F, h) \cap \mathrm{VI}(C, A) \neq \emptyset$. Then the sequences $\{x_n\}, \{y_n\}, \{y_{n,i}\}$ for all i, $\{u_n\}$ are bounded.*

Proof. Let us observe, first of all that, if $p \in \Omega$, then

$$\|y_{n,1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

For all from $i = 2$ to $i = N$, by induction, one proves that

$$\|y_{n,i} - p\| \leq \beta_{n,i} \|\|u_n - p\| + (1 - \beta_{n,i})\|y_{n,i-1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

Thus we obtain that for every $i = 1, \dots, N$,

$$\text{(3.2)} \qquad \|y_{n,i} - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

Let $z_n = P_C(y_{n,N} - \lambda_n A y_{n,N})$ and $w_n = P_C(y_n - \lambda_n A y_n)$ for every $n \geq 1$. Since $I - \lambda_n A$ is nonexpansive and $p = P_C(p - \lambda_n A p)$ (due to (2.2)), we have

$$\begin{aligned} \|z_n - p\| &= \|P_C(y_{n,N} - \lambda_n A y_{n,N}) - P_C(p - \lambda_n A p)\| \\ &\leq \|(y_{n,N} - \lambda_n A y_{n,N}) - (p - \lambda_n A p)\| \\ &\leq \|y_{n,N} - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \end{aligned}$$

Moreover,

$$\begin{aligned}\|y_n - p\| &= \|\alpha_n(f(y_{n,N}) - p) + (1-\alpha_n)(Tz_n - p)\| \\ &\le \alpha_n\|f(y_{n,N}) - p\| + (1-\alpha_n)\|z_n - p\| \\ &\le \alpha_n\|f(y_{n,N}) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\le \alpha_n\rho\|y_{n,N} - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &\le \alpha_n\rho\|x_n - p\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\ &= (1-(1-\rho)\alpha_n)\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &= (1-(1-\rho)\alpha_n)\|x_n - p\| + (1-\rho)\alpha_n\tfrac{\|f(p)-p\|}{1-\rho} \\ &\le \max\left\{\|x_n - p\|, \tfrac{\|f(p)-p\|}{1-\rho}\right\},\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\beta_n)(y_n - p) + \beta_n(Tw_n - p)\| \\ &\le (1-\beta_n)\|y_n - p\| + \beta_n\|w_n - p\| \\ &\le (1-\beta_n)\|y_n - p\| + \beta_n\|y_n - p\| \\ &\le \max\left\{\|x_n - p\|, \tfrac{\|f(p)-p\|}{1-\rho}\right\}.\end{aligned}$$

By induction, we get

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\rho}\right\}, \quad \forall n \ge 1.$$

This implies that $\{x_n\}$ is bounded and so are $\{Ay_{n,N}\}, \{Ay_n\}, \{z_n\}, \{w_n\}, \{u_n\}$, $\{y_n\}, \{y_{n,i}\}$ for each $i = 1, \ldots, N$. Since $\|Tz_n - p\| \le \|x_n - p\|$ and $\|Tw_n - p\| \le \|y_n - p\|$, $\{Tz_n\}$ and $\{Tw_n\}$ are also bounded. □

Lemma 3.2. *Let us suppose that $\Omega \ne \emptyset$. Moreover, let us suppose that the following hold:*

(H1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(H2) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$;
(H3) $\sum_{n=1}^{\infty} |\beta_{n,i} - \beta_{n-1,i}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\alpha_n} = 0$ *for each* $i = 1, \ldots, N$;
(H4) $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|r_n - r_{n-1}|}{\alpha_n} = 0$;
(H5) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n} = 0$;
(H6) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n} = 0$.

Then $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, *i.e.,* $\{x_n\}$ *is asymptotically regular.*

Proof. From (3.1), we have

$$\begin{cases} y_n = \alpha_n f(y_{n,N}) + (1-\alpha_n)Tz_n, \\ y_{n-1} = \alpha_{n-1} f(y_{n-1,N}) + (1-\alpha_{n-1})Tz_{n-1}, \quad \forall n \ge 1. \end{cases}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} = (1-\alpha_n)(Tz_n - Tz_{n-1}) &+ (\alpha_n - \alpha_{n-1})(f(y_{n-1,N}) - Tz_{n-1}) \\ &+ \alpha_n(f(y_{n,N}) - f(y_{n-1,N})). \end{aligned}$$

Since

$$\begin{aligned}\|z_n - z_{n-1}\| &\leq \|(y_{n,N} - \lambda_n A y_{n,N}) - (y_{n-1,N} - \lambda_{n-1} A y_{n-1,N})\| \\ &\leq \|(y_{n,N} - \lambda_n A y_{n,N}) - (y_{n-1,N} - \lambda_n A y_{n-1,N})\| \\ &\quad + |\lambda_{n-1} - \lambda_n| \|A y_{n-1,N}\| \\ &\leq \|y_{n,N} - y_{n-1,N}\| + |\lambda_{n-1} - \lambda_n| \|A y_{n-1,N}\|,\end{aligned}$$

we have

(3.3)

$$\begin{aligned}\|y_n - y_{n-1}\| &\leq (1-\alpha_n)\|z_n - z_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|f(y_{n-1,N}) - T z_{n-1}\| \\ &\quad + \alpha_n \rho \|y_{n,N} - y_{n-1,N}\| \\ &\leq (1-\alpha_n)(\|y_{n,N} - y_{n-1,N}\| + |\lambda_{n-1} - \lambda_n| \|A y_{n-1,N}\|) \\ &\quad + |\alpha_n - \alpha_{n-1}| \|f(y_{n-1,N}) - T z_{n-1}\| + \alpha_n \rho \|y_{n,N} - y_{n-1,N}\| \\ &\leq (1-(1-\rho)\alpha_n)\|y_{n,N} - y_{n-1,N}\| + M_1(|\lambda_{n-1} - \lambda_n| + |\alpha_n - \alpha_{n-1}|),\end{aligned}$$

where $\|A y_{n,N}\| + \|f(y_{n,N}) - T z_n\| \leq M_1, \forall n \geq 1$ for some $M_1 \geq 0$.

Furthermore, from (3.1) we have

$$\begin{cases} x_{n+1} = (1-\beta_n) y_n + \beta_n T w_n, \\ x_n = (1-\beta_{n-1}) y_{n-1} + \beta_{n-1} T w_{n-1}. \end{cases}$$

Also, simple calculations show that

$$x_{n+1} - x_n = (1-\beta_n)(y_n - y_{n-1}) + \beta_n (T w_n - T w_{n-1}) + (\beta_n - \beta_{n-1})(T w_{n-1} - y_{n-1}).$$

Since

$$\begin{aligned}\|w_n - w_{n-1}\| &\leq \|(y_n - \lambda_n A y_n) - (y_{n-1} - \lambda_{n-1} A y_{n-1})\| \\ &\leq \|(y_n - \lambda_n A y_n) - (y_{n-1} - \lambda_n A y_{n-1})\| + |\lambda_{n-1} - \lambda_n| \|A y_{n-1}\| \\ &\leq \|y_n - y_{n-1}\| + |\lambda_{n-1} - \lambda_n| \|A y_{n-1}\|,\end{aligned}$$

it follows that

(3.4)

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1-\beta_n)\|y_n - y_{n-1}\| + \beta_n \|w_n - w_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|T w_{n-1} - y_{n-1}\| \\ &\leq (1-\beta_n)\|y_n - y_{n-1}\| + \beta_n (\|y_n - y_{n-1}\| \\ &\quad + |\lambda_{n-1} - \lambda_n| \|A y_{n-1}\|) + |\beta_n - \beta_{n-1}| \|T w_{n-1} - y_{n-1}\| \\ &\leq \|y_n - y_{n-1}\| + |\lambda_{n-1} - \lambda_n| \|A y_{n-1}\| + |\beta_n - \beta_{n-1}| \|T w_{n-1} - y_{n-1}\|.\end{aligned}$$

This together with (3.3) implies that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq (1-(1-\rho)\alpha_n)\|y_{n,N} - y_{n-1,N}\| + M_1(|\lambda_{n-1} - \lambda_n| \\ &\quad + |\alpha_n - \alpha_{n-1}|) + |\lambda_{n-1} - \lambda_n| \|A y_{n-1}\| \\ &\quad + |\beta_n - \beta_{n-1}| \|T w_{n-1} - y_{n-1}\| \\ &\leq (1-(1-\rho)\alpha_n)\|y_{n,N} - y_{n-1,N}\| + M_2(|\lambda_{n-1} - \lambda_n| + |\alpha_n - \alpha_{n-1}|) \\ &\quad + M_2(|\lambda_{n-1} - \lambda_n| + |\beta_n - \beta_{n-1}|) \\ &= (1-(1-\rho)\alpha_n)\|y_{n,N} - y_{n-1,N}\| + M_2(2|\lambda_{n-1} - \lambda_n| \\ &\quad + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|).\end{aligned}$$

where $\|A y_n\| + \|T w_n - y_n\| + M_1 \leq M_2, \forall n \geq 1$ for some $M_2 \geq 0$.

Meantime, by the definition of $y_{n,i}$ one obtains that, for all $i = N, \dots, 2$

$$\begin{aligned}\|y_{n,i} - y_{n-1,i}\| &\leq \beta_{n,i}\|u_n - u_{n-1}\| + \|S_i u_{n-1} - y_{n-1,i-1}\| |\beta_{n,i} - \beta_{n-1,i}| \\ &\quad + (1-\beta_{n,i})\|y_{n,i-1} - y_{n-1,i-1}\|.\end{aligned} \tag{3.5}$$

In the case $i = 1$, we have

$$\begin{aligned} \|y_{n,1} - y_{n-1,1}\| &\le \beta_{n,1}\|u_n - u_{n-1}\| + \|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| \\ &\quad +(1-\beta_{n,1})\|u_n - u_{n-1}\| \\ &= \|u_n - u_{n-1}\| + \|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|. \end{aligned} \tag{3.6}$$

Substituting (3.7) in all (3.6)-type one obtains for $i = 2, \ldots, N$

$$\begin{aligned} \|y_{n,i} - y_{n-1,i}\| &\le \|u_n - u_{n-1}\| + \textstyle\sum_{k=2}^{i} \|S_k u_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad +\|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|. \end{aligned} \tag{3.7}$$

This together with (3.5) implies that

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (1-(1-\rho)\alpha_n)[\|u_n - u_{n-1}\| \\ &\quad + \textstyle\sum_{k=2}^{N} \|S_k u_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad +\|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|] \\ &\quad +M_2(2|\lambda_{n-1} - \lambda_n| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\le (1-(1-\rho)\alpha_n)\|u_n - u_{n-1}\| \\ &\quad + \textstyle\sum_{k=2}^{N} \|S_k u_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad +\|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| \\ &\quad +M_2(2|\lambda_{n-1} - \lambda_n| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|). \end{aligned} \tag{3.8}$$

By Lemma 2.4, we know that

$$\|u_n - u_{n-1}\| \le \|x_n - x_{n-1}\| + L|1 - \frac{r_{n-1}}{r_n}| \tag{3.9}$$

where $L = \sup_{n\ge 1} \|u_n - x_n\|$. So, substituting (3.9) in (3.8) we obtain

$$\begin{aligned} \|x_{n+1} - x_n\| &\le (1-(1-\rho)\alpha_n)(\|x_n - x_{n-1}\| + L|1 - \tfrac{r_{n-1}}{r_n}|) \\ &\quad + \textstyle\sum_{k=2}^{N} \|S_k u_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad +\|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| \\ &\quad +M_2(2|\lambda_{n-1} - \lambda_n| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\le (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| + L\tfrac{|r_n - r_{n-1}|}{r_n} \\ &\quad + \textstyle\sum_{k=2}^{N} \|S_k u_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad +\|S_1 u_{n-1} - u_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| \\ &\quad +M_2(2|\lambda_{n-1} - \lambda_n| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|) \\ &\le (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| + M[\tfrac{|r_n - r_{n-1}|}{r_n} \\ &\quad + \textstyle\sum_{k=2}^{N} |\beta_{n,k} - \beta_{n-1,k}| \\ &\quad +|\beta_{n,1} - \beta_{n-1,1}| + |\lambda_{n-1} - \lambda_n| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|] \\ &\le (1-(1-\rho)\alpha_n)\|x_n - x_{n-1}\| + M[\tfrac{|r_n - r_{n-1}|}{b} \\ &\quad + \textstyle\sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\lambda_{n-1} - \lambda_n| \\ &\quad +|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}|], \end{aligned} \tag{3.10}$$

where $b > 0$ is a minorant for $\{r_n\}$ and $L + 2M_2 + \sum_{k=2}^{N} \|S_k u_n - y_{n,k-1}\| + \|S_1 u_n - u_n\| \le M, \forall n \ge 1$ for some $M \ge 0$. By hypotheses (H1)-(H6) and Lemma 2.6, we obtain the claim. □

Lemma 3.3. *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that $\{x_n\}$ is asymptotically regular. Then $\|x_n - y_n\| \to 0$ and $\|x_n - u_n\| = \|x_n - T_{r_n} x_n\| \to 0$ as $n \to \infty$.*

Proof. We recall that, by the firm nonexpansivity of T_{r_n}, a standard calculation (see [7]) shows that if $p \in \mathrm{EP}(F, h)$

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

Let $q \in \Omega$. Then by Lemma 2.2, we have from (3.2)

$$\begin{aligned}
\|y_n - q\|^2 &= \|\alpha_n(f(y_{n,N}) - q) + (1 - \alpha_n)(Tz_n - q)\|^2 \\
&\leq \alpha_n\|f(y_{n,N}) - q\|^2 + (1 - \alpha_n)\|Tz_n - q\|^2 \\
&\leq \alpha_n\|f(y_{n,N}) - q\|^2 + \|z_n - q\|^2 \\
&\leq \alpha_n\|f(y_{n,N}) - q\|^2 + \|y_{n,N} - q\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_{n,N} - Aq\|^2 \\
&\leq \alpha_n\|f(y_{n,N}) - q\|^2 + \|u_n - q\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_{n,N} - Aq\|^2 \\
&\leq \alpha_n\|f(y_{n,N}) - q\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2 \\
&\quad +\lambda_n(\lambda_n - 2\alpha)\|Ay_{n,N} - Aq\|^2,
\end{aligned}$$

and hence

(3.11)

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \|(1 - \beta_n)(y_n - q) + \beta_n(Tw_n - q)\|^2 \\
&= (1 - \beta_n)\|y_n - q\|^2 + \beta_n\|Tw_n - q\|^2 - \beta_n(1 - \beta_n)\|y_n - Tw_n\|^2 \\
&\leq (1 - \beta_n)\|y_n - q\|^2 + \beta_n\|w_n - q\|^2 - \beta_n(1 - \beta_n)\|y_n - Tw_n\|^2 \\
&\leq (1 - \beta_n)\|y_n - q\|^2 + \beta_n[\|y_n - q\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ay_n - Aq\|^2] \\
&\quad -\beta_n(1 - \beta_n)\|y_n - Tw_n\|^2 \\
&= \|y_n - q\|^2 + \beta_n\lambda_n(\lambda_n - 2\alpha)\|Ay_n - Aq\|^2 - \beta_n(1 - \beta_n)\|y_n - Tw_n\|^2 \\
&\leq \alpha_n\|f(y_{n,N}) - q\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2 \\
&\quad +\lambda_n(\lambda_n - 2\alpha)\|Ay_{n,N} - Aq\|^2 + \beta_n\lambda_n(\lambda_n - 2\alpha)\|Ay_n - Aq\|^2 \\
&\quad -\beta_n(1 - \beta_n)\|y_n - Tw_n\|^2.
\end{aligned}$$

So, we deduce that

$$\begin{aligned}
&\|x_n - u_n\|^2 + \lambda_n(2\alpha - \lambda_n)\|Ay_{n,N} - Aq\|^2 + \beta_n\lambda_n(2\alpha - \lambda_n)\|Ay_n - Aq\|^2 \\
&+\beta_n(1 - \beta_n)\|y_n - Tw_n\|^2 \\
&\leq \alpha_n\|f(y_{n,N}) - q\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \\
&= \alpha_n\|f(y_{n,N}) - q\|^2 + (\|x_n - q\| + \|x_{n+1} - q\|)(\|x_n - q\| - \|x_{n+1} - q\|) \\
&\leq \alpha_n\|f(y_{n,N}) - q\|^2 + (\|x_n - q\| + \|x_{n+1} - q\|)\|x_n - x_{n+1}\|.
\end{aligned}$$

By Lemmas 3.1 and 3.2 we know that both $\{x_n\}$ and $\{y_{n,N}\}$ are bounded, and that $\{x_n\}$ is asymptotically regular. Therefore, utilizing (H1) we obtain that

(3.12)

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|Ay_{n,N} - Aq\| = \lim_{n\to\infty} \|Ay_n - Aq\| = \lim_{n\to\infty} \|y_n - Tw_n\| = 0.$$

We note that $\|x_{n+1} - y_n\| = \beta_n\|Tw_n - y_n\| \to 0$ as $n \to \infty$. This together with $\|x_{n+1} - x_n\| \to 0$, implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0.$$

□

Remark 3.4. By the last lemma we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$, i.e., the sets of strong/weak cluster points of $\{x_n\}$ and $\{u_n\}$ coincide.

Of course, if $\beta_{n,i} \to \beta_n \neq 0$, as $n \to \infty$, for all index i, the assumptions of Lemma 3.2 are enough to assure that

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,i}} = 0, \quad \forall i \in \{1, \dots, N\}.$$

In the next lemma, we examine the case in which at least one sequence $\{\beta_{n,k_0}\}$ is a null sequence.

Lemma 3.5. *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that (H1) holds. Moreover, for an index $k_0 \in \{1, \dots, N\}$, $\lim_{n\to\infty} \beta_{n,k_0} = 0$ and the following hold:*

(H7) *for all i,*

$$\lim_{n\to\infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|\beta_n - \beta_{n-1}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|r_n - r_{n-1}|}{\alpha_n \beta_{n,k_0}} = \lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n \beta_{n,k_0}} = 0;$$

(H8) *there exists a constant $\kappa > 0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}}| < \kappa$ for all $n > 1$.*

Then

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} = 0.$$

Proof. We start by (3.10). Dividing both the terms by β_{n,k_0} we have

$$\frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}} \leq [1 - \alpha_n(1-\rho)]\frac{\|x_n - x_{n-1}\|}{\beta_{n,k_0}} + M\Big[\frac{|r_n - r_{n-1}|}{b\beta_{n,k_0}} + \frac{\sum_{k=1}^N |\beta_{n,k} - \beta_{n-1,k}|}{\beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\beta_{n,k_0}}\Big].$$

So, by (H8) we have

$$\begin{aligned}
\frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}} &\leq [1 - \alpha_n(1-\rho)]\frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + [1 - \alpha_n(1-\rho)]\|x_n - x_{n-1}\|\Big|\frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}}\Big| \\
&\quad + M\Big[\frac{|r_n - r_{n-1}|}{b\beta_{n,k_0}} + \frac{\sum_{k=1}^N |\beta_{n,k} - \beta_{n-1,k}|}{\beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\beta_{n,k_0}}\Big] \\
&\leq [1 - \alpha_n(1-\rho)]\frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + \|x_n - x_{n-1}\|\Big|\frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}}\Big| \\
&\quad + M\Big[\frac{|r_n - r_{n-1}|}{b\beta_{n,k_0}} + \frac{\sum_{k=1}^N |\beta_{n,k} - \beta_{n-1,k}|}{\beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\beta_{n,k_0}}\Big] \\
&\leq [1 - \alpha_n(1-\rho)]\frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + \alpha_n\kappa\|x_n - x_{n-1}\| \\
&\quad + M\Big[\frac{|r_n - r_{n-1}|}{b\beta_{n,k_0}} + \frac{\sum_{k=1}^N |\beta_{n,k} - \beta_{n-1,k}|}{\beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\beta_{n,k_0}}\Big] \\
&= [1 - \alpha_n(1-\rho)]\frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + \alpha_n(1-\rho)\cdot\frac{1}{1-\rho}\Big\{\kappa\|x_n - x_{n-1}\| \\
&\quad + M\Big[\frac{|r_n - r_{n-1}|}{b\alpha_n\beta_{n,k_0}} + \frac{\sum_{k=1}^N |\beta_{n,k} - \beta_{n-1,k}|}{\alpha_n\beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\alpha_n\beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n\beta_{n,k_0}}\Big]\Big\}.
\end{aligned}$$

Therefore, utilizing Lemma 2.6, from (H1), (H7) and the asymptotical regularity of $\{x_n\}$ (due to Lemma 3.2), we deduce that

$$\lim_{n\to\infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} = 0.$$

□

Lemma 3.6. *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ for each $i = 1, \ldots, N$. Moreover, suppose that (H1)-(H6) are satisfied. Then, for all i, $\|S_i u_n - u_n\| \to 0$ as $n \to \infty$.*

Proof. First of all, by Lemma 3.2 we know that $\{x_n\}$ is asymptotically regular. Let us show that for each $i \in \{1, \ldots, N\}$ one has $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$. Let $p \in \Omega$. When $i = N$, by Lemma 2.2 we have

$$\begin{aligned}
\|y_n - p\|^2 &\leq \alpha_n \|f(y_{n,N}) - p\|^2 + (1-\alpha_n)\|Tz_n - p\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|z_n - p\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|y_{n,N} - p\|^2 + \lambda_n(\lambda_n - 2\alpha)\|y_{n,N} - p\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|y_{n,N} - p\|^2 \\
&= \alpha_n \|f(y_{n,N}) - p\|^2 + \beta_{n,N}\|S_N u_n - p\|^2 + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2 \\
&\quad -\beta_{n,N}(1-\beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|u_n - p\|^2 - \beta_{n,N}(1-\beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|x_n - p\|^2 - \beta_{n,N}(1-\beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2.
\end{aligned}$$

So we have

$$\begin{aligned}
\beta_{n,N}(1-\beta_{n,N})\|S_N u_n - y_{n,N-1}\|^2 &\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\
&= \alpha_n \|f(y_{n,N}) - p\|^2 \\
&\quad +(\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_{n,N} \leq \limsup_{n\to\infty} \beta_{n,N} < 1$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (due to Lemma 3.3), it is known that $\{\|S_N u_n - y_{n,N-1}\|\}$ is a null sequence.

Let $i \in \{1, \ldots, N-1\}$. Then one has

$$\begin{aligned}
\|y_n - p\|^2 &\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|y_{n,N} - p\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \beta_{n,N}\|S_N u_n - p\|^2 + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \beta_{n,N}\|x_n - p\|^2 + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \beta_{n,N}\|x_n - p\|^2 \\
&\quad +(1-\beta_{n,N})[\beta_{n,N-1}\|S_{N-1} u_n - p\|^2 + (1-\beta_{n,N-1})\|y_{n,N-2} - p\|^2] \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + (\beta_{n,N} + (1-\beta_{n,N})\beta_{n,N-1})\|x_n - p\|^2 \\
&\quad + \textstyle\prod_{k=N-1}^{N}(1-\beta_{n,k})\|y_{n,N-2} - p\|^2,
\end{aligned}$$

and so, after $(N-i+1)$-iterations,

(3.13)

$$\begin{aligned}
\|y_n - p\|^2 &\leq \alpha_n \|f(y_{n,N}) - p\|^2 + (\beta_{n,N} + \textstyle\sum_{j=i+2}^{N}(\prod_{l=j}^{N}(1-\beta_{n,l}))\beta_{n,j-1})\|x_n - p\|^2 \\
&\quad + \textstyle\prod_{k=i+1}^{N}(1-\beta_{n,k})\|y_{n,i} - p\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + (\beta_{n,N} + \textstyle\sum_{j=i+2}^{N}(\prod_{l=j}^{N}(1-\beta_{n,l}))\beta_{n,j-1}) \\
&\quad \times\|x_n - p\|^2 + \textstyle\prod_{k=i+1}^{N}(1-\beta_{n,k}) \times [\beta_{n,i}\|S_i u_n - p\|^2 \\
&\quad +(1-\beta_{n,i})\|y_{n,i-1} - p\|^2 - \beta_{n,i}(1-\beta_{n,i})\|S_i u_n - y_{n,i-1}\|^2] \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|x_n - p\|^2 - \beta_{n,i}\textstyle\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i u_n - y_{n,i-1}\|^2.
\end{aligned}$$

Again we obtain that

$$\begin{aligned}
\beta_{n,i}\textstyle\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i u_n - y_{n,i-1}\|^2 &\leq \alpha_n \|f(y_{n,N}) - p\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\leq \alpha_n \|f(y_{n,N}) - p\|^2 \\
&\quad +(\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_{n,i} \le \limsup_{n\to\infty} \beta_{n,i} < 1$ for each $i = 1, \dots, N-1$, and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (due to Lemma 3.3), it is known that

$$\lim_{n\to\infty} \|S_i u_n - y_{n,i-1}\| = 0.$$

Obviously for $i = 1$, we have $\|S_1 u_n - u_n\| \to 0$.

To conclude, we have that

$$\|S_2 u_n - u_n\| \le \|S_2 u_n - y_{n,1}\| + \|y_{n,1} - u_n\| = \|S_2 u_n - y_{n,1}\| + \beta_{n,1}\|S_1 u_n - u_n\|$$

from which $\|S_2 u_n - u_n\| \to 0$. Thus by induction $\|S_i u_n - u_n\| \to 0$ for all $i = 2, \dots, N$ since it is enough to observe that

$$\begin{aligned} \|S_i u_n - u_n\| &\le \|S_i u_n - y_{n,i-1}\| + \|y_{n,i-1} - S_{i-1} u_n\| + \|S_{i-1} u_n - u_n\| \\ &\le \|S_i u_n - y_{n,i-1}\| + (1 - \beta_{n,i-1})\|S_{i-1} u_n - y_{n,i-2}\| + \|S_{i-1} u_n - u_n\|. \end{aligned}$$

□

Remark 3.7. As an example, we consider $N = 2$ and the sequences:

(a) $\lambda_n = \alpha - \frac{1}{n}, \quad \forall n > \frac{1}{\alpha}$;
(b) $\alpha_n = \frac{1}{\sqrt{n}}, \quad r_n = 2 - \frac{1}{n}, \quad \forall n > 1$;
(c) $\beta_n = \beta_{n,1} = \frac{1}{2} - \frac{1}{n}, \quad \beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}, \quad \forall n > 2$.

Then they satisfy the hypotheses of Lemma 3.6.

Lemma 3.8. *Let us suppose that $\Omega \ne \emptyset$ and $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose there exists $k \in \{1, \dots, N\}$ such that $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1, \dots, N\}$ the largest index such that $\beta_{n,k_0} \to 0$ as $n \to \infty$. Suppose that*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;
(ii) *if* $i \le k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;
(iii) *if* $\beta_{n,i} \to \beta_i \ne 0$ *then* β_i *lies in* $(0, 1)$.

Moreover, suppose that (H1), (H7) and (H8) hold. Then, for all i, $\|S_i u_n - u_n\| \to 0$ as $n \to \infty$.

Proof. First of all we note that if (H7) holds than also (H2)-(H6) are satisfied. So $\{x_n\}$ is asymptotically regular.

Let k_0 be as in the hypotheses. As in Lemma 3.6, for every index $i \in \{1, \dots, N\}$ such that $\beta_{n,i} \to \beta_i \ne 0$ (which leads to $0 < \liminf_{n\to\infty} \beta_{n,i} \le \limsup_{n\to\infty} \beta_{n,i} < 1$), one has $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$.

For all the other indexes $i \le k_0$, we can prove that $\|S_i u_n - y_{n,i-1}\| \to 0$ as $n \to \infty$ in a similar manner. By the relation (due to (3.11) and (3.13))

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \|y_n - p\|^2 \\ &\le \alpha_n \|f(y_{n,N}) - p\|^2 + \|x_n - p\|^2 - \beta_{n,i} \prod_{k=i}^{N} (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2, \end{aligned}$$

we immediately obtain that

$$\prod_{k=i}^{N} (1-\beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \le \frac{\alpha_n}{\beta_{n,i}} \|f(y_{n,N}) - p\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|) \frac{\|x_n - x_{n+1}\|}{\beta_{n,i}}.$$

By Lemma 3.5 or by hypothesis (ii) on the sequences, we have

$$\frac{\|x_n - x_{n+1}\|}{\beta_{n,i}} = \frac{\|x_n - x_{n+1}\|}{\beta_{n,k_0}} \cdot \frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0.$$

So, the conclusion follows. □

Remark 3.9. Let us consider $N = 3$ and the following sequences:

(a) $\alpha_n = \frac{1}{n^{1/2}}, \quad r_n = 2 - \frac{1}{n^2}, \quad \forall n > 1;$
(b) $\lambda_n = \alpha - \frac{1}{n^2}, \quad \forall n > \frac{1}{\alpha^{1/2}};$
(c) $\beta_{n,1} = \frac{1}{n^{1/4}}, \quad \beta_n = \beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}, \quad \beta_{n,3} = \frac{1}{n^{1/3}}, \quad \forall n > 1.$

It is easy to see that all hypotheses (i)-(iii), (H1), (H7) and (H8) of Lemma 3.8 are satisfied.

Remark 3.10. Under the hypotheses of Lemma 3.8, similarly to Lemma 3.6, one can see that

$$\lim_{n\to\infty} \|S_i u_n - y_{n,i-1}\| = 0, \quad \forall i \in \{2, \ldots, N\}.$$

Corollary 3.11. *Let us suppose that the hypotheses of either Lemma 3.6 or Lemma 3.8 are satisfied. Then* $\omega_w(x_n) = \omega_w(u_n) = \omega_w(y_n)$, $\omega_s(x_n) = \omega_s(u_n) = \omega_s(y_{n,1})$ *and* $\omega_w(x_n) \subset \Omega$.

Proof. By Remark 3.4, we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$.

First of all, let us show that

$$\lim_{n\to\infty} \|y_{n,N} - z_n\| = 0.$$

Indeed, let $q \in \Omega$. Then by the firm nonexpansivity of P_C, we get

$$\begin{aligned}
\|z_n - q\|^2 &= \|P_C(y_{n,N} - \lambda_n A y_{n,N}) - P_C(q - \lambda_n A q)\|^2 \\
&\le \langle y_{n,N} - \lambda_n A y_{n,N} - (q - \lambda_n A q), z_n - q\rangle \\
&= \tfrac{1}{2}\{\|(y_{n,N} - \lambda_n A y_{n,N}) - (q - \lambda_n A q)\|^2 + \|z_n - q\|^2 \\
&\quad -\|(y_{n,N} - \lambda_n A y_{n,N}) - (q - \lambda_n A q) - (z_n - q)\|^2\} \\
&\le \tfrac{1}{2}\{\|y_{n,N} - q\|^2 + \|z_n - q\|^2 - \|y_{n,N} - z_n\|^2 \\
&\quad +2\lambda_n\langle y_{n,N} - z_n, A y_{n,N} - Aq\rangle - \lambda_n^2\|A y_{n,N} - Aq\|^2\},
\end{aligned}$$

and so

(3.14)
$$\|z_n - q\|^2 \le \|y_{n,N} - q\|^2 - \|y_{n,N} - z_n\|^2 + 2\lambda_n\langle y_{n,N} - z_n, A y_{n,N} - Aq\rangle - \lambda_n^2\|A y_{n,N} - Aq\|^2.$$

Thus, we have

$$\begin{aligned}
\|y_n - q\|^2 &\le \alpha_n\|f(y_{n,N}) - q\|^2 + (1 - \alpha_n)\|z_n - q\|^2 \\
&\le \alpha_n\|f(y_{n,N}) - q\|^2 + \|z_n - q\|^2 \\
&\le \alpha_n\|f(y_{n,N}) - q\|^2 + \|y_{n,N} - q\|^2 - \|y_{n,N} - z_n\|^2 \\
&\quad +2\lambda_n\langle y_{n,N} - z_n, A y_{n,N} - Aq\rangle - \lambda_n^2\|A y_{n,N} - Aq\|^2.
\end{aligned}$$

This implies that

(3.15)
$$\begin{aligned}\|y_{n,N}-z_n\|^2 &\le \alpha_n\|f(y_{n,N})-q\|^2+\|y_{n,N}-q\|^2-\|y_n-q\|^2\\ &\quad +2\lambda_n\langle y_{n,N}-z_n, Ay_{n,N}-Aq\rangle-\lambda_n^2\|Ay_{n,N}-Aq\|^2\\ &\le \alpha_n\|f(y_{n,N})-q\|^2+(\|y_{n,N}-q\|+\|y_n-q\|)\|y_{n,N}-y_n\|\\ &\quad +2\lambda_n\langle y_{n,N}-z_n, Ay_{n,N}-Aq\rangle-\lambda_n^2\|Ay_{n,N}-Aq\|^2.\end{aligned}$$

Note that by Remark 3.10,

$$\lim_{n\to\infty}\|S_Nu_n-y_{n,N-1}\|=0.$$

Meantime, it is known that

$$\lim_{n\to\infty}\|S_Nu_n-u_n\|=\lim_{n\to\infty}\|u_n-x_n\|=\lim_{n\to\infty}\|x_n-y_n\|=0.$$

Hence we have

$$\lim_{n\to\infty}\|S_Nu_n-y_n\|=0. \tag{3.16}$$

Furthermore, it follows from (3.1) that

$$\lim_{n\to\infty}\|y_{n,N}-y_{n,N-1}\|=\lim_{n\to\infty}\beta_{n,N}\|S_Nu_n-y_{n,N-1}\|=0,$$

which together with $\lim_{n\to\infty}\|S_Nu_n-y_{n,N-1}\|=0$, yields

$$\lim_{n\to\infty}\|S_Nu_n-y_{n,N}\|=0. \tag{3.17}$$

Combining (3.16) and (3.17), we conclude that

$$\lim_{n\to\infty}\|y_n-y_{n,N}\|=0. \tag{3.18}$$

Therefore, from (3.12), (3.15) and (3.18) it immediately follows that

$$\lim_{n\to\infty}\|y_{n,N}-z_n\|=0. \tag{3.19}$$

Now we observe that

$$\|x_n-y_{n,1}\|\le\|x_n-u_n\|+\|y_{n,1}-u_n\|=\|x_n-u_n\|+\beta_{n,1}\|S_1u_n-u_n\|.$$

By Lemma 3.6, $\|S_1u_n-u_n\|\to 0$ as $n\to\infty$, and hence

$$\lim_{n\to\infty}\|x_n-y_{n,1}\|=0. \tag{3.20}$$

So we get $\omega_w(x_n)=\omega_w(y_{n,1})$ and $\omega_s(x_n)=\omega_s(y_{n,1})$.

Let $p\in\omega_w(x_n)$. Since $p\in\omega_w(u_n)$, by Lemma 3.6 and demiclosedness principle, we have $p\in\mathrm{Fix}(S_i)$ for all index i, i.e., $p\in\cap_i\mathrm{Fix}(S_i)$. Since

$$\begin{aligned}\|x_n-Tx_n\| &\le \|x_n-y_n\|+\|y_n-Tz_n\|+\|Tz_n-Ty_{n,N}\|+\|Ty_{n,N}-Tx_n\|\\ &\le \|x_n-y_n\|+\alpha_n\|f(y_{n,N})-Tz_n\|+\|z_n-y_{n,N}\|+\|y_{n,N}-x_n\|\\ &\le \|x_n-y_n\|+\alpha_n\|f(y_{n,N})-Tz_n\|+\|z_n-y_{n,N}\|\\ &\quad +\textstyle\sum_{k=2}^N\|y_{n,k}-y_{n,k-1}\|+\|y_{n,1}-x_n\|\\ &\le \|x_n-y_n\|+\alpha_n\|f(y_{n,N})-Tz_n\|+\|z_n-y_{n,N}\|\\ &\quad +\textstyle\sum_{k=2}^N\beta_{n,k}\|S_ku_n-y_{n,k-1}\|+\|y_{n,1}-x_n\|.\end{aligned}$$

So, utilizing Lemma 3.3 and Remark 3.10 we deduce from (3.19) and (3.20) that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

By deniclosedness principle, we have $p \in \mathrm{Fix}(T)$. In addition, by Lemmas 2.5 and 3.3 we know that $p \in \mathrm{EP}(F, h)$. Finally, by standard argument as in [21], we can show that $p \in \mathrm{VI}(C, A)$ and consequently, $p \in \Omega$. □

Theorem 3.12. *Let us suppose that $\Omega \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}$, $i = 1, \ldots, N$, be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ for all index i. Moreover, Let us suppose that (H1)-(H6) hold. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$, explicitly defined by scheme (3.1), all converge strongly to the unique solution $x^* \in \Omega$ of the variational inequality*

$$\langle f(x^*) - x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega. \tag{3.21}$$

Proof. Since the mapping $P_\Omega f$ is a ρ-contraction, it has a unique fixed point x^*; it is the unique solution of (3.21). Since (H1)-(H6) hold, the sequence $\{x_n\}$ is asymptotically regular (according to Lemma 3.2). By Lemma 3.3, $\|x_n - y_n\| \to 0$ and $\|x_n - u_n\| \to 0$ as $n \to \infty$. Moreover, utilizing Lemma 2.7 and the nonexpansivity of $(I - \lambda_n A)$, we have from (3.2) and (3.11)

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \|y_n - x^*\|^2 \\
&\leq \|\alpha_n(f(y_{n,N}) - f(x^*)) + (1-\alpha_n)(Tz_n - x^*)\|^2 \\
&\quad +2\alpha_n\langle f(x^*) - x^*, y_n - x^*\rangle \\
&\leq \alpha_n\rho\|y_{n,N} - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 \\
&\quad +2\alpha_n\langle f(x^*) - x^*, y_n - x^*\rangle \\
&= \alpha_n\rho\|y_{n,N} - x^*\|^2 \\
&\quad +(1-\alpha_n)\|P_C(I-\lambda_n A)y_{n,N} - P_C(I-\lambda_n A)x^*\| \\
&\quad +2\alpha_n\langle f(x^*) - x^*, y_n - x^*\rangle \\
&\leq \alpha_n\rho\|y_{n,N} - x^*\|^2 + (1-\alpha_n)\|y_{n,N} - x^*\| \\
&\quad +2\alpha_n\langle f(x^*) - x^*, y_n - x^*\rangle \\
&= [1-(1-\rho)\alpha_n]\|y_{n,N} - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, y_n - x^*\rangle \\
&\leq [1-(1-\rho)\alpha_n]\|x_n - x^*\|^2 + 2\alpha_n\langle f(x^*) - x^*, y_n - x^*\rangle \\
&= [1-(1-\rho)\alpha_n]\|x_n - x^*\|^2 \\
&\quad +(1-\rho)\alpha_n \cdot \tfrac{2}{1-\rho}\langle f(x^*) - x^*, y_n - x^*\rangle.
\end{aligned}$$

Now, let $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle f(x^*) - x^*, x_{n_k} - x^*\rangle. \tag{3.22}$$

By the boundedness of $\{x_n\}$, we may assume, without loss of generality, that $x_{n_k} \rightharpoonup z \in \omega_w(x_n)$. According to Corollary 3.11, we know that $\omega_w(x_n) \subset \Omega$ and hence $z \in \Omega$. Taking into consideration that $x^* = P_\Omega f(x^*)$ we obtain from (3.22) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(x^*) - x^*, y_n - x^*\rangle \\
&= \limsup_{n\to\infty}[\langle f(x^*) - x^*, x_n - x^*\rangle + \langle f(x^*) - x^*, y_n - x_n\rangle] \\
&= \limsup_{n\to\infty}\langle f(x^*) - x^*, x_n - x^*\rangle = \lim_{k\to\infty}\langle f(x^*) - x^*, x_{n_k} - x^*\rangle \\
&= \langle f(x^*) - x^*, z - x^*\rangle \leq 0.
\end{aligned}$$

In terms of Lemma 2.6 we derive $x_n \to x^*$ as $n \to \infty$. □

In a similar way, we can conclude another theorem as follows.

Theorem 3.13. *Let us suppose that $\Omega \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}$, $i = 1, \ldots, N$, be sequences in $(0,1)$ such that $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose that there exists*

$k \in \{1, \dots, N\}$ for which $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1, \dots, N\}$ the largest index for which $\beta_{n,k_0} \to 0$. Moreover, let us suppose that (H1), (H7) and (H8) hold and

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$*;*
(ii) *if* $i \le k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$*;*
(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ explicitly defined by scheme (3.1) all converge strongly to the unique solution $x^* \in \Omega$ of the variational inequality

$$\langle f(x^*) - x^*, z - x^* \rangle \le 0, \quad \forall z \in \Omega.$$

Remark 3.14. According to the above argument processes for Theorems 3.12 and 3.13, we can readily see that if in scheme (3.1), the iterative step $y_n = \alpha_n f(y_{n,N}) + (1 - \alpha_n) T P_C(y_{n,N} - \lambda_n A y_{n,N})$ is replaced by the iterative one $y_n = \alpha_n f(x_n) + (1 - \alpha_n) T P_C(y_{n,N} - \lambda_n A y_{n,N})$, then Theorems 3.12 and 3.13 remain valid.

Remark 3.15. Our Theorems 3.12 and 3.13 improve, extend, supplement and develop [26, [10, Theorems 3.1] and [14, Theorems 3.12 and 3.13] in the following aspects:

(a) The multi-step iterative scheme (3.1) of [14] is extended to develop our composite viscosity iterative scheme (3.1) by virtue of Jung's two-step iterative scheme (3.1) of [10] for the VI (1.1) and a nonexpansive mapping T;
(b) The argument techniques in our Theorems 3.12 and 3.13 are the combinations of the argument ones in [14, Theorem 3.12 and 3.13], and the argument ones in [10, Theorem 3.1];
(c) The problem of finding an element of $\mathrm{Fix}(T) \cap (\cap_i \mathrm{Fix}(S_i)) \cap \mathrm{EP}(F, h) \cap \mathrm{VI}(C, A)$ in our Theorems 3.12 and 3.13 is more general than the one of finding an element of $\mathrm{Fix}(T) \cap (\cap_i \mathrm{Fix}(S_i)) \cap \mathrm{EP}(F, h)$ in [14, Theorem 3.12 and 3.13] and the one of finding an element of $\mathrm{Fix}(T) \cap \mathrm{VI}(C, A)$ in [10, Theorem 3.1].

4. Applications

For a given nonlinear mapping $A : C \to H$, we consider the variational inequality (VI) of finding $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x} \rangle \ge 0, \quad \forall y \in C. \tag{4.1}$$

We will indicate with $\mathrm{VI}(C, A)$ the set of solutions of the VI (4.1).

Recall that if u is a point C, then the following relation holds:

$$u \in \mathrm{VI}(C, A) \;\Leftrightarrow\; u = P_C(I - \lambda A)u, \quad \forall \lambda > 0. \tag{4.2}$$

An operator $A : C \to H$ is said to be an α-inverse strongly monotone operator if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \ge \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

As an example, we recall that the α-inverse strongly monotone operators are firmly nonexpansive mappings if $\alpha \ge 1$ and that every α-inverse strongly monotone operator is also $\frac{1}{\alpha}$-Lipschitz continuous (see [23]).

Let us observe also that, if A is α-inverse strongly monotone, the mapping $P_C(I-\lambda A)$ are nonexpansive for all $\lambda > 0$ since they are compositions of nonexpansive mappings (see page 419 in [23]).

Let us consider $\tilde{S}_1, \dots, \tilde{S}_M$ a finite number of nonexpansive self-mappings on C and $A_1, \dots, A_N$ be a finite number of α-inverse strongly monotone operators. Let T be a nonexpansive self-mapping on C with fixed points. Let us consider the following mixed problem of finding $x^* \in \mathrm{Fix}(T) \cap \mathrm{EP}(F,h) \cap \mathrm{VI}(C,A)$ such that

$$
(4.3)\quad \begin{cases}
\langle (I-\tilde{S}_1)x^*, y-x^*\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T)\cap \mathrm{EP}(F,h)\cap \mathrm{VI}(C,A),\\
\langle (I-\tilde{S}_2)x^*, y-x^*\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T)\cap \mathrm{EP}(F,h)\cap \mathrm{VI}(C,A),\\
\dots\\
\langle (I-\tilde{S}_M)x^*, y-x^*\rangle \geq 0, \quad \forall y \in \mathrm{Fix}(T)\cap \mathrm{EP}(F,h)\cap \mathrm{VI}(C,A),\\
\langle A_1x^*, y-x^*\rangle \geq 0, \quad \forall y \in C,\\
\langle A_2x^*, y-x^*\rangle \geq 0, \quad \forall y \in C,\\
\dots\\
\langle A_Nx^*, y-x^*\rangle \geq 0, \quad \forall y \in C.
\end{cases}
$$

Let us call (SVI) the set of solutions of the $(M+N)$-system. This problem is equivalent to finding a common fixed point of T, $\{P_{\mathrm{Fix}(T)\cap \mathrm{EP}(F,h)\cap \mathrm{VI}(C,A)}\tilde{S}_i\}_{i=1}^N$, $\{P_C(I-\lambda A_i)\}_{i=1}^M$. So we claim that

Theorem 4.1. *Let us suppose that $\Omega = \mathrm{Fix}(T)\cap(\mathrm{SVI})\cap\mathrm{EP}(F,h)\cap\mathrm{VI}(C,A) \neq \emptyset$. Fix $\lambda > 0$. Let $\{\alpha_n\}, \{\beta_{n,i}\}$, $i = 1, \dots, (M+N)$, be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty}\beta_{n,i} \leq \limsup_{n\to\infty}\beta_{n,i} < 1$ for all index i. Moreover, Let us suppose that (H1)-(H6) hold. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ explicitly defined by scheme*

$$
(4.4)\quad \begin{cases}
F(u_n,y)+h(u_n,y)+\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \forall y \in C,\\
y_{n,1} = \beta_{n,1}P_{\mathrm{Fix}(T)\cap\mathrm{EP}(F,h)\cap\mathrm{VI}(C,A)}\tilde{S}_1u_n + (1-\beta_{n,1})u_n,\\
y_{n,i} = \beta_{n,i}P_{\mathrm{Fix}(T)\cap\mathrm{EP}(F,h)\cap\mathrm{VI}(C,A)}\tilde{S}_iu_n + (1-\beta_{n,i})y_{n,i-1}, \quad i = 2,\dots,M,\\
y_{n,M+j} = \beta_{n,M+j}P_C(I-\lambda A_j)u_n + (1-\beta_{n,M+j})y_{n,M+j-1}, \quad j = 1,\dots,N,\\
y_n = \alpha_n f(y_{n,M+N}) + (1-\alpha_n)TP_C(y_{n,M+N}-\lambda_n Ay_{n,M+N}),\\
x_{n+1} = (1-\beta_n)y_n + \beta_n TP_C(y_n - \lambda_n Ay_n), \quad \forall n \geq 1,
\end{cases}
$$

all converge strongly to the unique solution $x^* \in \Omega$ of the variational inequality

$$\langle f(x^*)-x^*, z-x^*\rangle \leq 0, \quad \forall z \in \Omega.$$

Theorem 4.2. *Let us suppose that $\Omega \neq \emptyset$. Fix $\lambda > 0$. Let $\{\alpha_n\}, \{\beta_{n,i}\}$, $i = 1, \dots, (M+N)$, be sequences in $(0,1)$ and $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose that there exists $k \in \{1,\dots,M+N\}$ such that $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1,\dots,M+N\}$ be the largest index for which $\beta_{n,k_0} \to 0$. Moreover, let us suppose that (H1), (H7) and (H8) hold and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;

(ii) *if* $i \leq k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;

(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0,1)$.

Then the sequences $\{x_n\}, \{y_n\}$ *and* $\{u_n\}$ *explicitly defined by scheme (4.4) all converge strongly to the unique solution* $x^* \in \Omega$ *of the variational inequality*

$$\langle f(x^*) - x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega.$$

Remark 4.3. If we choose $A = A_1 = \cdots = A_N = 0$ in system (4.3), we obtain a system of hierarchical fixed point problems introduced by Mainge and Moudafi [17, 18].

On the other hand, recall that a mapping $S : C \to C$ is called κ-strictly pseudocontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

If $\kappa = 0$, then S is nonexpansive. Put $A = I - S$, where $S : C \to C$ is a κ-strictly pseudocontractive mapping. Then A is $\frac{1-\kappa}{2}$-inverse strongly monotone; see [10].

Utilizing Theorems 3.12 and 3.13, we first give the following strong convergence theorems for finding a common element of the solution set $\mathrm{EP}(F, h)$ of the EP (1.8) and the common fixed point set $\mathrm{Fix}(T) \cap (\cap_i \mathrm{Fix}(S_i)) \cap \mathrm{Fix}(S)$ of a finite family of nonexpansive mappings $T, S_i : C \to C$, $i = 1, \ldots, N$, and a κ-strictly pseudocontractive mapping S.

Theorem 4.4. *Let* $\alpha = \frac{1-\kappa}{2}$. *Let us suppose that* $\Omega = \mathrm{Fix}(T) \cap (\cap_i \mathrm{Fix}(S_i)) \cap \mathrm{Fix}(S) \cap \mathrm{EP}(F, h) \neq \emptyset$. *Let* $\{\alpha_n\}, \{\beta_{n,i}\}$, $i = 1, \ldots, N$, *be sequences in* $(0, 1)$ *such that* $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ *for all index* i. *Moreover, Let us suppose that (H1)-(H6) hold. Then the sequences* $\{x_n\}, \{y_n\}$ *and* $\{u_n\}$ *generated explicitly by*

$$(4.5) \qquad \begin{cases} F(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1}) u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, \quad i = 2, \ldots, N, \\ y_n = \alpha_n f(y_{n,N}) + (1 - \alpha_n) T((1 - \lambda_n) y_{n,N} + \lambda_n S y_{n,N}), \\ x_{n+1} = (1 - \beta_n) y_n + \beta_n T((1 - \lambda_n) y_n + \lambda_n S y_n), \quad \forall n \geq 1, \end{cases}$$

all converge strongly to the unique solution $x^* \in \Omega$ *of the variational inequality*

$$\langle f(x^*) - x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega.$$

Proof. In Theorem 3.12, put $A = I - S$. Then A is $\frac{1-\kappa}{2}$-inverse strongly monotone. Hence we have that $\mathrm{Fix}(S) = \mathrm{VI}(C, A)$, $P_C(y_{n,N} - \lambda_n A y_{n,N}) = (1 - \lambda_n) y_{n,N} + \lambda_n S y_{n,N}$ and $P_C(y_n - \lambda_n A y_n) = (1 - \lambda_n) y_n + \lambda_n S y_n$. Thus, in terms of Theorems 3.12, we obtain the desired result. □

Theorem 4.5. *Let* $\alpha = \frac{1-\kappa}{2}$. *Let us suppose that* $\Omega = \mathrm{Fix}(T) \cap (\cap_i \mathrm{Fix}(S_i)) \cap \mathrm{Fix}(S) \cap \mathrm{EP}(F, h) \neq \emptyset$. *Let* $\{\alpha_n\}, \{\beta_{n,i}\}$, $i = 1, \ldots, N$, *be sequences in* $(0, 1)$ *such that* $\beta_{n,i} \to \beta_i$ *for all* i *as* $n \to \infty$. *Suppose that there exists* $k \in \{1, \ldots, N\}$ *for which* $\beta_{n,k} \to 0$ *as* $n \to \infty$. *Let* $k_0 \in \{1, \ldots, N\}$ *the largest index for which* $\beta_{n,k_0} \to 0$. *Moreover, let us suppose that (H1), (H7) and (H8) hold and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;

(ii) *if* $i \leq k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;

(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$ *and* $\{u_n\}$ *generated explicitly by (4.5), all converge strongly to the unique solution* $x^* \in \Omega$ *of the variational inequality*

$$\langle f(x^*) - x^*, z - x^* \rangle \leq 0, \quad \forall z \in \Omega.$$

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Manuscript received December 20, 2013
revised March 16, 2013

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