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EXTRAPOLATED SIMULTANEOUS SUBGRADIENT PROJECTION METHOD FOR VARIATIONAL INEQUALITY OVER THE INTERSECTION OF CONVEX SUBSETS

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Abstract. Many convex optimization problems in the Euclidean space can be formulated as a variational inequality over a subset of points satisfying a system of convex inequalities. In this article we propose a method for solving this problem. In the method we combine a hybrid descent idea presented in I. Yamada and N. Ogura, Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mapping, *Numer. Funct. Anal. and Optimiz.* **25** (2004) 619–655 and an extrapolated simultaneous subgradient projections introduced in L. T. Dos Santos, A parallel subgradient projections method for the convex feasibility problem, *J. Comp. and Applied Math.* **18** (1987) 307–320. The method does not require computation of the metric projection and can be simply performed. The method provides long steps which seem to be advantageous for the behavior of the method.

1. Introduction

Let $\mathbb{R}^n$ be equipped with an inner product $\langle \cdot, \cdot \rangle$ and with the corresponding norm $\| \cdot \|$, $C \subset \mathbb{R}^n$ be a nonempty, closed and convex subset and $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous operator. The variational inequality problem $\mathrm{VIP}(F, C)$ is to find a point $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C.$$

Denote by $\mathrm{Sol}(F, C)$ the set of all solutions of $\mathrm{VIP}(F, C)$. Throughout this article we suppose that F is Lipschitz continuous and strongly monotone which guarantee the existence and the uniqueness of a solution (see, e.g., [23, Theorem 46.C]). Many optimization problems can be presented as special cases of $\mathrm{VIP}(F, C)$, e.g. the problem of finding an element of C with minimal norm, differentiable convex constrained minimization, complementarity problem, Nash equilibrium problem (see, e.g., [13] for a preview of problems which can be translated to VIPs). In this article we suppose that $C = \bigcap_{i=1}^m C_i \neq \emptyset$, where $C_i := \{x \in \mathbb{R}^n : c_i(x) \leq 0\}$ and $c_i : \mathbb{R}^n \to \mathbb{R}$ is convex, $i \in I := \{1, 2, \ldots, m\}$. This situation occurs, e.g., in the differentiable convex constrained minimization

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & c_i(x) \leq 0, i = 1, 2, \ldots, m, \end{array}$$

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which is equivalent to VIP$(\nabla f, C)$. A classical iterative method for solving VIP(F,C) is the *gradient projection method*

$$\begin{array}{ll} x^0 \in \mathbb{R}^n & -\text{arbitrary} \\ x^{k+1} = P_C(x^k - \lambda F x^k), & \end{array}$$

where P_C denotes the metric projection onto C and $\lambda > 0$ is sufficiently small (see [15]). The iteration can be easily translated to the following one

$$u^{k+1} = P_C u^k - \lambda F P_C u^k.$$

Unfortunately, both above iterations are unpractical in our case, because, in general, the metric projection $P_C x$ is hard to compute, but the method requires to compute it in every iteration. Following Lions [18] and Bauschke [1], Yamada proposed a method for solving VIP(F,C) which applies cyclically nonexpansive operators T_i with $\operatorname{Fix} T_i = C_i$, $i \in I$,

$$u^{k+1} = T_{i_k} u^k - \lambda_k F T_{i_k} u^k,$$

where $i_k = k(\operatorname{mod} m) + 1$ is a cyclic control,

$$\lim_k \lambda_k = 0, \sum_{k=0}^{\infty} \lambda_k = +\infty, \quad \sum_{k=0}^{\infty} |\lambda_k - \lambda_{k+m}| < +\infty,$$

and

$$C = \operatorname{Fix}(T_m T_{m-1} \dots T_2 T_1) = \operatorname{Fix}(T_1 T_m \dots T_3 T_2) = \dots = \operatorname{Fix}(T_{m-1} T_{m-2} \dots T_1 T_m)$$

(see [21, Theorem 3.3]). Suzuki proved that the latter assumption is equivalent to $C = \operatorname{Fix}(T_m T_{m-1} ... T_2 T_1)$ (see [19, Theorem 2]). A similar method under different assumptions on λ_k was proposed by Xu and Kim in [20]. If the subsets C_i have not simple structure allowing an easy computation of P_{C_i}, the method proposed by Yamada is also unpractical, because, the known constructions of nonexpansive operators T_i with $\operatorname{Fix} T_i = C_i$ are compositions of metric projections P_{C_i} and relaxations of projections onto closed convex supersets of C_i (see [14]). Recently, several methods for VIP(F,C) were proposed which apply quasi-nonexpansive operators instead of nonexpansive ones (see [7–9, 16, 22]). In this article we suppose that $C_i = \{x \in \mathbb{R}^n : c_i(x) \leq 0\}$, where $c_i : \mathbb{R}^n \to \mathbb{R}$ are convex functions, $i \in I$, and propose a method which employs subgradient projections P_{c_i} relative to c_i, $i \in I$. Contrary to the metric projection $P_{C_i} x$, the subgradient projection $P_{c_i} x$, where $x \in \mathbb{R}^n$, is easier to compute. Therefore, the method is preferred in the case, when C_i has a structure which makes a computation of $P_{C_i} x$ difficult or even impossible. In the method, we combine a hybrid steepest descent idea (see [8, 9, 16, 21, 22]) and an extrapolated simultaneous subgradient projection method introduced by Dos Santos in [12] and developed in [4, Sect. 4.3], [17, Sect. 3], [10], [6] and [5, Section 4.9]. The latter provides long steps which seem to be advantageous in the behavior of the method latter. The paper is organized as follows. In Section 2, we recall some definitions and fact which we will use in the sequel. In Section 3, we describe in detail the method mentioned above, and in Section 4, we present the main result of the paper (Theorem 4.3), where we prove the convergence of sequences generated by our method.

2. Preliminaries

Let $U : \mathbb{R}^n \to \mathbb{R}^n$. A point $z \in \mathbb{R}^n$ satisfying $Uz = z$ is called a *fixed point* of U. A subset $\operatorname{Fix} U := \{z \in \mathbb{R}^n : Uz = z\}$ is called the *fixed point set* of U. An operator $U_\alpha : \mathrm{Id} + \alpha(U - \mathrm{Id})$, where $\alpha \geq 0$ and Id denotes the identity operator, is called an *α-relaxation* of U. It is clear that $\operatorname{Fix} U_\alpha = \operatorname{Fix} U$ for any $\alpha > 0$.

2.1. **VIP over the intersection of convex subsets.** Let $c_i : \mathbb{R}^n \to \mathbb{R}$ be convex and $C_i := \{x \in \mathbb{R}^n : c_i(x) \leq 0\}$, $i \in I := \{1, 2, \ldots, m\}$. Suppose that $C := \bigcap_{i \in I} C_i \neq \emptyset$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be *η-strongly monotone* and *κ-Lipschitz continuous*, where $0 < \eta \leq \kappa$, i.e.,

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2 \text{ for all } x, y \in \mathbb{R}^n$$

and

$$\|Fx - Fy\| \leq \kappa \|x - y\| \text{ for all } x, y \in \mathbb{R}^n.$$

Consider the following variational inequality problem $\mathrm{VIP}(F, C)$:

$$\text{find } x^* \in C \text{ satisfying } \langle F(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C.$$

As mentioned in the previous section, the above assumptions guarantee the existence and the uniqueness os a solution of $\mathrm{VIP}(F, C)$.

2.2. **Quasi-nonexpansive and approximately shrinking operators.** Let $U : \mathbb{R}^n \to \mathbb{R}^n$ be an operator having a fixed point. We say that U is *quasi-nonexpansive* (QNE) if $\|Ux - z\| \leq \|x - z\|$ for all $x \in \mathbb{R}^n$ and all $z \in \operatorname{Fix} U$. The subset of fixed points of a QNE operator is closed and convex (see [3, Proposition 2.6(ii)]). We say that U is *γ-strongly quasi-nonexpansive* (γ-SQNE), where $\gamma \geq 0$, if

$$\|Ux - z\|^2 \leq \|x - z\|^2 - \gamma \|Ux - x\|^2$$

for all $x \in \mathbb{R}^n$ and all $z \in \operatorname{Fix} U$. If $\gamma_1 < \gamma_2$ and U is γ_2-SQNE then it is γ_1-SQNE. If $\gamma > 0$ then T is called strongly quasi-nonexpansive (SQNE). We say that U is a *cutter* if

$$\langle z - Ux, x - Ux \rangle \leq 0$$

for all $x \in \mathbb{R}^n$ and all $z \in \operatorname{Fix} U$. The operator U is a cutter if and only if its α-relaxation is $\frac{2-\alpha}{\alpha}$-SQNE, where $\alpha \in (0, 2]$ (see [11, Proposition 2.3(ii)] and [5, Theorem 2.1.39]). In particular, U is a cutter if and only if U is 1-SQNE. Furthermore, U is QNE if and only if $\frac{1}{2}(U + \mathrm{Id})$ is a cutter (see [3, Proposition 2.3(v)$\Leftrightarrow$(vi)] or [5, Corollary 2.1.33(ii)]). The subset

$$\Delta_m := \Big\{ w = (\omega_1, \omega_2, \ldots, \omega_m) \in \mathbb{R}^m : \omega_i \geq 0, i = 1, 2, \ldots, m, \text{ and} \sum_{i=1}^m \omega_i = 1 \Big\}$$

is called the *standard simplex*. An element of Δ_m is called a *weight*. A function $w : \mathbb{R}^n \to \Delta_m$ is called a *weight function*.

Definition 2.1 (cf. [5, Definition 2.1.25])**.** Let $U_i : \mathbb{R}^n \to \mathcal{H}$, $i \in I$, be a family of operators. We say that a weight function $w : \mathbb{R}^n \to \Delta_m$ is *appropriate*, if for any $x \notin \bigcap_{i \in I} \operatorname{Fix} U_i$ there exists $j \in I$ such that

$$\omega_j(x) \parallel U_j x - x \parallel \neq 0. \tag{2.1}$$

The proposition below follows from [5, Theorems 2.1.26 and 2.1.50].

Proposition 2.2. *Let $U_i : \mathbb{R}^n \to \mathbb{R}^n$ be cutters, $i \in I := \{1, 2, \dots, m\}$, with $\bigcap_{i\in I} \operatorname{Fix} U_i \neq \emptyset$. and let $w = (\omega_1, \omega_2, \dots, \omega_m) : \mathbb{R}^n \to \Delta_m$ be an appropriate weight function. Then a convex combination $U := \sum_{\in I} \omega_i U_i$ is a cutter. Moreover, $\operatorname{Fix} U = \bigcap_{i\in I} \operatorname{Fix} U_i$.*

A comprehensive review of the properties of QNE and SQNE operators can be found in [5, Chapter 2].

Below we present one of several equivalent definitions of an approximately shrinking operator (cf. [9, Proposition 3.2])

Definition 2.3. We say that a QNE operator $U : \mathbb{R}^n \to \mathbb{R}^n$ is *approximately shrinking* if for any bounded sequence $\{x^k\}_{k=0}^{\infty} \subseteq \mathbb{R}^n$ and for any $\eta > 0$ there are $\gamma > 0$ and $k_0 \geq 0$ such that for all $k \geq k_0$ it holds

$$\|Ux^k - x^k\| < \gamma \Longrightarrow d(x^k, \operatorname{Fix} U) < \eta. \tag{2.2}$$

2.3. **Subgradient projection and its properties.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying $S(f, 0) := \{x \in \mathbb{R}^n : f(x) \leq 0\} \neq \emptyset$. Let $g_f(x)$ be a *subgradient* of f at x, i.e., $g_f(x) \in \partial f(x)$, where

$$\partial f(x) := \{g \in \mathbb{R}^n : \langle g, y - x\rangle \leq f(y) - f(x) \text{ for all } y \in \mathbb{R}^n\}$$

is a *subdifferential* of f at $x \in \mathbb{R}^n$. The existence of $g_f(x)$ follows from [2, Corollary 7.9]. The operator $P_f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$P_f x = \begin{cases} x - \frac{f(x)_+}{\|g_f(x)\|^2} g(x) & \text{if } g_f(x) \neq 0, \\ x & \text{otherwise,} \end{cases}$$

where $f(x)_+ := \max\{0, f(x)\}$, is called a *subgradient projection* relative to f. Note that, in general, $g_f(x)$ is not uniquely defined, therefore $P_f x$ depends on the current selection of $g_f(x) \in \partial f(x)$.

Below we recall some properties of a subgradient projection P_f.

Proposition 2.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying $\{x \in \mathbb{R}^n : f(x) \leq 0\} \neq \emptyset$. Then*

(i) *P_f is a cutter and $\operatorname{Fix} P_f = S(f, 0)$,*
(ii) *P_f is approximately shrinking.*

Proof. For (i) see [5, Corollary 4.2.6 and Lemma 4.2.5] and for (ii) see [8, Lemma 24]. □

Proposition 2.5. *Let $U_i : \mathbb{R}^n \to \mathbb{R}$ be a subgradient projection relative to a convex function $c_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I := \{1, 2, \dots, m\}$, $C := \bigcap_{i\in I} C_i \neq \emptyset$ and $U : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $Ux = \sum_{i\in I} \omega_i(x) U_i(x)$, where $w = (\omega_1, \omega_2, \dots, \omega_m) : \mathbb{R}^n \to \Delta_m$ is a weight function. Further, let $x \in \mathbb{R}^n$ and $z \in C$. Then*

$$\|Ux - x\| \geq \frac{1}{2R} \sum_{i\in I} \omega_i(x) \|U_i x - x\|^2 \tag{2.3}$$

for any $R > 0$ such that $\|x - z\| \leq R$.

Proof. Because a subgradient projection P_i is a cutter, $i \in I$, it is strongly quasi-nonexpansive. Therefore, the proposition is a special case of [9, Proposition 4.5]. □

3. Extrapolated simultaneous subgradient projection method

Let $c_i : \mathbb{R}^n \to \mathbb{R}$ be convex, $C_i := \{x \in \mathbb{R}^n : c_i(x) \leq 0\}$ and $U_i := P_{c_i}$, $i \in I := \{1, 2, \ldots, m\}$. Suppose that $C := \bigcap_{i\in I} C_i \neq \emptyset$. Let $w = (\omega_1, \omega_2, \ldots, \omega_m) : \mathbb{R}^n \to \Delta_m$ be an appropriate weight function (see Definition 2.1). Define $U : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Ux := \sum_{i=1}^{m} \omega_i(x) U_i x$$

and a *step size function* $\sigma_w : \mathbb{R}^n \to \mathbb{R}$ by

$$\sigma_w(x) := \begin{cases} 1 & \text{if } x \in \bigcap_{i=1}^m C_i \\ \frac{\sum_{i=1}^m \omega_i(x)\|U_i x - x\|^2}{\left\|\sum_{i=1}^m \omega_i(x)(U_i x - x)\right\|^2} & \text{otherwise.} \end{cases}$$

By (2.3), we have $\operatorname{Fix} U = \bigcap_{i\in I} C_i$ and σ_w is well defined. Furthermore, the convexity of the function $\|\cdot\|^2$ yields that $\sigma_w(x) \geq 1$ for all $x \in \mathbb{R}^n$. Define an operator $T_w : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T_w x := \sigma_w(x) Ux = \sigma_w(x) \sum_{i=1}^{m} \omega_i(x) U_i x.$$

We call the operator T_w an extrapolated simultaneous subgradient projection. For $\alpha \in [0, 2]$ the relaxation $T_{w,\alpha}$ of T_w is given by $T_{w,\alpha} := \operatorname{Id} + \alpha(T_w - \operatorname{Id})$. Let $\delta \in (0, 1]$ and suppose that the weight function w satisfies the following condition

$$\omega_j(x) \geq \delta \text{ for some } j \in \operatorname{Argmax}\{\|U_i x - x\| : i \in I\} \tag{3.1}$$

for all $x \notin C$. Then w is a special case of a regular weight function (see [5, Definition 5.8.2]). It is clear that w is appropriate.

Consider the following method for solving $\operatorname{VIP}(F, C)$

$$x^{k+1} = T_k x^k - \lambda_k F T_k x^k, \tag{3.2}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is η-strongly monotone and κ-Lipschitz continuous, $0 < \eta \leq \kappa$, $\{\lambda_k\}_{k=0}^{\infty} \subset [0, 2\eta/\kappa^2]$ is a sequence satisfying

$$\lim_k \lambda_k = 0 \text{ and } \sum_{k=0}^{\infty} \lambda_k = +\infty, \tag{3.3}$$

$T_k : \mathbb{R}^n \to \mathbb{R}^n$ are quasi-nonexpansive operators defined by

$$T_k := T_{w^k, \alpha_k}, \tag{3.4}$$

with ω^k being a sequence of weight functions satisfying (3.1) for some constant $\delta \in (0, 1]$ and $\alpha_k \in [\varepsilon, 2-\varepsilon]$ for some constant $\varepsilon \in (0, 1]$. It is clear that $\bigcap_{k\geq 0} \operatorname{Fix} T_k \supset C$. We can the method an *extrapolated simultaneous subgradient projection* (ESSP) method for $\operatorname{VIP}(F, C)$. The method is a special case of a *generalized hybrid steepest descent* (GHSD) method (see [8]). We can also write

$$x^{k+1} = x^k - \lambda_k F(x^k + \alpha_k \sigma_k (V_k x^k - x^k)), \tag{3.5}$$

where $V_k := \sum_{i=1}^m \omega_i^k U_i$, $U_i = P_{c_i}$, $i \in I$, and

$$\sigma_k = \begin{cases} 1 & \text{if } x^k \in \bigcap_{i=1}^m C_i \\ \frac{\sum_{i=1}^m \omega_i^k \|U_i x^k - x^k\|^2}{\|\sum_{i=1}^m \omega_i^k U_i x^k - x^k\|^2} & \text{otherwise.} \end{cases} \tag{3.6}$$

Proposition 3.1. *Let T_k be defined by (3.4), $k \geq 0$. Then* $\operatorname{Fix} T_k = C$ *and T_k is $\frac{\varepsilon}{2}$-SQNE.*

Proof. Because U_i are 1-SQNE, $i \in I$, the operator $V_k := \sum_{i=1}^m \omega_i^k U_i$ is also 1-SQNE (see [5, Theorem 2.1.50]). Therefore, T_k is $\frac{2-\alpha_k}{\alpha_k}$-SQNE (see [5, Theorem 4.9.1]). By $\frac{2-\alpha_k}{\alpha_k} \geq \frac{\varepsilon}{2}$, T_k is $\frac{\varepsilon}{2}$-SQNE. Furthermore, $\operatorname{Fix} V_k = C$ (see [5, Theorem 2.1.26(i)]). Consequently,

$$\|T_k x - x\| = \alpha_k \sigma_k \|V_k x - x\| \geq \varepsilon \|V_k x - x\|$$

and $\operatorname{Fix} T_k = C$. □

4. Convergence results

Before we formulate our main result, we recall some general convergence theorem. We start with the following

Definition 4.1. Let $T_k : \mathbb{R}^n \to \mathbb{R}^n$ be ρ_k-strongly quasi-nonexpansive, where $\rho_k \geq 0$, $k \geq 0$. We say that method (3.2) is *approximately shrinking* with respect to C if for any $\eta > 0$ there are $\gamma > 0$ and $k_0 \geq 0$, such that for all $k \geq k_0$ it holds

$$\rho_k \|T_k x^k - x^k\|^2 < \gamma \Longrightarrow d(x^k, C) < \eta. \tag{4.1}$$

Sufficient conditions for the convergence of sequences generated by the GHSD method (3.2) are given in the following result, which is a special case of [8, Theorem 12].

Proposition 4.2. *Let $\{x^k\}_{k=0}^\infty$ be generated by GHSD method (3.2), where T_k are ρ_k-SQNE with $\rho_k \geq 0$, $k \geq 0$. If the method is approximately shrinking with respect to C, then $\{x^k\}_{k=0}^\infty$ converges to a unique solution of VIP(F, C).*

Now we can formulate the main result of the paper.

Theorem 4.3. *The ESSP method (3.5) is approximately shrinking with respect to C. Consequently, any sequence $\{x^k\}_{k=0}^\infty$ generated by the method converges to a unique solution of* VIP(F, C).

Proof. The convexity of the function $\|\cdot\|^2$ yields that $\sigma_k \geq 1$. Therefore,

$$\|T_k x^k - x^k\| = \alpha_k \sigma_k \left\|V_k x^k - x^k\right\| \geq \varepsilon \left\|V_k x^k - x^k\right\|. \tag{4.2}$$

By [8, Lemma 9], x^k is bounded. Let $R > 0$ be such that $\left\|x^k - z\right\| \leq R$ for all $k \geq 0$ and for some $z \in C$. Let $\eta > 0$. Because the family $\{C_i : i \in I\}$ is boundedly regular (see [2, Proposition 5.4]), there are $\delta_0 > 0$ and $k_0 \geq 0$ such that

$$\max_{i \in I} d(x^k, C_i) < \delta_0 \Longrightarrow d(x^k, C) < \eta \tag{4.3}$$

for all $k \geq k_0$. Because U_i is AS (see Proposition 2.4), there is $\eta_i > 0$ and $k_i \geq 0$ such that for all $k \geq k_i$ it holds

$$\left\|U_i x^k - x^k\right\| \leq \eta_i \Longrightarrow d(x^k, \operatorname{Fix} U_i) \leq \delta_0, \tag{4.4}$$

$i \in I$. Let $k' = \max\{k_0, k_1, \dots, k_m\}$ and let $k \geq k'$. By Proposition 3.1, T_k is $\frac{\varepsilon}{2}$-SQNE. Let $j_k \in \operatorname{Argmax}\{\left\|U_i x^k - x^k\right\| : i \in I\}$ be such that $\omega_{j_k}^k \geq \delta$ (see (3.1)). Further, let $\gamma = \min_{i\in I} \frac{\delta\varepsilon^2\eta_i^2}{4R}$ and suppose that $\frac{\varepsilon}{2}\|T_k x^k - x^k\| < \gamma$. This, together with (4.2) and Proposition 2.5 yield

$$\begin{aligned}
\left\|U_i x^k - x^k\right\|^2 &\leq \max_{i\in I}\left\|U_i x^k - x^k\right\|^2 = \left\|U_{j_k} x^k - x^k\right\|^2 \\
&\leq \frac{1}{\delta}\omega_{j_k}^k \left\|U_{j_k} x^k - x^k\right\|^2 \leq \frac{1}{\delta}\sum_{i\in I}\omega_i^k \left\|U_i x^k - x^k\right\|^2 \\
&\leq \frac{2R}{\delta}\left\|V_k x^k - x^k\right\| \leq \frac{2R}{\delta\varepsilon}\|T_k x^k - x^k\| \\
&\leq \frac{4R\gamma}{\delta\varepsilon^2} \leq \eta_i^2,
\end{aligned}$$

$i \in I$, consequently, $\left\|U_i x^k - x^k\right\| \leq \eta_i$, $i \in I$. Now (4.4) and (4.3) give $d(x^k, C) < \eta$, i.e., the method is AS with respect to C. By Proposition 4.2, the sequence $\{x^k\}_{k=0}^\infty$ converges to a unique solution of VIP(F, C). □

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