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ITERATIVE ALGORITHMS FOR EQUILIBRIUM PROBLEMS BASED ON PROXIMAL-LIKE METHODS

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Abstract. In this paper, we present two algorithms, namely Algorithm 1.2 and Algorithm 1.3, with most violated constraint control strategy used in the projection method for solving equilibrium problems. Compared with the projection methods, both algorithms are designed to avoid computing the projection to the nonlinear level set. Under certain conditions, the convergences of the Algorithm 1.2 and Algorithm 1.3 are established, and the linear convergence rate of Algorithm 1.3 is obtained. At last, some examples to illustrate the convergence performance of Algorithm 1.2 and Algorithm 1.3 are given .

1. Introduction

Let K be a nonempty closed convex subset of $\mathbb{R}^n$ and let $f : K \times K \to \mathbb{R}$ be a continuous function satisfying the following two properties:

P1. $f(x, x) = 0$ for all $x \in K$.

P2. $f(x, \cdot) : K \to \mathbb{R}$ is convex for all $x \in K$.

The equilibrium problem (for short, EP) considered here is formulated as follows:

$$\text{Find} \quad \bar{x} \in K \quad \text{such that} \quad f(\bar{x}, y) \geq 0 \quad \text{for all} \quad y \in K. \tag{1.1}$$

The equilibrium problem provides a very general mathematical model and many problems such as optimization problems, Nash equilibria problems, complementarity problems, fixed point problems, variational inequality problems and convex vector optimization problems can be recast into the form (1.1); see, for example, [3, 15, 20, 22] and the references therein.

For each $y \in K$, the level set $L_f(y)$ of $f(y, \cdot)$ is defined by

$$L_f(y) := \{x \in K : f(y, x) \leq 0\}.$$

Obviously, $L_f(y)$ is closed and convex for each $y \in K$. As an auxiliary problem for solving EP, we consider the following convex feasibility problem (for short, CFP):

$$\text{Find} \quad \bar{x} \in K \quad \text{such that} \quad \bar{x} \in \cap_{y \in K} L_f(y). \tag{1.2}$$

This problem has a wide applications in various fields of classical mathematics and modern physical science (e.g. image processing, computer tomograph and radiation

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therapy treatment planning) and has been studied extensively; see for example [4, 13, 15, 16, 17] and the references therein. The relationship between the solution sets of *EP* and *CFP* is described as in the following proposition, which is taken from [15, Theorem 2.1]. Recall that f is pseudomonotone if the following implication holds for any $x, y \in K$:

$$f(x, y) \geq 0 \Longrightarrow f(y, x) \leq 0.$$

Proposition 1.1. *The solution set of CFP* (1.2) *is a subset of the solution set of EP* (1.1)*. The converse is also true if f is pseudomonotone.*

Based on this relationship, some algorithms for solving (1.1) can be designed via finding a solution of problem (1.2). The most famous algorithms for solving problem (1.2) are the so-called projection-type methods, which are usually classified in two categories: successive and simultaneous (cf. [15]). Here we consider the successive projection method to solve problem (1.2). For this purpose, we use $P_Z : \mathbb{R}^n \to Z \subseteq \mathbb{R}^n$ to denote the orthogonal projection onto Z. Give initial point x^0. Then the iteration sequence $\{x^k\}$ can be constructed as follows:

$$x^{k+1} = x^k + \lambda_k (P_{L_f(y^k)}(x^k) - x^k),$$

where $\{\lambda_k\} \subset [\alpha, 2-\alpha]$ with $\alpha \in (0, 1)$ is a sequence of exogenously given relaxation parameters and the sequence $\{y^k\} \subseteq K$ is chosen to satisfy some control strategy, which is usually one of the three control strategies descried in the following:

(1) *Remotest set control:* This is obtained by determining each $y^k \in K$ such that

$$\|x^k - P_{L_f(y^k)}(x^k)\| = \sup_{y \in K} \|x^k - P_{L_f(y)}(x^k)\| \quad \text{for each} \quad k \in \mathbb{N}. \tag{1.3}$$

(2) *Approximately remotest set control:* This is obtained by choosing $\{y^k\} \subseteq K$ such that

$$\lim_{k \to \infty} \|x^k - P_{L_f(y^k)}(x^k)\| = 0 \Rightarrow \lim_{k \to \infty} \sup_{y \in K} \|x^k - P_{L_f(y)}(x^k)\| = 0. \tag{1.4}$$

(3) *Most violated constraint control:* This control strategy measures the proximity of x^k to a set in the family not through the Euclidean distance, but through the values of the function $f(\cdot, x^k) : K \to \mathbb{R}$, that is, each y^k is chosen so that

$$f(y^k, x^k) = \max_{y \in K} f(y, x^k) \quad \text{for each} \quad k \in \mathbb{N}. \tag{1.5}$$

As pointed out in [15], the three control strategies described above require some sort of maximization in y in order to choose y^k in each step. Existence of such maximizer, and/or efficient methods for finding it, demand some compactness assumption on K or coercivity of $f(\cdot, x)$. Furthermore, note that evaluating $f(y, x^k)$ is generally simpler than computing the distance $\|x^k - P_{L_f(y)}(x^k)\|$. The authors in [15] considered a modified version of the successive projection methods, where the maximization step in each iteration is limited to a compact set, but without imposing any compactness assumption on K (nor coercivity on $f(\cdot, x)$). More precisely, Iusem and Sosa in [15] presented the following successive projection-like algorithm:

Algorithm IS.

Step 0: Give sequences $\{\lambda_k\} \subseteq [\alpha, 1]$ with $\alpha \in (0,1)$ and $\{\varepsilon_k\} \subset \mathbb{R}_+$ with $\lim_{k\to\infty} \varepsilon_k = 0$;
Step 1: Select $x^0 \in K$, put $\rho_0 := \|x^0\|$ and $k := 0$;
Step 2: Set $C_k := \{x \in K| \quad \|x\| \le \rho_k + 1\}$;
Step 3: Find $y^k \in C_k$ such that

$$\max_{y\in C_k} f(y, x^k) \le f(y^k, x^k) + \varepsilon_k; \tag{1.6}$$

Step 4: Compute x^{k+1} as

$$x^{k+1} := x^k + \lambda_k \left(P_{L_f(y^k)}(x^k) - x^k \right); \tag{1.7}$$

Step 5: Update ρ_k through

$$\rho_{k+1} := \max\{\rho_k, \|x^{k+1}\|\}; \tag{1.8}$$

Step 6: Set $k := k + 1$, and turn to **Step 2**.

Under the assumption that $\cap_{k=0}^{\infty} L_f(y^k) \ne \emptyset$, they proved the convergence of the algorithm. Note that in Algorithm IS, one has to compute the projection $P_{L_f(y^k)}(x^k)$ at each iteration, which, as is well-known, is very difficult in general. This motivates us present two modified versions of Algorithm IS that avoid the computations of the projections $P_{L_f(y^k)}(x^k)$. As usual, for any $y \in K$, we use $[f(y, \cdot)]_+$ to denote the convex function $[f(y, \cdot)]_+ : K \to \mathbb{R}$ defined by

$$[f(y, x)]_+ := \max\{0, f(y, x)\} \quad \text{for each} \quad x \in K. \tag{1.9}$$

Furthermore, we assume for Algorithms 1.2 and 1.3 that sequences $\{\alpha_k\} \subseteq [\alpha_1, \alpha_2]$ with $0 < \alpha_1 < \alpha_2 < +\infty$ and $\{\varepsilon_k\} \subset \mathbb{R}_+$ with $\lim_{k\to\infty} \varepsilon_k = 0$.

Algorithm 1.2.

Step 0: Select $x^0 \in K$, put $\rho_0 := \|x^0\|$ and $k := 0$;
Step 1: Set $C_k := \{x \in K| \quad \|x\| \le \rho_k + 1\}$;
Step 2: Find $y^k \in C_k$ to satisfy (1.6);
Step 3: Compute x^{k+1} as

$$x^{k+1} := \arg\min_{x\in K} \left\{ [f(y^k, x)]_+ + \frac{1}{2\alpha_k} \|x - x^k\|^2 \right\}; \tag{1.10}$$

Step 4: Update ρ_k through (1.8);
Step 5: Set $k := k + 1$, and turn to **Step 1**.

A different control strategy for the sequence $\{y_k\}$ is used in **Algorithm 1.3** below.

Algorithm 1.3.

Step 0: Select $\eta \in (0, 1)$, $x^0 \in K$ and put $k := 0$;
Step 1: Find $y^k \in K$ such that

$$f(y^k, x^k) \ge \eta \sup_{y\in K} f(y, x^k); \tag{1.11}$$

Step 2: Compute x^{k+1} by (1.10);

Step 3: Set $k := k + 1$, and turn to **Step 1**.

The next section contains some necessary notations and preliminary results. We will establish the convergence results for Algorithms 1.2 and 1.3 in Sections 3 and 4, respectively. Moreover a linear convergence result for Algorithm 1.3 is presented in Section 4. Numerical experiments for Algorithms 1.2 and 1.3 are provided in the last section.

2. Notations and preliminaries

The notation used in the present paper is standard (cf. [27]). Let $x \in \mathbb{R}^n$ and $r > 0$. We use $\mathbf{B}(x, r)$ and $\bar{\mathbf{B}}(x, r)$ to denote the open and closed balls with center x and radius r, respectively. By $\langle \cdot, \cdot \rangle$ we shall denote the inner product in $\mathbb{R}^n$. Let $Z \subseteq \mathbb{R}^n$. The indicator function δ_Z and the normal cone $N_Z(z_0)$ of Z at $z_0 \in Z$ are respectively defined by

$$\delta_Z(z) := \begin{cases} 0, & z \in Z, \\ +\infty, & \text{otherwise}, \end{cases}$$

and

$$N_Z(z_0) := \{x^* \in \mathbb{R}^n : \langle x^*, z - z_0 \rangle \le 0 \quad \text{for all } z \in Z\}.$$

As usual, we use $\mathrm{d}(\cdot, Z) : \mathbb{R}^n \to \mathbb{R}$ and $P_Z : \mathbb{R}^n \to Z$ to denote the distance function and the orthogonal projection onto Z, respectively, that is,

$$\mathrm{d}(x, Z) := \inf_{y \in Z} \|x - y\|$$

and

$$P_Z(x) := \{y \in Z : \|x - y\| = \min_{z \in Z} \|x - z\|\}.$$

Then, the following property of the orthogonal projection holds (cf. [1]):

$$x_0 \in P_Z(x) \Longleftrightarrow \langle x - x_0, y - x_0 \rangle \le 0 \quad \text{for each} \quad y \in Z. \tag{2.1}$$

Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. The effective domain of g is defined by

$$\operatorname{dom} g := \{x \in X : \ g(x) < +\infty\},$$

and the subdifferential of g at $x \in \operatorname{dom} g$ is defined by

$$\partial g(x) := \{x^* \in \mathbb{R}^n : \ g(x) + \langle x^*, y - x \rangle \le g(y) \quad \text{for each } y \in \mathbb{R}^n\}.$$

Clearly, by the first order optimality condition (see [27, Theorem 2.5.7]), the following equivalence holds:

$$x_0 \text{ is a minimizer of } g \text{ if and only if } 0 \in \partial\, g(x_0). \tag{2.2}$$

Moreover, if $h : X \to \mathbb{R} \cup \{+\infty\}$ is a proper convex function such that $\operatorname{dom} g \cap \operatorname{dom} h \neq \emptyset$, then

$$\partial g(a) + \partial h(a) \subseteq \partial (g + h)(a) \quad \text{for each } a \in \operatorname{dom} g \cap \operatorname{dom} h. \tag{2.3}$$

Furthermore, if g or h is continuous at some point of $\operatorname{dom} g \cap \operatorname{dom} h$, then

$$\partial (g + h)(x) = \partial g(x) + \partial h(x) \quad \text{for each } x \in \operatorname{dom} g \cap \operatorname{dom} h. \tag{2.4}$$

Let $x \in \operatorname{dom} g$ and $y \in \operatorname{dom} h$. If $x^* \in \partial g(x)$ and $y^* \in \partial h(y)$, then by the monotonicity of the subdifferential (cf. [26]), we have

$$\langle x^* - y^*, x - y\rangle \geq 0. \tag{2.5}$$

Recall that a sequence $\{x_k\}$ is Fejér monotone w.r.t a set S if

$$\|x_{k+1} - \bar{x}\| \leq \|x_k - \bar{x}\| \quad \text{for each} \quad \bar{x} \in S \text{ and } k = 0, 1, \ldots.$$

Then the following lemma holds (cf. [1, Theorem 2.16]):

Lemma 2.1. *Suppose the sequence $\{x_k\}$ is Fejér monotone w.r.t a set S. Then the following assertions hold.*

(i) *$\{x_k\}$ is bounded and* $\mathrm{d}(x_{k+1}, S) \leq \mathrm{d}(x_k, S)$.

(ii) *If there exits a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ such that $\{x_{k_m}\}$ is convergent to a point $x \in S$, then $\{x_k\}$ is convergent to x.*

Consider the following convex inequality system:

$$x \in K, \quad F(x) \leq 0, \tag{2.6}$$

where K is a closed convex subset of $\mathbb{R}^n$ as in the previous section and $F : K \to \mathbb{R}$ is a continuous convex function. Let K_F denote the solution set of the inequality system (2.6).

Definition 2.2. The inequality system (2.6) is said to have

(i) an (global) error bound if there exists $\gamma > 0$ such that

$$\mathrm{d}(x, K_F) \leq \gamma[F(x)]_+. \tag{2.7}$$

holds for any $x \in K$.

(ii) bounded error bounds if and only if, for any $\rho > 0$, there exists $\gamma > 0$ such that (2.7) holds for any $x \in K \cap \mathbf{B}(0, \rho)$.

Clearly, the inequality system (2.6) has an error bound implies that it has bounded error bounds. The following proposition provides a useful characterization for error bounds; see for example [2, Theorem 2.2].

Proposition 2.3. *Let $\gamma > 0$ and $\rho > 0$. Then* (2.7) *holds for each $x \in K \cap \mathbf{B}(0, \rho)$ if and only if the following inclusion holds for each $z \in K_F \cap \mathbf{B}(0, \rho)$:*

$$\mathbf{B}(0, 1) \cap N_{K_F}(z) \subseteq \gamma\partial[F(\cdot)_+ + \delta_K(\cdot)](z). \tag{2.8}$$

3. Convergence analysis of Algorithm 1.2

Unless explicitly stated otherwise, let K and f be as in Section 1; namely, K is a closed and convex subset of $\mathbb{R}^n$ and $f : K \times K \to \mathbb{R}$ is a continuous function satisfying P1 and P2. Recall that, for each $y \in K$, the convex function $[f(y, \cdot)]_+ : K \to \mathbb{R}$ is defined by (1.9). The following two lemmas are taken from [8, Lemma 4.1] and [8, Lemma 4.2], respectively.

Lemma 3.1. *Let $y^*, \xi \in K$ and $\alpha > 0$. Suppose that*

$$x^+ \in \arg\min_{x \in K} \left\{ [f(y^*, x)]_+ + \frac{1}{2\alpha}\|x - \xi\|^2 \right\}. \tag{3.1}$$

Then, for each $x \in K$, we have

$$[f(y^*, x)]_+ + \frac{1}{2\alpha}\|x - \xi\|^2 \geq [f(y^*, x^+)]_+ + \frac{1}{2\alpha}\|x^+ - \xi\|^2 + \frac{1}{2\alpha}\|x - x^+\|^2. \tag{3.2}$$

Lemma 3.2. *Let $y^* \in K$ and $\alpha > 0$. Then the following equivalence holds:*

$$x^* \in \arg\min_{x \in K} \left\{ [f(y^*, x)]_+ + \frac{1}{2\alpha}\|x - x^*\|^2 \right\} \Longleftrightarrow x^* \in \arg\min_{x \in K} [f(y^*, x)]_+ = L_f(y^*). \tag{3.3}$$

For convenience, we set $L := \cap_{k=0}^{\infty} L_f(y^k)$. The following lemma will be useful for our study.

Lemma 3.3. *Let $\{x^k\}$ and $\{y^k\}$ be sequences generated by Algorithm 1.2 or Algorithm 1.3. Suppose that $L \neq \emptyset$. Then the sequence $\{x_k\}$ is Fejér monotone w.r.t L and*

$$\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0. \tag{3.4}$$

Proof. Let $\bar{x} \in L$ and $k \in \mathbb{N}$. Then we have that

$$[f(y^k, \bar{x})]_+ \leq [f(y^k, x)]_+ \quad \text{for each} \quad x \in K. \tag{3.5}$$

Using (1.10) and applying Lemma 3.1 (to $\{y^k, x^k, \bar{x}, x^{k+1}, \alpha_k\}$ in place of $\{y^*, \xi, x, x^+, \alpha\}$), we have that

$$[f(y^k, \bar{x})]_+ + \frac{1}{2\alpha_k}\|\bar{x} - x^k\|^2 \geq [f(y^k, x^{k+1})]_+ + \frac{1}{2\alpha_k}\|x^{k+1} - x^k\|^2 + \frac{1}{2\alpha_k}\|\bar{x} - x^{k+1}\|^2. \tag{3.6}$$

Since $\alpha_k > 0$ and $[f(y^k, \bar{x})]_+ \leq [f(y^k, x^{k+1})]_+$ by (3.5), it follows that

$$\|\bar{x} - x^k\|^2 \geq \|x^{k+1} - x^k\|^2 + \|\bar{x} - x^{k+1}\|^2. \tag{3.7}$$

Hence the sequence $\{x_k\}$ is Fejér monotone w.r.t L as $\bar{x} \in L$ and $k \in \mathbb{N}$ are arbitrary; moreover (3.4) is seen to hold. The proof is complete. □

The following lemma is taken from [15, Theorem 2.5].

Lemma 3.4. *If there exist an open set $U \subset \mathbb{R}^n$ and $\bar{x} \in K \cap U$ such that $f(\bar{x}, y) \geq 0$ for all $y \in K \cap U$, then $\bar{x}$ solves EP.*

The following theorem provides some convergence properties of Algorithm 1.2.

Theorem 3.5. *Let $\{x^k\}$ and $\{y^k\}$ be sequences generated by Algorithm 1.2. Then the following assertions hold.*

(i) *If $L \neq \emptyset$, then $\{x^k\}$ converges to a solution of EP.*
(ii) *If EP lacks solutions, then $\{x^k\}$ is not convergent.*

Proof. Let $\{k_m\} \subseteq \mathbb{N}$ be a subsequence and consider the following conditions:

$$x^{k_m} \to x^*, \; y^{k_m} \to y^*, \; \alpha_{k_m} \to \bar{\alpha} \tag{3.8}$$

and

$$\|x^{k_m+1} - x^{k_m}\| \to 0 \quad \text{and} \quad x^{k_m+1} \to x^*. \tag{3.9}$$

We first verify the following implication:

$$(3.10) \qquad (3.8) \text{ and } (3.9) \Longrightarrow x^k \to x^* \in L \text{ and } x^* \text{ solves EP.}$$

To do this, suppose that both (3.8) and (3.9) hold. Note by (1.10) that

$$[f(y^{k_m}, x^{k_m+1})]_+ + \frac{1}{2\alpha_{k_m}}\|x^{k_m+1} - x^{k_m}\|^2 \le [f(y^{k_m}, x)]_+ + \frac{1}{2\alpha_{k_m}}\|x - x^{k_m}\|^2 \quad \text{for each} \quad x \in K.$$

Then taking limits and making use of (3.8) and (3.9), we conclude from the continuity of f that

$$(3.11) \qquad x^* = \arg\min_{x \in K} \left\{ [f(y^*, x)]_+ + \frac{1}{2\bar{\alpha}}\|x - x^*\|^2 \right\}.$$

This together with Lemma 3.2 implies that $x^* \in L_f(y^*)$, that is, $f(y^*, x^*) \le 0$.

Noting that $\{\rho_k\}$ is bounded, one sees that $\bar{\rho} := \sup\{\rho_k\}$ is well-defined. Take $\delta \in (0, \frac{1}{2})$ and consider the open ball $U(\delta) := \mathbf{B}(0, \bar{\rho} + 1 - \delta)$. It follows from Step 4 of Algorithm 1.2 that

$$\|x^k\| \le \rho_k \le \bar{\rho} < \bar{\rho} + 1 - \delta \quad \text{for each} \quad k \in \mathbb{N}.$$

This means that $x^* \in U(\delta)$ and so $x^* \in K \cap U(\delta)$. Since the sequence $\{\rho_k\}$ is nondecreasing, it follows from the definition of $\bar{\rho}$ that there exists $N \in \mathbb{N}$ such that

$$(3.12) \qquad \rho_k \ge \bar{\rho} - \delta \quad \text{for each} \quad k \ge N.$$

Consequently, we have that

$$x^k \in K \cap U(\delta) \subseteq K \cap \bar{B}(0, \rho_k + 1) = C_k \quad \text{for each} \quad k \ge N.$$

Now fix $y \in K \cap U(\delta)$. It follows from Step 2 of Algorithm 1.2 that

$$(3.13) \qquad f(y, x^k) \le \max_{y \in C_k} f(y, x^k) \le f(y^k, x^k) + \varepsilon_k \quad \text{for each} \quad k \ge N.$$

Note that $\varepsilon_k \to 0$ and f is continuous. Then, passing to the limit, we obtain that

$$(3.14) \qquad f(y, x^*) \le f(y^*, x^*) \le 0 \quad \text{for each} \quad y \in K \cap U(\delta).$$

Hence $x^* \in \cap_{y \in K \cap U(\delta)} L_f(y)$. Thus, by Proposition 1.1,

$$(3.15) \qquad f(x^*, y) \ge 0 \quad \text{for each} \quad y \in K \cap U(\delta).$$

This together with Lemma 3.4 implies that x^* is a solution of *EP*. Furthermore, since $\delta \in (0, \frac{1}{2})$ is arbitrary and f is continuous, it follows from (3.14) that

$$(3.16) \qquad f(y, x^*) \le 0 \quad \text{for each} \quad y \in K \cap \bar{\mathbf{B}}(0, \bar{\rho} + 1).$$

Therefore $f(y^k, x^*) \le 0$ for all $k \in \mathbb{N}$ because $y^k \in C_k \subseteq K \cap \bar{\mathbf{B}}(0, \bar{\rho} + 1)$, that is, $x^* \in L$. Thus, by the fact $x^{k_m} \to x^*$, Lemmas 2.1(ii) and 3.3, we see that $x^k \to x^*$. Hence we complete the proof of implication (3.10).

Now we are back to the proof of the theorem. To show assertion (i), suppose that $L \neq \emptyset$ and let $\bar{x} \in L$. Then, by Lemma 3.3, the sequence $\{\|\bar{x} - x^k\|\}$ converges and so $\{x^k\}$ is bounded. It follows from Step 2 and Step 4 of Algorithm 1.2 that $\{\rho_k\}$ and $\{y^k\}$ are also bounded. Thus, we can choose a subsequence $\{k_m\}$ of $\{k\}$ such that (3.8) holds. By Lemma 3.3, one sees that (3.9) holds. Hence the implication

(3.10) is applicable to conclude that $x^k \to x^*$ and x^* is a solution of *EP*. This completes the proof of assertion (i).

To show assertion (ii), we assume that *EP* lacks solutions. Suppose on the contrary that $\{x^k\}$ converges to x^*. Then, by Step 2 of Algorithm 1.2, the sequence $\{y^k\}$ is bounded. Thus we can select a subsequence $\{k_m\} \subseteq \mathbb{N}$ such that (3.8) and (3.9) hold. Then the implication (3.10) means that x^* is a solution of *EP*, which is a contradiction. The proof is complete. □

4. Convergence analysis of Algorithm 1.3

This section is devoted to establishing the linear convergence result for Algorithm 1.3. The main tool used in the present section is the notion of error bounds for convex inequality systems, which was originally introduced by Hoffman [11] for linear inequality systems. This notion has become one of the most important and useful tools for studying convergence analysis of algorithms and the sensitivity analysis to solve optimization problems, and has been studied extensively; see [6, 11, 18, 24, 25] and references therein for details.

Let $\{y^k\} \subseteq K$ and consider the following parameter inequality system:

$$x \in K, \quad f(y^k, x) \le 0. \tag{4.1}$$

We introduce in the following definition the notions of uniform error bounds for the parameter inequality system (4.1).

Definition 4.1. Let $\{y^k\} \subseteq K$. The parameter inequality system (4.1) is said to have

(i) a uniform error bound if there exists $\gamma > 0$ such that

$$\mathrm{d}(x, L_f(y^k)) \le \gamma [f(y^k, x)]_+ \tag{4.2}$$

for any y^k and $x \in K$.

(ii) uniform bounded error bounds if, for any $\rho > 0$, there exists $\gamma > 0$ such that (4.2) holds for any y^k and $x \in K \cap \mathbf{B}(0, \rho)$.

Recall that $L = \cap_{k=0}^{\infty} L_f(y^k)$. Before proving the main theorem, we first verify two lemmas.

Lemma 4.2. *Let $\{x^k\}$ and $\{y^k\}$ be sequences generated by Algorithm 1.3. Suppose that $L \neq \emptyset$ and that the parameter inequality system* (4.1) *has uniform bounded error bounds. Then there exists $k_0 \in \mathbb{N}$ such that*

$$x^{k+1} = P_{L_f(y^k)}(x^k) \quad \textit{for each} \quad k \ge k_0. \tag{4.3}$$

Proof. Since $L \neq \emptyset$, it follows from Lemma 3.3 that the sequence $\{x^k\}$ is Fejér monotone w.r.t L and (3.4) holds. Take $\bar{x} \in L$, then $\rho := 3 \max\{\sup_k \|x^k\|, \|\bar{x}\|\} < +\infty$. Since the parameter inequality system (4.1) has uniform bounded error bounds, there exits $\gamma > 0$ such that (4.2) holds for each y^k and each $x \in K \cap \mathbf{B}(0, \rho)$. Fix $k \in \mathbb{N}$. By Step 2 of Algorithm 1.3, it follows form (2.2) that

$$0 \in \partial([f(y^k, \cdot)]_+ + \delta_K + \frac{1}{2\alpha_k} \| \cdot - x^k \|^2)(x_{k+1}),$$

that is,

$$0 \in \partial([f(y^k,\cdot)]_+ + \delta_K)(x_{k+1}) + \frac{1}{\alpha_k}(x^{k+1} - x^k).$$

Denote $\omega_k := \frac{1}{\alpha_k}(x^k - x^{k+1})$. Then

$$\omega_k \in \partial([f(y^k,\cdot)]_+ + \delta_K)(x^{k+1}) \tag{4.4}$$

and by (3.4), there exists $k_0 \in \mathbb{N}$ such that

$$\|\omega_k\| < \frac{1}{\gamma} \quad \text{for each} \quad k \geq k_0. \tag{4.5}$$

Below we first show that

$$x^{k+1} \in L_f(y^k) \quad \text{for each} \quad k \geq k_0. \tag{4.6}$$

To do this, suppose on the contrary that there exists $k \geq k_0$ such that $x^{k+1} \notin L_f(y^k)$. Denote $\bar{x}^{k+1} := P_{L_f(y^k)}(x^{k+1})$. Then,

$$\begin{aligned}\|\bar{x}^{k+1}\| &= \|P_{L_f(y^k)}(x^{k+1})\| \\ &\leq \|P_{L_f(y^k)}(x^{k+1}) - P_{L_f(y^k)}(\bar{x})\| + \|P_{L_f(y^k)}(\bar{x})\| \\ &\leq \|x^{k+1} - \bar{x}\| + \|\bar{x}\| \leq \|x^{k+1}\| + 2\|\bar{x}\|,\end{aligned}$$

which implies that $\bar{x}^{k+1} \in K \cap \mathbf{B}(0,\rho)$. By (2.1), we have that

$$\langle x^{k+1} - \bar{x}^{k+1}, x - \bar{x}^{k+1}\rangle \leq 0 \quad \text{for each} \quad x \in L_f(y^k). \tag{4.7}$$

This implies that $x^{k+1} - \bar{x}^{k+1} \in N_{L_f(y^k)}(\bar{x}^{k+1})$. Let $z := \frac{x^{k+1} - \bar{x}^{k+1}}{\gamma\|x^{k+1} - \bar{x}^{k+1}\|}$. Then

$$z \in \frac{1}{\gamma}\mathbf{B}(0,1) \cap N_{L_f(y^k)}(\bar{x}^{k+1}). \tag{4.8}$$

Hence, by Proposition 2.3, we have that

$$z \in \partial([f(y^k,\cdot)]_+ + \delta_K)(\bar{x}^{k+1}).$$

Note that $\omega_k \in \partial([f(y^k,\cdot)]_+ + \delta_K)(x^{k+1})$. Then, by (2.5), we conclude that

$$\langle \omega_k - z, x^{k+1} - \bar{x}^{k+1}\rangle \geq 0, \tag{4.9}$$

which implies

$$\langle z, x^{k+1} - \bar{x}^{k+1}\rangle \leq \langle \omega_k, x^{k+1} - \bar{x}^{k+1}\rangle \leq \|\omega_k\| \cdot \|x^{k+1} - \bar{x}^{k+1}\|. \tag{4.10}$$

Thus, by the definition of z, we see that

$$\|\omega_k\| \geq \langle z, \frac{x^{k+1} - \bar{x}^{k+1}}{\|x^{k+1} - \bar{x}^{k+1}\|}\rangle = \frac{1}{\gamma}, \tag{4.11}$$

which contradicts with (4.5). Therefore, (4.6) holds.

Let $k \geq k_0$. Then $x^{k+1} \in L_f(y^k)$ by (4.6). By step 2 of Algorithm 1.3, we see that

$$[f(y^k,x)]_+ + \frac{1}{2\alpha_k}\|x - x^k\|^2 \geq [f(y^k,x^{k+1})]_+ + \frac{1}{2\alpha_k}\|x^{k+1} - x^k\|^2 \quad \text{for each} \quad x \in K. \tag{4.12}$$

Note that $x^{k+1} \in L_f(y^k)$ and that $\alpha_k > 0$. It follows from (4.12) that

$$\|x - x^k\|^2 \geq \|x^{k+1} - x^k\|^2 \quad \text{for each} \quad x \in L_f(y^k), \tag{4.13}$$

which implies that

$$x^{k+1} = \arg\min_{x \in L_f(y^k)} \|x - x^k\| = P_{L_f(y^k)}(x^k).$$

The proof is complete. □

Lemma 4.3. *Let $\{x^k\}$ and $\{y^k\}$ be two sequences generated by Algorithm 1.3. Suppose that the conditions of Lemma 4.2 hold. Then, for sufficiently large k, the following relations hold:*

$$\|x^{k+1} - x^k\|^2 = \|P_{L_f(y^k)}(x^k) - x^k\|^2 \leq \mathrm{d}^2(x^k, L) - \mathrm{d}^2(x^{k+1}, L). \tag{4.14}$$

Proof. By Lemma 4.2, there exists $k_0 \in \mathbb{N}$ such that (4.3) holds. Let $k \geq k_0$. Then

$$\begin{aligned}
\mathrm{d}^2(x^k, L) - \mathrm{d}^2(x^{k+1}, L) &\geq \|x^k - P_L(x^k)\|^2 - \|x^{k+1} - P_L(x^k)\|^2 \\
&= \|x^k - P_L(x^k)\|^2 - \|P_{L_f(y^k)}(x^k) - x^k + x^k - P_L(x^k)\|^2 \\
&= \|P_{L_f(y^k)}(x^k) - x^k\|^2 \\
&\quad -2\langle P_{L_f(y^k)}(x^k) - x^k, P_{L_f(y^k)}(x^k) - P_L(x^k)\rangle \\
&\geq \|P_{L_f(y^k)}(x^k) - x^k\|^2 \\
&= \|x^{k+1} - x^k\|^2,
\end{aligned}$$

where the last inequality holds by (2.1). Hence, (4.14) holds and the proof is complete. □

To establish the linear convergence result for Algorithm 1.3, we still need the following assumption, which was used in [19, page 451].

P3. For any $\rho > 0$ there exists $l_\rho > 0$ such that

(4.15)

$$|f(y, x') - f(y, x'')| \leq l_\rho \|x' - x''\| \quad \text{for any } x', x'' \in K \cap \mathbf{B}(0, \rho) \text{ and } y \in K.$$

Now we are ready to give the main theorem of the present section.

Theorem 4.4. *Let $\{x^k\}$ and $\{y^k\}$ be two sequences generated by Algorithm 1.3. Suppose that $L := \cap_{k=0}^{\infty} L_f(y^k) \neq \emptyset$ and assumption* P3 *holds. Suppose further that the parameter inequality system* (4.1) *has uniform bounded error bounds and that the inequality system* (2.6) *with $F(\cdot) := \sup_{k\in\mathbb{N}} f(y^k, \cdot)$ has bounded error bounds. Then the sequence $\{x^k\}$ converges linearly to a solution of EP, that is, there exist constants $C > 0$ and $q \in (0, 1)$ such that*

$$\|x^k - x^*\| \leq Cq^k \quad \textit{for each} \quad k \in \mathbb{N}. \tag{4.16}$$

Proof. By Lemma 3.3, the sequence $\{x_k\}$ is Fejér monotone w.r.t L so $\{x^k\}$ is bounded by Lemma 2.1(i). Let $\rho := \sup\{\|x^k\|\} + 1$. Then $\rho < +\infty$. Since the inequality system (2.6) with $F(\cdot) := \sup_{k\in\mathbb{N}} f(y^k, \cdot)$ has bounded error bounds, it follows that there exists $\gamma > 0$ such that

$$\mathrm{d}(x^k, L) \leq \gamma \sup_{j\in\mathbb{N}} [f(y^j, x^k)]_+ \leq \gamma \sup_{y\in K} [f(y, x^k)]_+ \leq \frac{\gamma}{\eta} f(y^k, x^k), \tag{4.17}$$

where the last inequality holds by Step 1 of Algorithm 1.3. On the other hand, by Lemma 4.2, there exists $k_0 \in \mathbb{N}$ such that (4.3) holds. Let $k \geq k_0$ and note that $x^k, x^{k+1} \in \mathbf{B}(0, \rho)$. Then, by (4.3) and assumption P3, there exits l_ρ such that

$$f(y^k, x^k) \leq |f(y^k, x^k) - f(y^k, x^{k+1})| \leq l_\rho \|x^{k+1} - x^k\|. \tag{4.18}$$

This together with (4.17) implies that

$$\mathrm{d}(x^k, L) \leq \frac{\gamma l_\rho}{\eta} \|x^{k+1} - x^k\|. \tag{4.19}$$

Hence, by Lemma 3.3, we see that

$$\lim_{k\to\infty} \mathrm{d}(x^k, L) = 0. \tag{4.20}$$

Consider

$$\Omega_k := \bigcap_{m=0}^{k} \mathbf{B}(P_L(x^m), \mathrm{d}(x^m, L)). \tag{4.21}$$

Note by Lemma 3.3 that the sequence $\{x^m\}$ is Fejér monotone w.r.t L. Then

$$\|x^k - P_L(x^m)\| \leq \|x^m - P_L(x^m)\| = \mathrm{d}(x^m, L) \quad \text{for each} \quad m \leq k. \tag{4.22}$$

This implies that $x^k \in \Omega_k$. Hence, Ω_k is a nonempty and bounded closed convex set with the properties $\Omega_{k+1} \subseteq \Omega_k$. Therefore, $\cap_{k=0}^{\infty} \Omega_k \neq \emptyset$. Taking $x^* \in \cap_{k=0}^{\infty} \Omega_k$. Then, $x^* \in \mathbf{B}(P_L(x^k), \mathrm{d}(x^k, L))$ and hence

$$\|x^k - x^*\| \leq \|x^k - P_L(x^k)\| + \|P_L(x^k) - x^*\| \leq 2\mathrm{d}(x^k, L). \tag{4.23}$$

This together with (4.20) implies that $\lim_{k\to\infty} \|x^k - x^*\| = 0$. Moreover, by Step 1 of Algorithm 1.3 and (4.18),

$$\eta f(y, x^k) \leq \eta \sup_{y\in K} f(y, x^k) \leq f(y^k, x^k) \leq l_\rho \|x^{k+1} - x^k\| \quad \text{for each} \quad y \in K. \tag{4.24}$$

Taking limits on both sides of (4.24), we conclude from the continuity of f that $f(y, x^*) \leq 0$ for each $y \in K$, that is, $x^* \in \cap_{y\in K} L_f(y)$. This means that x^* is a solution of *CFP*. Thus, by Proposition 1.1, x^* is a solution of *EP*.

Let $\lambda := \frac{\gamma l_\rho}{\eta}$. Without loss of generality, we assume that $\lambda > 1$. Denote $C := 2\mathrm{d}(x_0, L)$ and $q := \frac{\sqrt{\lambda^2-1}}{\lambda}$. Below we show that (4.16) holds. To do this, by (4.19) and Lemma 4.3, we have

$$\frac{1}{\lambda^2}\mathrm{d}^2(x^k, L) \leq \|x^{k+1} - x^k\|^2 \leq \mathrm{d}^2(x^k, L) - \mathrm{d}^2(x^{k+1}, L).$$

This implies that

$$\mathrm{d}(x^{k+1}, L) \leq \left(1 - \frac{1}{\lambda^2}\right)^{\frac{1}{2}} \mathrm{d}(x^k, L) = \left(\frac{\sqrt{\lambda^2 - 1}}{\lambda}\right) \mathrm{d}(x^k, L).$$

Thus, by induction, one has that

$$\mathrm{d}(x^k, L) \leq \mathrm{d}(x^0, L) \left(\frac{\sqrt{\lambda^2 - 1}}{\lambda}\right)^k.$$

Combining with (4.23), we get

$$\|x^k - x^*\| \le 2\mathrm{d}(x^0, L)\left(\frac{\sqrt{\lambda^2-1}}{\lambda}\right)^k = Cq^k.$$

Hence, (4.16) holds and the proof is complete. □

Remark 4.5. Note that if the solution set of *CFP* is empty, Algorithm 1.2, Algorithm 1.3 and Algorithm IS may not be convergent. For example, let $K = [0, 2]$ and define $f(x, y) := (x - y)^2$ for each $(x, y) \in K \times K$. Then the solution set of *CFP* is empty, but the solution set of *EP* is K. In this case, all of the three algorithms oscillate between two points.

5. Numerical experiments

In this section, by applying Matlab (version R2011b), we report some numerical examples to test Algorithm 1.2 and Algorithm 1.3. For comparison purpose, we also test Algorithm IS of Iusem and Sosa [15]. Throughout the computational experiments, the parameters ε_k and η appearing in these algorithms are set as $\varepsilon_k \equiv 0$ and $\eta = 0.95$.

Example 5.1. Let

$$K := \left\{ x \in \mathbb{R}^5 \,\middle|\, \sum_{i=1}^{5} x_i \ge 10, x_i \ge 0, i = 1, \ldots, 5 \right\}. \tag{5.1}$$

Define $f : K \times K \to \mathbb{R}$ by

$$f(x, y) := \langle Mx + 10c(x) + q, y - x\rangle \quad \text{for each} \quad (x, y) \in K \times K, \tag{5.2}$$

where

$$M := \begin{pmatrix} 0.726 & -0.949 & 0.266 & -1.193 & -0.504 \\ 1.645 & 0.678 & 0.333 & -0.217 & -1.443 \\ -1.016 & -0.225 & 0.769 & 0.934 & 1.007 \\ 1.063 & 0.567 & -1.144 & 0.550 & -0.548 \\ -0.259 & 1.453 & -1.073 & 0.509 & 1.026 \end{pmatrix}, \quad q := \begin{pmatrix} 5.308 \\ 0.008 \\ -0.938 \\ 1.024 \\ -1.312 \end{pmatrix}, \tag{5.3}$$

and

$$c(x) := (\max\{x_1 - 2, 0\}, \ldots, \max\{x_5 - 2, 0\})^T \quad \text{for each } x \in K.$$

Then $x^* = (2, 2, 2, 2, 2)^T$ is a solution for *EP*. Furthermore, by definition, one can check that f is strongly monotone and so x^* is the unique solution by [12, Corollary 3.2.] as the *EP* is a variational inequality problem. Choose $x^0 := (1, 2, 3, 4, 5)^T$. Numerical results with the starting point are listed in Tables 1 and 2; while comparisons of Algorithms 1.2 and 1.3 with Algorithm IS are illustrated in Figures 1 and 2 below.

Example 5.2. Let

$$K := \left\{ x \in \mathbb{R}^{10} \,\middle|\, \|x\|_2 \le 100, x_i \ge -9, i = 1, \ldots, 10 \right\}. \tag{5.4}$$

TABLE 1. Errors $\|x^k - x^*\|$ for Algorithms 1.2 and IS.

k	Alg.1.2			Alg.IS	
	$\alpha_k \equiv 0.01$	$\alpha_k \equiv 0.5$	$\alpha_k \equiv 100$	$\lambda_k \equiv 0.5$	$\lambda_k \equiv 1$
1	3.5503	2.7340	3.6728	3.2949	2.7341
2	3.3111	1.4891	2.4162	2.4568	1.4892
3	3.0844	1.0946	1.4317	1.8549	1.0019
4	2.8699	5.1560e-01	7.7386e-01	1.4936	7.4009e-01
5	2.6713	3.5614e-01	3.4526e-01	1.2569	4.9564e-01
6	2.4885	2.0866e-01	2.1723e-01	1.0491	3.5022e-01
7	2.3539	1.4815e-01	1.3176e-01	9.0485e-01	2.5577e-01
10	1.9444	3.9883e-02	2.4972e-02	5.8358e-01	9.0874e-02
20	1.2044	3.7463e-04	7.9685e-05	1.1251e-01	3.0936e-03
30	9.4952e-01	3.7927e-07	1.8253e-07	2.4514e-02	3.7593e-05
40	7.7826e-01	6.4590e-09	2.3059e-09	5.9900e-03	4.9637e-07

TABLE 2. Errors $\|x^k - x^*\|$ for Algorithms 1.3 and IS.

k	Alg.1.3			Alg.IS	
	$\alpha_k \equiv 0.01$	$\alpha_k \equiv 0.5$	$\alpha_k \equiv 100$	$\lambda_k \equiv 0.5$	$\lambda_k \equiv 1$
1	3.6178	1.8333	2.8749	3.2949	2.7341
2	3.3755	1.3859	1.9970	2.4568	1.4892
3	3.1457	9.6625e-01	8.3107e-01	1.8549	1.0019
4	2.9282	7.1984e-01	4.2486e-01	1.4936	7.4009e-01
5	2.7255	4.1155e-01	2.2608e-01	1.2569	4.9564e-01
6	2.5390	3.0624e-01	1.4950e-01	1.0491	3.5022e-01
7	2.3678	1.8373e-01	9.2021e-02	9.0485e-01	2.5577e-01
10	1.9381	4.6477e-02	1.2608e-02	5.8358e-01	9.0874e-02
20	1.2263	4.8397e-04	4.0673e-05	1.1251e-01	3.0936e-03
30	9.7785e-01	1.5438e-07	1.4640e-07	2.4514e-02	3.7593e-05
40	8.0065e-01	3.1454e-09	2.2201e-09	5.9900e-03	4.9637e-07

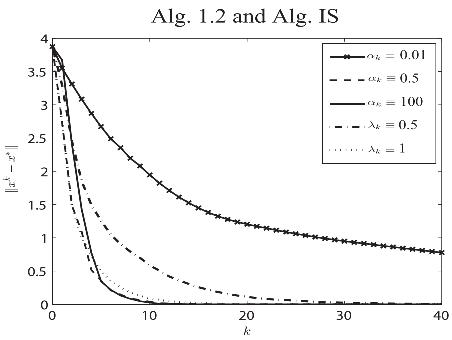

FIGURE 1. Comparison of Alg. 1.2 with IS

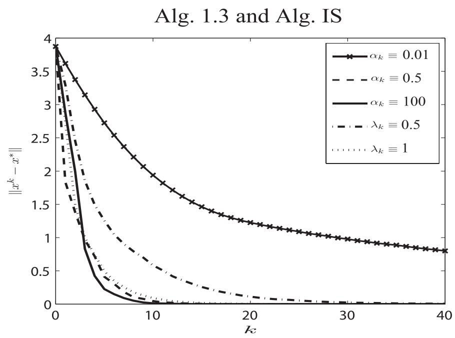

FIGURE 2. Comparison of Alg. 1.3 with IS

Define $f : K \times K \to \mathbb{R}$ by

(5.5)
$$f(x, y) := \max_{1\le i\le 10}\{a_i^T(y - x)\} + \max_{1\le i\le 10}\{b_i^T(y - x)\} \quad \text{for each} \quad (x, y) \in K \times K,$$

where $a_i, b_i \in \mathbb{R}^{10}, 1 \le i \le 10$, are chosen by

$$[a_1, a_2, \ldots, a_{10}] = A = \begin{pmatrix} 0.8 & 1.5 & 0.3 & 1.4 & 1.3 & 0.7 & 0.9 & 1.1 & 1.0 & 0.8 \\ 1.3 & 0.8 & 0.5 & 0.2 & 0.1 & 0.2 & 0.1 & 0 & 0.8 & 0.7 \\ 0.5 & 1.1 & 0.1 & 1.1 & 0.4 & 1.5 & 0.1 & 0.1 & 0.6 & 1.3 \\ 0.7 & 1.5 & 0.8 & 1.1 & 0.1 & 0.5 & 1.2 & 1.0 & 0.1 & 1.2 \\ 0.1 & 0.4 & 0.5 & 0.8 & 0.7 & 0.4 & 1.4 & 0.9 & 1.2 & 1.1 \\ 0.3 & 0.6 & 0.3 & 0.3 & 0 & 0.1 & 0.8 & 0.8 & 0.5 & 0.1 \\ 1.0 & 0.7 & 0.3 & 0.9 & 1.3 & 0.4 & 0.2 & 1.1 & 0.9 & 0.1 \\ 0.5 & 1.1 & 1.4 & 0.4 & 0.3 & 0.1 & 1.2 & 1.1 & 1.1 & 0.1 \\ 1.3 & 1.2 & 1.0 & 0.2 & 0.1 & 0.8 & 0.5 & 1.2 & 0.2 & 1.2 \\ 0.2 & 0.2 & 0.7 & 0.3 & 0.5 & 1.1 & 0.4 & 0.4 & 0.2 & 1.4 \end{pmatrix}$$

and

$$[b_1, b_2, \ldots, b_{10}] = A^T.$$

Then, we can check by definition that $x^* = (-9, -9, \ldots, -9)$ is the unique solution of *CFP*. Choose $x^0 := (0, 0, 0, 0, 0)^T$. Numerical results with this starting point are listed in Tables 3 and 4; while comparisons of Algorithms 1.2 and 1.3 with Algorithm IS are illustrated in Figures 3 and 4 below.

TABLE 3. Errors $\|x^k - x^*\|$ for Algorithms 1.2 and IS.

k	Alg.1.2			Alg.IS	
	$\alpha_k \equiv 0.1$	$\alpha_k \equiv 10$	$\alpha_k \equiv 100$	$\lambda_k \equiv 0.5$	$\lambda_k \equiv 1$
1	2.7863e+01	2.7502e+01	2.7502e+01	2.7981e+01	2.7502e+01
2	2.7302e+01	2.6547e+01	2.6547e+01	2.7024e+01	2.6547e+01
3	2.6849e+01	2.5303e+01	2.4999e+01	2.6472e+01	2.5293e+01
4	2.6360e+01	2.4066e+01	2.3243e+01	2.5517e+01	2.4058e+01
5	2.5814e+01	2.3001e+01	2.1885e+01	2.4842e+01	2.2931e+01
6	2.5262e+01	2.1710e+01	2.0680e+01	2.4234e+01	2.1688e+01
7	2.4768e+01	2.0444e+01	1.8968e+01	2.3681e+01	2.0319e+01
16	2.0140e+01	4.6319	1.9309e-01	1.8059e+01	4.3691
17	1.9645e+01	1.4614	0	1.7353e+01	2.2605
18	1.9153e+01	0	0	1.6560e+01	0
30	1.3480e+01	0	0	5.3413	0

TABLE 4. Errors $\|x^k - x^*\|$ for Algorithms 1.3 and IS.

k	Alg.1.3			Alg.IS	
	$\alpha_k \equiv 0.1$	$\alpha_k \equiv 10$	$\alpha_k \equiv 100$	$\lambda_k \equiv 0.5$	$\lambda_k \equiv 1$
1	2.7863e+01	0	0	2.7981e+01	2.7502e+01
2	2.7267e+01	0	0	2.7024e+01	2.6547e+01
3	2.6673e+01	0	0	2.6472e+01	2.5293e+01
4	2.6079e+01	0	0	2.5517e+01	2.4058e+01
5	2.5487e+01	0	0	2.4842e+01	2.2931e+01
6	2.4897e+01	0	0	2.4234e+01	2.1688e+01
7	2.4308e+01	0	0	2.3681e+01	2.0319e+01
16	1.9103e+01	0	0	1.8059e+01	4.3691
17	1.8539e+01	0	0	1.7353e+01	2.2605
18	1.7979e+01	0	0	1.6560e+01	0
30	1.1733e+01	0	0	5.3413	0

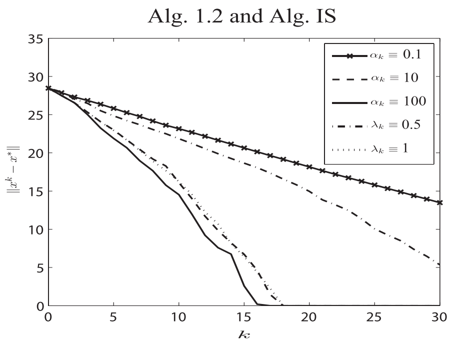

FIGURE 3. Comparison of Alg. 1.2 with IS

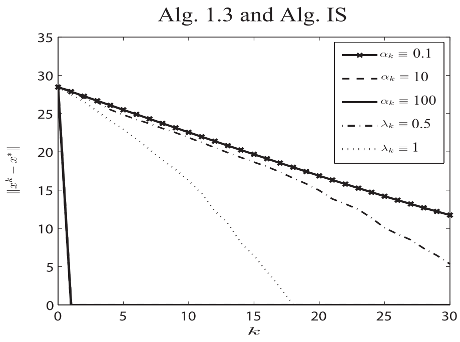

FIGURE 4. Comparison of Alg. 1.3 with IS

From the numerical results, we conclude the following conclusions:

• Performances of our Algorithms 1.2 and 1.3 depend on the choices of the parameters $\{\alpha_k\}$. In particular, the larger the parameters $\{\alpha_k\}$, the better the behavior of Algorithms 1.2 and Algorithm 1.3.

• For the choices of parameters $\{\alpha_k\}$ that are far from zero, the performances of our Algorithms 1.2 and 1.3 are better than Algorithm IS.

• Algorithm 1.3 would have very good behavior for some special EP, such as ones determined by polyhedral functions, and it could terminate in a finite numbers of iterations as shown by Example 5.2.

REFERENCES

[1] H. H. Bauschke and J. M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Review **38** (1996), 367–426.

[2] J. V. Burke and M. C. Ferris, *Weak sharp minima in mathematical programming*, SIAM J. Control Optim. **31** (1993), 1340–1359.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Stud. **63** (1994), 123–145.

[4] P. L. Combettes, *Hilbertian convex feasibility problem: convergence of projection methods*, Appl. Math. Optim. **35** (1997), 311–330.

[5] Y. Censor and S. Zenios, *Parallel Optimization, Theory, Algorithms and Applications*, Oxford University Press, Oxford, 1997.

[6] S. Deng, *Global error bounds for convex inequality systems in Banach spaces*, SIAM J. Control Optim. **36** (1998), 1240–1249.

[7] M. C. Ferris, *Finite termination of the proximal point algorithm*, Math. Program. **50** (1991), 359–366.

[8] S. D. Flåm and A. S. Antipin, *Equilibrium programming using proximal-like algorithms*, Math. Program. **78** (1997), 29–41.

[9] L. G. Gubin, B. T. Polyak and E. V. Raik, *The method of projections for finding the common point of convex sets*, Comput. Math. Math. Phys. **7** (1967), 1–24.

[10] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms II*, Springer, Berlin, 1993.

[11] A. J. Hoffman, *On approximate solutions of systems of linear inequalities*, J. Res. Natl. Bur. Stand. **49** (1952), 263–265.

[12] P. T. Harker and J. S. Pang, *Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications*, Math. Program. **48** (1990), 161–220.

[13] A. N. Iusem, G. Kassay and W. Sosa, *On certain conditions for the existence of solutions of equilibrium problems*, Math. Program. **116** (2009), 259–273.
[14] A. N. Iusem and M. Nasri, *Inexact proximal point methods for equilibrium problems in Banach spaces*, Numer. Funct. Anal. Optim. **28** (2007), 1279–1308.
[15] A. N. Iusem and W. Sosa, *Iterative algorithms for equilibrium problems*, Optim. **52** (2003), 301–316.
[16] A. N. Iusem and W. Sosa, *New existence results for equilibrium problems*, Nonlinear Anal. **52** (2003), 621–635.
[17] I. V. Konnov, *Application of the proximal point method to nonmonotone equilibrium problems*, *J. Optim. Theory Appl.* **119** (2003), 317–333.
[18] A. Kruger, V. N. Huynh and M. Théra, *Stability of error bounds for convex constraint systems in Banach spaces*, SIAM J. Optim. **20** (2010), 3280–3296.
[19] B. S. Mordukhovich, *Variational Analysis and Generalized Differentiation, I: Basic Theory*, Springer, Berlin, 2006.
[20] L. D. Muu and W. Oettli, *Convergence of an adaptive penalty scheme for finding constrained equilibria*, Nonlinear Anal. **18** (1992), 1159–1166.
[21] B. S. Mordukhovich, B. Panicucci, M. Pappalardo and M. Passacantando, *Hybrid proximal methods for equilibrium problems*, Optim. Lett. **6** (2012), 1535–1550.
[22] T. T. V. Nguyen, J. J. Strodiot and V. H. Nguyen, *A bundle method for solving equilibrium problems*, Math. Program. **116** (2009), 529–552.
[23] K. Taji, M. Fukushima and T. Ibaraki, *A globally convergent Newton method for solving strongly monotone variational inequalities*, Math. Program. **58** (1993), 369–383.
[24] J. S. Pang, *Error bounds in mathematical programming*, Math. Program. **79** (1997), 299–332.
[25] S. M. Robinson, *An application of error bounds for convex programming in a linear space*, SIAM J. Control Optim. **13** (1975), 271–273.
[26] S. Simons, *Subdifferentials are locally maximal monotone*, Bull. Austral. Math. Soc. **47** (1993), 465–471.
[27] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific, New Jersey, 2002.

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