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MULTIPLICITY OF SOLUTIONS FOR A CLASS OF NONLINEAR NONHOMOGENEOUS ELLIPTIC EQUATIONS

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Dedicated to Professor Simeon Reich on the occasion of his 65th birthday

Abstract. We consider nonlinear, nonhomogeneous Dirichlet problems driven by the sum of a $p-$Laplacian ($p > 2$) and a Laplacian, with a reaction term which has space dependent zeros of constant sign. We prove three muliplicity theorems for such equations providing precise sign information for all solutions. In the first multiplicity theorem, we do not impose any growth condition on the reaction near $\pm\infty$. In the other two, we assume that the reaction is $(p-1)-$ linear and resonant with respect to principal eigenvalue of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$. Our approach uses variational methods based on the critical point theory, together with suitable truncation and comparison techniques and Morse theory (critical groups).

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2- boundary $\partial\Omega$. In this paper, we study the following nonlinear Dirichlet problem

$$-\triangle_p u(z) - \triangle u(z) = f(z, u(z)) \text{ in } \Omega,\ u\mid_{\partial\Omega}= 0,\ 2 < p < \infty. \tag{1.1}$$

Here $\triangle_p$ denotes the the $p-$Laplace differential operator, defined by

$$\triangle_p u = div\left(\|Du\|_{\mathbb{R}^N}^{p-2} Du\right), \text{ for all } u \in W_0^{1,p}(\Omega).$$

The reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous). We assume that $f(z, .)$ has $z-$dependent zeros of constant sign.

Our aim is to obtain multiplicity results providing precise sign information for all the solutions produced. Specifically, we prove a multiplicity theorem in which no growth control at $\pm\infty$ is imposed on $f(z, .)$, and two multiplicity theorems in which $f(z, .)$ exhibits a $(p-1)-$linear growth near $\pm\infty$ and is resonant with respect to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$ of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$.

Recently, equations involving the sum of a $p-$Laplacian and a Laplacian (a $(p, 2)-$equation for short), were investigated by Aizicovici-Papageorgiou-Staicu [1],

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Cingolani-Degiovanni [8], Cingolani-Vannella [9] and Sun [17]. Cingolani-Degiovanni [8] and Cingolani-Vannella [9] prove only existence theorems, while Sun [17] proves a "three solutions theorem" under the condition that $f \in C^1\left(\overline{\Omega} \times \mathbb{R}\right)$, and does not produce nodal solutions. Nodal solutions are obtained in Aizicovici-Papageorgiou-Staicu [1] for coercive nonresonant problems. Here, the multiplicity results of Section 4 concern indefinite (i.e., noncoercive) resonant equations. We mention that $(p, 2)-$equations arise in quantum physics in the search of solitons (see Benci-D'Avenia-Fortunato-Pisani [6]). We stress that in contrast to the equations driven by the $p-$Laplacian, the differential operator in (1.1) is not homogeneous. The lack of homogeneity is the source of serious difficulties especially when we look for nodal solutions.

Our approach uses variational methods based on the critical point theory coupled with suitable truncation and comparison techniques and Morse theory (critical groups). In the next section, we briefly recall the main mathematical tools which we will use in the sequel.

2. Mathematical background

In the analysis of problem (1.1), in addition to the Sobolev space $W_0^{1,p}(\Omega)$, we will use the Banach space

$$C_0^1\left(\overline{\Omega}\right) := \left\{u \in C^1\left(\overline{\Omega}\right) : u\mid_{\partial\Omega} = 0\right\}.$$

This is an ordered Banach space with positive cone

$$C_+ = \left\{u \in C_0^1\left(\overline{\Omega}\right) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior, given by

$$int\ C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n} < 0 \text{ for all } z \in \partial\Omega\right\},$$

where $n(.)$ denotes the outward unit normal on $\partial\Omega$.

Let $f_0 : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function with subcritical growth in $x \in \mathbb{R}$, i.e.,

$$|f_0(z,x)| \leq \alpha(z)\left(1 + |x|^{r-1}\right) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $\alpha \in L^\infty(\Omega)_+$, $p \leq r < p^*$, where p^* is the critical Sobolev exponent, i.e.,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N. \end{cases}$$

Let $F_0(z,x) = \int_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\psi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_0(z,u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

The next result relates local $C_0^1\left(\overline{\Omega}\right)$ and $W_0^{1,p}(\Omega)$ minimizers of ψ_0 and can be found in Aizicovici-Papageorgiou-Staicu [4]. (We recall that $C_0^{1,\beta}\left(\overline{\Omega}\right)$, with $\beta \in (0,1)$, stands for the subspace of all functions of $C_0^1\left(\overline{\Omega}\right)$ whose first-order partial derivatives are Hölder continuous with exponent β).

Proposition 2.1. *If* $u_0 \in W_0^{1,p}(\Omega)$ *is a local* $C_0^1\left(\overline{\Omega}\right)-$ *minimizer of* ψ_0 *(i.e., there exists* $\rho_0 > 0$ *such that* $\psi_0(u_0) \leq \psi_0(u_0 + h)$ *for all* $h \in C_0^1\left(\overline{\Omega}\right)$ *with* $\|h\|_{C_0^1(\overline{\Omega})} \leq \rho_0$*), then* $u_0 \in C_0^{1,\beta}\left(\overline{\Omega}\right)$ *for some* $\beta \in (0,1)$ *and it is a local* $W_0^{1,p}(\Omega)-$ *minimizer of* ψ_0 *(i.e., there exists* $\rho_1 > 0$ *such that* $\psi_0(u_0) \leq \psi_0(u_0 + h)$ *for all* $h \in W_0^{1,p}(\Omega)$ *with* $\|h\| \leq \rho_1$*).*

Hereafter, by $\|.\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$, and by $\|.\|_q$ we denote the norm of $L^q(\Omega)$ or $L^q\left(\Omega, \mathbb{R}^N\right)$, $1 \leq q \leq \infty$. By virtue of Poincaré's inequality, we have

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also, by $\|.\|$ we denote the $\mathbb{R}^N$-norm. However, no confusion is possible, since it will be clear from the context which norm is used.

For every $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then for every $u \in W_0^{1,p}(\Omega)$, we define $u^{\pm}(.) = u(.)^{\pm}$. We know that

$$u^{\pm} \in W_0^{1,p}(\Omega), \ |u| = u^+ + u^-, \ u = u^+ - u^-.$$

If $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example a Carathéodory function), then we define

$$N_h(u)(.) = h(., u(.)) \text{ for all } u \in W_0^{1,p}(\Omega).$$

(the Nemytskii operator corresponding to the function $h(.,.)$). By $|.|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

Let $h_1,\ h_2 \in L^\infty(\Omega)$. We write $h_1 \prec h_2$ if, for any compact set $K \subset \Omega$, we can find $\varepsilon = \varepsilon(K) > 0$ such that

$$h_1(z) + \varepsilon \leq h_2(z) \text{ for a.a. } z \in K.$$

Evidently, if $h_1,\ h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$.

The next proposition is essentially due to Arcoya-Ruiz [5], where only the $p-$Laplacian ($1 < p < \infty$) is present. A minor modification of their proof in order to accommodate the linear term $-\triangle u$ leads to the following strong comparison principle:

Proposition 2.2. *If* $\xi \geq 0$, $h_1,\ h_2 \in L^\infty(\Omega)$, $h_1 \prec h_2$ *and* $u, v \in C_0^1\left(\overline{\Omega}\right)$ *are solutions of*

$$-\triangle_p u(z) - \triangle u(z) + \xi |u(z)|^{p-2} u(z) = h_1(z) \text{ in } \Omega,$$
$$-\triangle_p v(z) - \triangle v(z) + \xi |v(z)|^{p-2} v(z) = h_2(z) \text{ in } \Omega,$$

with $v \in int\ C_+$, *then* $v - u \in int\ C_+$.

Let $(X, \|.\|)$ be a Banach space and X^* be its topological dual. By $\langle .,. \rangle$ we denote the duality brackets for the pair (X^*, X). Also $\xrightarrow{w}$ will designate weak convergence in X.

Given $\varphi \in C^1(X)$, we say that φ satisfies the *Cerami condition* (*$C-$condition*, for short), if the following is true:

"every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$

admits a strongly convergent subsequence."

This compactness-type condition, which in general is weaker than the more common Palais-Smale condition leads to a deformation theorem from which one may derive the minimax theory of certain critical values of $\varphi \in C^1(X)$. In particular, we have the following result, known in the literature as the "mountain pass theorem".

Theorem 2.3. *If $\varphi \in C^1(X)$ satisfies the $C-$ condition, $x_0, x_1 \in X$ and $\rho > 0$ are such that $\|x_1 - x_0\| > \rho > 0$, $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} =: \eta_\rho$, and $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\varphi(\gamma(t))$, where*

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geq \eta_\rho$ and c is a critical value of φ (i.e., there exists $x^* \in X$ such that $\varphi'(x^*) = 0$ and $\varphi(x^*) = c$).

Let (Y_2, Y_1) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}- relative singular homology group with integer coefficients for the pair (Y_1, Y_2).

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\};$$
$$K_\varphi = \{x \in X : \varphi'(x) = 0\};$$
$$K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

If $x \in X$ is an isolated critical point of φ with $\varphi(x) = c$ (i.e., $x \in K_\varphi^c$), then the *critical groups of* φ at x are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x\}), \text{ for all } k \geq 0,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$.

The excision property of the singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and $\inf\varphi(K_\varphi) > -\infty$. Let $c < \inf\varphi(K_\varphi)$. The *critical groups of* φ *at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0.$$

The second deformation theorem (see, for example, Gasinski-Papageorgiou [12], p.628) implies that this definition is independent of the choice of the level $c < \inf\varphi(K_\varphi)$.

Suppose K_φ is finite. We set

$$M(t, x) = \sum_{k\geq 0} rank\ C_k(\varphi, x)\, t^k \text{ for all } t \in \mathbb{R}, \text{ all } x \in K_\varphi$$

and

$$P(t, \infty) = \sum_{k\geq 0} rank\ C_k(\varphi, \infty)\, t^k \text{ for all } t \in \mathbb{R}.$$

The *Morse relation* says that

$$\sum_{x\in K_\varphi} M(t, x) = P(t, \infty) + (1 + t)\, Q(t) \text{ for all } t \in \mathbb{R}, \tag{2.1}$$

where $Q(t) = \sum_{k\geq 0} \xi_k t^k$ is a formal series with nonnegative integer coefficients.

Let $r \in (1, \infty)$ and let $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* \left(\frac{1}{r} + \frac{1}{r'} = 1\right)$, be the nonlinear map defined by

$$\langle A_r(u), y\rangle = \int_\Omega \|Du\|^{r-2} (Du, Dy)_{\mathbb{R}^N}\, dz \text{ for all } u,\ y \in W_0^{1,r}(\Omega). \tag{2.2}$$

If $r = 2$, then we write $A := A_2 \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)$.

The following result is well-known (see, e.g., [2]).

Proposition 2.4. *If $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ is the nonlinear map defined by (2.2), then A_r is continuous, bounded (i.e., it maps bounded sets to bounded sets), strictly monotone (hence maximal monotone), and of type $(S)_+$ (i.e., if $\{u_n\}_{n\geq 1}$ is such that $u_n \xrightarrow{w} u$ in $W_0^{1,r}(\Omega)$ and*

$$\limsup_{n\to\infty} \langle A_r(u_n), u_n - u\rangle \leq 0,$$

one has $u_n \to u$ in $W_0^{1,r}(\Omega)$).

Finally let us recall some basic facts about the spectrum of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$ with $1 < p < \infty$; see e.g., [12]. So, let $m \in L^\infty(\Omega)$, $m \geq 0$, $m \neq 0$ and consider the following nonlinear weighted eigenvalue problem:

$$-\triangle_p u(z) = \widehat{\lambda} m(z) |u(z)|^{p-2} u(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{2.3}$$

A number $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$, if problem (2.3) has a nontrivial solution $\widehat{u}$. Then $\widehat{u}$ is called an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$.

The smallest eigenvalue of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$ exists and is denoted by $\widehat{\lambda}_1(p,m)$. We know that $\widehat{\lambda}_1(p,m) > 0$, is isolated (i.e., there exists $\varepsilon > 0$ such that the interval $\left(\widehat{\lambda}_1(p,m), \widehat{\lambda}_1(p,m) + \varepsilon\right)$ contains no other eigenvalues), simple (i.e., if $\widehat{u}$ and $\widehat{v}$ are both eigenfunctions corresponding to the eigenvalue $\widehat{\lambda}_1(p,m)$, then $\widehat{u} = \xi\widehat{v}$ for some $\xi \in \mathbb{R}\setminus\{0\}$), and it admits the following variational characterization

$$\widehat{\lambda}_1(p,m) = \inf\left\{\frac{\|Du\|_p^p}{\int_\Omega m|u|^p\, dz} : u \in W_0^{1,p}(\Omega),\ u \neq 0\right\}. \tag{2.4}$$

The infimum in (2.4) is attained on the corresponding one dimensional eigenspace.

By $\widehat{u}_{1,p}(m)$ we denote the L^p- normalized (i.e., $\|\widehat{u}_{1,p}(m)\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(p,m)$.

From (2.4) we see that $\widehat{u}_{1,p}(m)$ does not change sign, and so, we may assume that $\widehat{u}_{1,p}(m) \geq 0$. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski-Papageorgiou ([12], p.737-738)) imply that

$$\widehat{u}_{1,p}(m) \in int\ C_+.$$

The Ljusternik-Schnirelmann minimax scheme provides an increasing sequence of distinct eigenvalues $\left\{\widehat{\lambda}_{k,p}(m)\right\}_{k\geq 1}$ of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$ such that $\widehat{\lambda}_{k,p}(m) \to +\infty$.

If $p = 2$ (linear eigenvalue problem) or $N = 1$ (ordinary differential equations), then this sequence exhausts the eigenvalues of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$. If $p \neq 2$ and $N \geq 2$, then we do not know if this is true.

We mention that $\widehat{\lambda}_{1,p}(m)$ is the only eigenvalue with eigenfunctions of constant sign.

If $m = 1$, then we write $\widehat{\lambda}_1(p) := \widehat{\lambda}_{1,p}(m) > 0$ and $\widehat{u}_{1,p} := \widehat{u}_{1,p}(m) \in int\ C_+$.

If $p = 2$ (linear eigenvalue problem), then we have the following orthogonal direct sum decomposition

$$H_0^1(\Omega) = \overline{\bigoplus_{k\geq 1} E\left(\widehat{\lambda}_k(2)\right)},$$

where $E\left(\widehat{\lambda}_k(2)\right)$ denotes the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k(2)$.

The elements of $E\left(\widehat{\lambda}_k(2)\right)$ $(k \geq 1)$ exhibit the so-called *Unique Continuation Property* (UCP for short), i.e., if $u \in E\left(\widehat{\lambda}_k(2)\right)$ and u vanishes on a set of positive measure, then $u \equiv 0$. By standard regularity theory, we have $E\left(\widehat{\lambda}_k(2)\right) \subseteq C_0^1\left(\overline{\Omega}\right)$ for all $k \geq 1$.

3. Multiplicity with no growth conditions at $\pm\infty$

In this section, we prove a three solutions theorem for equations in which no growth restriction is imposed on the reaction $x \to f(z,x)$. Instead, we assume that $f(z,.)$ admits $z-$dependent zeros of constant sign. More precisely, the hypotheses on the nonlinearity $f(z,x)$ are the following:

$\mathbf{H}^1(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$, and:

(i) for every $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$|f(z,x)| \leq a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) there exist functions $w_\pm \in W_0^{1,p}(\Omega) \cap C\left(\overline{\Omega}\right)$ and constants c_-, c_+ such that:

$w_-(z) \leq c_- < 0 < c_+ \leq w_+(z)$ for all $z \in \Omega$,

$f(z, w_+(z)) \leq 0 \leq f(z, w_-(z))$ for a.a. $z \in \Omega$,

$A_p(w_-) + A(w_-) \leq 0 \leq A_p(w_+) + A(w_+)$ in $W^{-1,p'}(\Omega)$;

(iii) there exists an integer $m \geq 2$ and a function $\eta \in L^\infty(\Omega)$ such that:

$$\widehat{\lambda}_m(2) \leq \eta(z) \leq \widehat{\lambda}_{m+1}(2) \text{ for a.a. } z \in \Omega,$$

$$\widehat{\lambda}_m(2) \neq \eta,\ \widehat{\lambda}_{m+1}(2) \neq \eta,$$

$$\lim_{x\to 0} \frac{f(z,x)}{x} = \eta(z) \text{ uniformly for a.a. } z \in \Omega.$$

Remarks: Hypotheses $\mathbf{H}^1(f)(ii)$ and (iii) imply that $f(z,.)$ has $z-$dependent zeros of constant sign. If

$$f(z,c_-)=0=f(z,c_+) \text{ for a.a. } z\in\Omega,$$

then hypothesis $\mathbf{H}^1(f)(ii)$ is satisfied with $w_+(z)=c_+$ and $w_-(z)=c_-$ for a.a. $z\in\Omega$. Hypothesis $\mathbf{H}^1(f)(iii)$ is a nonuniform nonresonance condition at zero with respect to any nonprincipal eigenvalue of $\left(-\triangle, H_0^1(\Omega)\right)$.

Note that no growth condition at $\pm\infty$ is imposed on $f(z,.)$.

Proposition 3.1. *If hypotheses* $\mathbf{H}^1(f)$ *hold, then problem* (1.1) *has at least three nontrivial distinct solutions*

$$u_0\in int\ C_+,\ v_0\in -int\ C_+,\ y_0\in C_0^1\left(\overline{\Omega}\right)\setminus\{0\},$$

with

$$w_-(z)<v_0(z)\le y_0(z)\le u_0(z)<w_+(z) \ \textit{for all } z\in\overline{\Omega}.$$

Proof. First, we produce the positive solution. To this end, we introduce the following truncation of $f(z,.)$:

$$\widehat{f}_+(z,x)=\begin{cases}0 & \text{if } x<0,\\ f(z,x) & \text{if } 0\le x\le w_+(z),\\ f(z,w_+(z)) & \text{if } w_+(z)<x.\end{cases} \tag{3.1}$$

This is a Carathéodory function. We set $\widehat{F}_+(z,x)=\int\limits_0^x\widehat{f}_+(z,s)\,ds$ and consider the C^1-functional $\widehat{\varphi}_+:W_0^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\widehat{\varphi}_+(u)=\frac{1}{p}\|Du\|_p^p+\frac{1}{2}\|Du\|_2^2-\int\limits_\Omega\widehat{F}_+(z,u(z))\,dz \text{ for all } u\in W_0^{1,p}(\Omega).$$

From (3.1) it is clear that $\widehat{\varphi}_+$ is coercive. Moreover, using the Sobolev embedding theorem, we see that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem we can find $u_0\in W_0^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0)=\inf\left\{\widehat{\varphi}_+(u):u\in W_0^{1,p}(\Omega)\right\}. \tag{3.2}$$

By virtue of hypothesis $\mathbf{H}^1(f)(iii)$, given $\varepsilon>0$, we can find $\delta=\delta(\varepsilon)\in(0,c_+]$ such that

$$\frac{1}{2}(\eta(z)-\varepsilon)x^2\le F(z,x) \text{ for a.a. } z\in\Omega,\ \text{all } |x|\le\delta, \tag{3.3}$$

where $F(z,x)=\int_0^x f(z,s)\,ds$. Let $t\in(0,1)$ be small such that $t\widehat{u}_{1,2}(z)\le\delta$ for all $z\in\overline{\Omega}$. Then

$$\begin{aligned}\widehat{\varphi}_+(t\widehat{u}_{1,2})&=\frac{t^p}{p}\|D\widehat{u}_{1,2}\|_p^p+\frac{t^2}{2}\|D\widehat{u}_{1,2}\|_2^2-\int\limits_\Omega\widehat{F}_+(z,\widehat{u}_{1,2}(z))\,dz\\ &\le\frac{t^p}{p}\|D\widehat{u}_{1,2}\|_p^p+\frac{t^2}{2}\left[\int\limits_\Omega\left(\widehat{\lambda}_1(2)-\eta(z)\right)\widehat{u}_{1,2}^2dz\right]+\frac{t^2\varepsilon}{2}\end{aligned}$$

(see (3.3) and (2.4)). Note that

$$\xi := \int_\Omega \left(\widehat{\lambda}_1(2) - \eta(z)\right) \widehat{u}_{1,2}^2 dz > 0$$

and so, choosing $\varepsilon \in (0, \xi)$ and taking $t \in (0,1)$ even smaller if necessary (recall that $p > 2$), we have

$$\widehat{\varphi}_+(t\widehat{u}_{1,2}) < 0.$$

Then $\widehat{\varphi}_+(u_0) < 0 = \widehat{\varphi}_+(0)$ (see (3.2)), hence

$$u_0 \neq 0.$$

From (3.2) we have

$$\widehat{\varphi}_+'(u_0) = 0,$$

and this implies

$$A_p(u_0) + A(u_0) = N_{\widehat{f}_+}(u_0). \tag{3.4}$$

On (3.4) we act with $-u_0^- \in W_0^{1,p}(\Omega)$ and obtain

$$\left\|Du_0^-\right\|_p^p + \left\|Du_0^-\right\|_2^2 = 0 \text{ (see (3.1))},$$

hence

$$u_0 \geq 0, u_0 \neq 0.$$

Also on (3.4) we act with $(u_0 - w_+)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\left\langle A_p(u_0), (u_0 - w_+)^+ \right\rangle + \left\langle A(u_0), (u_0 - w_+)^+ \right\rangle \\
&= \int_\Omega \widehat{f}_+(z, u_0)(u_0 - w_+)^+ dz \\
&= \int_\Omega f(z, w_+)(u_0 - w_+)^+ dz \text{ (see (3.1))} \\
&\leq \left\langle A_p(w_+), (u_0 - w_+)^+ \right\rangle + \left\langle A(w_+), (u_0 - w_+)^+ \right\rangle \text{ (see } \mathbf{H}^1(f)(ii)\text{)},
\end{aligned}$$

hence

$$\begin{aligned}
\int_{\{u_0 > w_+\}} &\left(\|Du_0\|^{p-2} Du_0 - \|Dw_+\|^{p-2} Dw_+, Du_0 - Dw_+\right)_{\mathbb{R}^N} dz \\
&+ \left\|D(u_0 - w_+)^+\right\|_2^2 \leq 0
\end{aligned}$$

and we get

$$u_0 \leq w_+ \text{ (see Proposition 2.4)}.$$

So, we have proved that

$$u_0 \in [0, w_+] = \left\{u \in W_0^{1,p}(\Omega) : 0 \leq u(z) \leq w_+(z) \text{ a.e. in } \Omega.\right\}$$

Hence, (3.4) becomes

$$A_p(u_0) + A(u_0) = N_f(u_0) \text{ (see (3.1))},$$

therefore

$$-\triangle_p u_0(z) - \triangle u_0(z) = f(z, u_0(z)) \text{ a.e. in } \Omega, \ u_0|_{\partial\Omega} = 0.$$

From Ladyzhenskaya-Uraltseva ([13], p. 286), we have that $u_0 \in L^\infty(\Omega)$. Then we apply the regularity result of Lieberman ([14], p.320) and infer that

$$u_0 \in C_+ \setminus \{0\}.$$

Hypotheses $\mathbf{H}^1(f)(i)$ and (iii) imply that there exists $\xi_* > 0$ such that

$$f(z,x) + \xi_* x^{p-1} \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } x \in [0, \|w_+\|_\infty].$$

Hence

$$\triangle_p u_0(z) + \triangle u_0(z) \leq \xi_* u_0(z)^{p-1} \geq 0 \text{ a.e. in } \Omega.$$

From the strong maximum principle of Pucci-Serrin ([16], p.34), we have

$$u_0(z) > 0 \text{ for all } z \in \Omega.$$

So, we can apply the boundary point theorem of Pucci-Serrin ([16], p.120) and conclude that

$$u_0 \in int\ C_+.$$

Let

$$a(y) = \|y\|^{p-2} y + y \text{ for all } y \in \mathbb{R}^N.$$

Then $a \in C^1\left(\mathbb{R}^N\right)$ (recall that $p > 2$) and we have

$$\nabla a(y) = \|y\|^{p-2}\left[I_N + (p-2)\frac{y \otimes y}{\|y\|^2}\right] + I_N \text{ for } y \neq 0.$$

Here I_N denotes the $N \times N$ identity matrix and for $a, b \in \mathbb{R}^N$, by $a \otimes b$ we denote the tensor $[a_i b_j]_{i,j=1}^N$. Then for all $\xi \in \mathbb{R}^N$ we have

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \|\xi\|^2.$$

Note that

$$div\ a(Du) = \triangle_p u + \triangle u \text{ for all } u \in W_0^{1,p}(\Omega).$$

Since

$$A_p(u_0) + A(u_0) - N_f(u_0) = 0 \leq A_p(w_+) + A(w_+) - N_f(w_+) \text{ in } W^{-1,p'}(\Omega)$$

(see hypothesis $\mathbf{H}^1(f)(ii)$) and $u_0 \neq w_+$, from the tangency principle of Pucci-Serrin ([16], p.35) we have

$$u_0(z) < w_+(z) \text{ for all } z \in \overline{\Omega}.$$

Similarly, to produce the negative solution, we introduce the following truncation of $f(z,.)$:

$$\widehat{f}_-(z,x) = \begin{cases} f(z, w_-(z)) & \text{if } x < w_-(z), \\ f(z,x) & \text{if } w_-(z) \leq x \leq 0, \\ 0 & \text{if } 0 < x. \end{cases}$$

This is a Carathéodory function. We set $\widehat{F}_-(z,x) = \int_0^x \widehat{f}_-(z,s)\,ds$ and consider the C^1-functional $\widehat{\varphi}_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{F}_-(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Working with $\widehat{\varphi}_-$ as above, we produce another nontrivial constant sign solution $v_0 \in -intC_+$, such that

$$w_-(z) < v_0(z) \text{ for all } z \in \overline{\Omega}.$$

Now, to produce a third nontrivial solution, we truncate $f(z,.)$ at $\{v_0(z), u_0(z)\}$. So, we introduce the following Carathéodory function:

$$f_0(z,x) = \begin{cases} f(z, v_0(z)) & \text{if } x < v_0(z), \\ f(z,x) & \text{if } v_0(z) \le x \le u_0(z), \\ f(z, u_0(z)) & \text{if } u_0(z) < x. \end{cases} \tag{3.5}$$

We set $F_0(z,x) = \int\limits_0^x f_0(z,s)\,ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int\limits_\Omega F_0(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also, let $f_0^{\pm}(z,x) = f_0(z, \pm x^{\pm})$, $F_0^{\pm}(z,x) = \int\limits_0^x f_0^{\pm}(z,s)\,ds$ and consider the C^1-functionals $\varphi_0^{\pm} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0^{\pm}(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int\limits_\Omega F_0^{\pm}(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

As in the first part of the proof, we can show that

$$K_{\varphi_0} \subseteq [v_0, u_0],\ K_{\varphi_0^+} \subseteq [0, u_0],\ K_{\varphi_0^-} \subseteq [v_0, 0].$$

In fact, we can assume that $K_{\varphi_0^+} = \{0, u_0\}$, $K_{\varphi_0^-} = \{v_0, 0\}$. Indeed, if $\widehat{u}_0 \in K_{\varphi_0^+} \setminus \{0, u_0\}$, then $\widehat{u}_0 \in int\ C_+$ (by nonlinear regularity and the nonlinear maximum principle) is the third nontrivial solution of (1.1) and so, we are done.

Claim 1: $u_0 \in int\ C_+$ and $v_0 \in -int\ C_+$ are local minimizers of φ_0.

From (3.5) it is clear that φ_0^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\varphi_0^+(\widehat{u}_0) = \inf\left\{\varphi_0^+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{3.6}$$

As before, via hypothesis $\mathbf{H}^1(f)(iii)$), we have

$$\varphi_0^+(\widehat{u}_0) < 0 = \varphi_0^+(0), \text{ hence } \widehat{u}_0 \ne 0.$$

From (3.6) we have $\widehat{u}_0 \in K_{\varphi_0^+} = \{0, u_0\}$, hence $\widehat{u}_0 = u_0$. Note that

$$\varphi_0^+\mid_{C_+} = \varphi_0\mid_{C_+},$$

hence $u_0 \in int\ C_+$ is a local $C_0^1\left(\overline{\Omega}\right)-$minimizer of φ_0, therefore u_0 is a local $W_0^{1,p}(\Omega)-$minimizer of φ_0 (see Proposition 2.1).

Similarly for $v_0 \in -int\ C_+$, using this time the functional φ_0^-.

We may assume that K_{φ} is finite (or otherwise we already have an infinity of solutions) and that $\varphi_0(v_0) \leq \varphi_0(u_0)$. (The analysis is similar if the opposite inequality holds). By virtue of Claim 1, we can find $\rho \in (0,1)$ small such that

$$\varphi_0(v_0) \leq \varphi_0(u_0) < \inf\{\varphi_0(u) : \|u - u_0\| = \rho\} = \eta_\rho, \|u_0 - v_0\| > \rho \tag{3.7}$$

(see Aizicovici-Papageorgiou-Staicu [2], proof of Proposition 29). The functional φ_0 is coercive (see (3.5)). So, it satisfies the C-condition. This fact and (3.7) permit the use of Theorem 2.3 (the mountain pass theorem). So, we can find $y_0 \in W_0^{1,p}(\Omega)$ such that

$$y_0 \in K_{\varphi_0} \text{ and } \eta_\rho \leq \varphi_0(y_0). \tag{3.8}$$

From (3.7) and (3.8) we see that $y_0 \notin \{v_0, u_0\}$. Also, since $y_0 \in K_{\varphi_0}$ (see (3.8)), we have $y_0 \in [v_0, u_0]$ and so, y_0 is a solution of (1.1) (see (3.5)) and by the nonlinear regularity theory (see [13], [14]) we conclude that $y_0 \in C_0^1(\overline{\Omega})$. Moreover, since y_0 is a critical point of φ_0 of mountain pass type, we have

$$C_1(\varphi_0, y_0) \neq 0. \tag{3.9}$$

We need to show that $y_0 \neq 0$. This will follow as a consequence of (3.9) and the following Claim.

Claim 2: $C_k(\varphi_0, 0) = \delta_{k,d_m}\mathbb{Z}$ for all $k \geq 0$, with $d_m = \dim \bigoplus_{i=1}^{m} E\left(\widehat{\lambda}_i(2)\right) \geq 2$.

Here and in what follows $\delta_{k,j}$ (for $k,\ j \in \mathbb{Z}_+$) denotes the Kronecker delta.

Let $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the C^2-functional defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \frac{1}{2}\int_\Omega \eta(z) u^2(z)\, dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

For $\rho > 0$, let $\overline{B}_\rho^c = \left\{u \in C_0^1(\overline{\Omega}) : \|u\|_{C_0^1(\overline{\Omega})} \leq \rho\right\}$. By virtue of hypothesis $\mathbf{H}^1(f)(iii)$, given $\varepsilon > 0$, we can find $\rho = \rho(\varepsilon) \in (0,1)$ such that $\|\varphi_0 - \psi\|_{C_0^1(\overline{B}_\rho^c)} \leq \varepsilon$, hence

$$C_k\left(\varphi_0 |_{C_0^1(\overline{\Omega})}, 0\right) = C_k\left(\psi |_{C_0^1(\overline{\Omega})}, 0\right) \text{ for all } k \geq 0. \tag{3.10}$$

Since $C_0^1(\overline{\Omega})$ is dense in $W_0^{1,p}(\Omega)$, from Palais [15], we have

$$\begin{aligned} &C_k\left(\varphi_0 |_{C_0^1(\overline{\Omega})}, 0\right) = C_k(\varphi_0, 0) \text{ and} \\ &C_k\left(\psi |_{C_0^1(\overline{\Omega})}, 0\right) = C_k(\psi, 0) \text{ for all } k \geq 0. \end{aligned} \tag{3.11}$$

From Cingolani-Vannella ([9], p.273), we have

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0, \text{ with } d_m = \dim \bigoplus_{i=1}^{m} E\left(\widehat{\lambda}_i(2)\right) \geq 2. \tag{3.12}$$

Then from (3.10) (3.11) and (3.12), it follows that

$$C_k(\varphi_0, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0.$$

This proves Claim 2.

From (3.9) and Claim 2, we infer that $y_0 \neq 0$. So, $y_0 \in C_0^1\left(\overline{\Omega}\right)$ is the third nontrivial solution of (1.1). ☐

In fact we can show that the third nontrivial solution can be choosen to be nodal. The strategy is the following. First, we produce extremal nontrivial constant sign solutions, i.e., a smallest nontrivial positive solution u_+ and a biggest nontrivial negative solution v_-. Then, we concentrate on the order interval

$$[v_-, u_+] := \left\{u \in W_0^{1,p}(\Omega) : v_-(z) \leq u(z) \leq u_+(z) \text{ for a.a. } z \in \Omega\right\}.$$

Using truncation and comparison techniques together with variational methods, we produce a nontrivial critical point y_0 of φ_0 in $[v_-, u_+]$, distinct from v_-, u_+. Then by virtue of the extremality of v_- and u_+, y_0 must be nodal.

In what follows, we implement this strategy. By virtue of hypotheses $\mathbf{H}^1(f)(i),(iii)$, given $\varepsilon \in (0, \min\{-c_-, c_+\})$, we can find $c_\varepsilon > 0$ such that

$$f(z,x)\,x \geq (\eta(z) - \varepsilon)\,x^2 - c_\varepsilon |x|^p \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho, \tag{3.13}$$

with $\rho = \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$. Then, we introduce the following Carathéodory function:

(3.14)
$$h_\varepsilon(z,x) = \begin{cases} (\eta(z) - \varepsilon)\,w_-(z) - c_\varepsilon |w_-(z)|^{p-2}\,w_-(z) & \text{if } x < w_-(z), \\ (\eta(z) - \varepsilon)\,x - c_\varepsilon |x|^{p-2}\,x & \text{if } w_-(z) \leq x \leq w_+(z), \\ (\eta(z) - \varepsilon)\,w_+(z) - c_\varepsilon w_+(z)^{p-1} & \text{if } w_+(z) < x. \end{cases}$$

We consider the following auxiliary Dirichlet problem

$$-\triangle_p u(z) - \triangle u(z) = h_\varepsilon(z,x) \text{ in } \Omega, \ u\,|_{\partial\Omega} = 0. \tag{3.15}$$

Proposition 3.2. *For $\varepsilon \in (0, \min\{-c_-, c_+\})$ small, problem (3.15) has a unique nontrivial positive solution $u_* \in int\ C_+$ and a unique nontrivial negative solution $v_* \in -int\ C_+$.*

Proof. First, we establish the existence and uniqueness of the positive solution. So, let $h_\varepsilon^+(z,x) = h_\varepsilon(z,x^+)$. We set $H_\varepsilon^+(z,x) = \int_0^x h_\varepsilon^+(z,s)\,ds$ and consider the C^1-functional $\sigma_\varepsilon^+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\varepsilon^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega H_\varepsilon^+(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

It is clear from (3.14) that σ_ε^+ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore, we can find $u_* \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\varepsilon^+(u_*) = \inf\left\{\sigma_\varepsilon^+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{3.16}$$

As in the proof of Proposition 3.1, for $\varepsilon > 0$ small, we have

$$\sigma_\varepsilon^+(u_*) < 0 = \sigma_\varepsilon^+(0), \text{ hence } u_* \neq 0.$$

From (3.16) it follows

$$\left(\sigma_\varepsilon^+\right)'(u_*) = 0,$$

hence

$$A_p(u_*) + A(u_*) = N_{h_\varepsilon^+}(u_*). \tag{3.17}$$

Acting on (3.17) with $-u_*^- \in W_0^{1,p}(\Omega)$, we infer that

$$u_* \geq 0, u_* \neq 0.$$

Also on (3.17), we act with $(u_* - w_+)^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}
&\left\langle A_p(u_*), (u_* - w_+)^+ \right\rangle + \left\langle A(u_*), (u_* - w_+)^+ \right\rangle \\
&= \int_\Omega h_\varepsilon^+(z, u_*)(u_* - w_+)^+ dz \\
&= \int_\Omega \left[(\eta(z) - \varepsilon) w_+(z) - c_\varepsilon w_+(z)^{p-1} \right] (u_* - w_+)^+ dz \text{ (see (3.14))} \\
&\leq \int_\Omega f(z, w_+)(u_* - w_+)^+ dz \text{ (see (3.13))} \\
&\leq \left\langle A_p(w_+), (u_* - w_+)^+ \right\rangle + \left\langle A(w_+), (u_* - w_+)^+ \right\rangle \text{ (see } \mathbf{H}^1(f)(ii)\text{)},
\end{aligned}$$

hence

$$u_* \leq w_+$$

(as before, see the proof of Proposition 3.1). Therefore, we conclude that

$$u_* \in [0, w_+] = \left\{ u \in W_0^{1,p}(\Omega) : 0 \leq u(z) \leq w_+(z) \text{ a.e. in } \Omega \right\}.$$

Then (3.17) becomes

$$A_p(u_*) + A(u_*) = (\eta - \varepsilon) u_* - c_\varepsilon u_*^{p-1} \text{ (see (3.14))},$$

hence u_* solves (3.15) and $u_* \in C_+ \setminus \{0\}$ (by nonlinear regularity). In fact we have

$$\triangle_p u_*(z) + \triangle u_*(z) \leq c_\varepsilon u_*(z)^{p-1} \text{ a.e. in } \Omega,$$

hence $u_* \in int\ C_+$ (see Pucci-Serrin ([16], pp. 34 and 120)).

So, we have established the existence of a nontrivial positive solution $u_* \in int\ C_+$ for problem (3.15). Next we examine the uniqueness of u_*. To this end, we introduce the integral functional $\gamma_+ : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\gamma_+(u) = \begin{cases} \frac{1}{p} \left\| Du^{\frac{1}{2}} \right\|_p^p + \frac{1}{2} \left\| Du^{\frac{1}{2}} \right\|_2^2 & \text{if } u \geq 0, u^{\frac{1}{2}} \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $u_1,\ u_2 \in dom\ \gamma_+$, $t \in [0,1]$ and set $y = (tu_1 + (1-t) u_2)^{\frac{1}{2}}$. Let $v_1 = u_1^{\frac{1}{2}}$, $v_2 = u_2^{\frac{1}{2}}$. From Lemma 4 of Benguria-Brezis-Lieb [7] (see also Lemma 1 of Diaz-Saa [10]), we have

$$\|Dy(z)\| \leq \left(t \|Dv_1(z)\|^2 + (1-t) \|Dv_2(z)\|^2 \right)^{\frac{1}{2}} \text{ a.e in } \Omega.$$

Let

$$G_0(t) = \frac{1}{p} t^p + \frac{1}{2} t^2,\ t \geq 0.$$

Evidently $t \to G_0(t)$ is increasing. So, we have

$$G_0(\|Dy(z)\|) \le G_0\left(\left(t\|Dv_1(z)\|^2 + (1-t)\|Dv_2(z)\|^2\right)^{\frac{1}{2}}\right) \text{ a.e in } \Omega.$$

Note that $t \to G_0\left(t^{\frac{1}{2}}\right) = \frac{1}{p}t^{\frac{p}{2}} + \frac{1}{2}t$, $t \ge 0$, is convex (recall that $p > 2$). Therefore

$$\begin{aligned}&G_0\left(\left(t\|Dv_1(z)\|^2 + (1-t)\|Dv_2(z)\|^2\right)^{\frac{1}{2}}\right)\\ &\le tG_0(\|Dv_1(z)\| + (1-t)G_0(\|Dv_2(z)\|)) \text{ a.e in } \Omega.\end{aligned}$$

We set $G(y) = G_0(\|y\|)$ for all $y \in \mathbb{R}^N$. Then

$$G(Dy(z)) \le tG\left(Du_1(z)^{\frac{1}{2}}\right) + (1-t)G\left(Du_2(z)^{\frac{1}{2}}\right) \text{ a.e in } \Omega,$$

hence γ_+ is convex.

Also, via Fatou's Lemma, we can check that γ_+ is lower semicontinuous, and of course $dom\ \gamma_+ \neq \varnothing$.

Let $u \in W_0^{1,p}(\Omega)$ be a nontrivial positive solution of (3.15). Then, from the first part of the proof we have $u \in [0, w_+] \cap int\ C_+$. Hence $u^2 \ge 0$ and $\left(u^2\right)^{\frac{1}{2}} = u \in W_0^{1,p}(\Omega)$, i.e., $u^2 \in dom\ \gamma_+$. For $h \in C_0^1\left(\overline{\Omega}\right) \setminus \{0\}$, we have $u^2 + \lambda h \in dom\ \gamma_+$ for all $\lambda \in (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ small. This means that γ_+ is Gateaux differentiable at u^2 in the direction h, and by the chain rule, we have

$$\begin{aligned}\gamma_+'\left(u^2\right)(h) &= \frac{1}{2}\int_\Omega \frac{-\triangle_p u - \triangle u}{u} h dz\\ &= \frac{1}{2}\int_\Omega \frac{h_\varepsilon(z,u)}{u} h dz \text{ (see } (3.15))\\ &= \frac{1}{2}\int_\Omega \frac{(\eta(z)-\varepsilon)u - c_\varepsilon u^{p-1}}{u} h dz \text{ (recall that } u \in [0, w_+] \text{ and see } (3.14))\\ &= \frac{1}{2}\int_\Omega \left[(\eta(z) - \varepsilon) - c_\varepsilon u^{p-2}\right] h dz.\end{aligned} \tag{3.18}$$

Similarly, if $y \in W_0^{1,p}(\Omega)$ is another nontrivial positive solution of (3.15), then again we have $y \in [0, w_+] \cap int\ C_+$ and as above, we obtain

$$\gamma_+'\left(y^2\right)(h) = \frac{1}{2}\int_\Omega \left[(\eta(z) - \varepsilon) - c_\varepsilon y^{p-1}\right] h dz. \tag{3.19}$$

Since γ_+ is convex, γ_+' is monotone and so, we have

$$\begin{aligned}0 &\le \left\langle \gamma_+'\left(u^2\right) - \gamma_+'\left(y^2\right), u^2 - y^2\right\rangle_{L^1(\Omega)}\\ &= \frac{c_\varepsilon}{2}\int_\Omega \left(y^{p-2} - u^{p-2}\right)\left(u^2 - y^2\right) dz \text{ (see } (3.18), (3.19)),\end{aligned}$$

hence

$$y = u \text{ (since } p > 2).$$

This proves the uniqueness of $u_* \in [0, w_+] \cap int\ C_+$.

Similarly, we consider $h_\varepsilon^-(z,u) = h_\varepsilon(z,-u^-)$, $H_\varepsilon^-(z,x) = \int_0^x h_\varepsilon^-(z,s)\,ds$ and the C^1-functionals $\sigma_\varepsilon^- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\varepsilon^-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega H_\varepsilon^-(z,u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Reasoning as above, we prove the existence and uniqueness of a nontrivial negative solution $v_* \in -int\ C_+$. ☐

This proposition leads to the existence of extremal nontrivial constant sign solutions for problem (1.1).

Proposition 3.3. *If hypotheses* $\mathbf{H}^1(f)$ *hold, then problem* (1.1) *has a smallest nontrivial positive solution* $u_+ \in [0,w_+] \cap int\ C_+$ *and a biggest nontrivial negative solution* $v_- \in [w_-,0] \cap (-int\ C_+)$.

Proof. Let $\mathcal{S}_+$ be the set of nontrivial solutions of (1.1) in the order interval $[0,w_+]$. From Proposition 3.1 and its proof it follows that

$$\mathcal{S}_+ \neq \varnothing, \text{ and } \mathcal{S}_+ \subset int\ C_+.$$

Claim: If $\widetilde{u} \in \mathcal{S}_+$, then $u_* \leq \widetilde{u}$.

We introduce the Carathéodory function

$$\tau_\varepsilon^+(z,x) = \begin{cases} 0 & \text{if } x < 0, \\ (\eta(z) - \varepsilon)x - c_\varepsilon x^{p-1} & \text{if } 0 \leq x \leq \widetilde{u}(z), \\ (\eta(z) - \varepsilon)\widetilde{u}(z) - c_\varepsilon \widetilde{u}(z)(z)^{p-1} & \text{if } \widetilde{u}(z) < x. \end{cases} \tag{3.20}$$

Here $\varepsilon > 0$ is small, as in Proposition 3.2. We set $T_\varepsilon^+(z,x) = \int_0^x \tau_\varepsilon^+(z,s)\,ds$ and consider the C^1-functional $\xi_\varepsilon^+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\xi_\varepsilon^+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega T_\varepsilon^+(z,u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (3.20) it is clear that ξ_ε^+ is coercive and sequentially weakly lower semicontinuous. So, we can find $\widehat{y}_0 \in W_0^{1,p}(\Omega)$ such that

$$\xi_\varepsilon^+(\widehat{y}_0) = \inf\left\{\xi_\varepsilon^+(u) : u \in W_0^{1,p}(\Omega)\right\}. \tag{3.21}$$

As in the proof of Proposition 3.1, using hypothesis $\mathbf{H}^1(f)(iii)$, we show that

$$\xi_\varepsilon^+(\widehat{y}_0) < 0 = \xi_\varepsilon^+(0), \text{ hence } \widehat{y}_0 \neq 0.$$

From (3.21), we have

$$\left(\xi_\varepsilon^+\right)'(\widehat{y}_0) = 0,$$

hence

$$A_p(\widehat{y}_0) + A(\widehat{y}_0) = N_{\tau_\varepsilon^+}(\widehat{y}_0). \tag{3.22}$$

On (3.22), first we act with $-\widehat{y_0}^{-} \in W_0^{1,p}(\Omega)$ and obtain

$$\left\|D\widehat{y_0}^{-}\right\|_p^p + \frac{1}{2}\left\|D\widehat{y_0}^{-}\right\|_2^2 = 0 \text{ (see (3.20))},$$

hence

$$\widehat{y}_0 \geq 0,\ \widehat{y}_0 \neq 0.$$

Also, acting on (3.22) with $(\widehat{y}_0 - \widetilde{u})^+ \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
&\left\langle A_p(\widehat{y}_0), (\widehat{y}_0 - \widetilde{u})^+\right\rangle + \left\langle A(\widehat{y}_0), (\widehat{y}_0 - \widetilde{u})^+\right\rangle \\
&= \int_\Omega \tau_\varepsilon^+(z, \widehat{y}_0)(\widehat{y}_0 - \widetilde{u})^+ dz \\
&= \int_\Omega \left[(\eta(z) - \varepsilon)\widetilde{u} - c_\varepsilon \widetilde{u}^{p-1}\right](\widehat{y}_0 - \widetilde{u})^+ dz \text{ (see (3.20))} \\
&\leq \int_\Omega f(z, \widetilde{u})(\widehat{y}_0 - \widetilde{u})^+ dz \text{ (see (3.13))} \\
&= \left\langle A_p(\widetilde{u}), (\widehat{y}_0 - \widetilde{u})^+\right\rangle + \left\langle A(\widetilde{u}), (\widehat{y}_0 - \widetilde{u})^+\right\rangle,
\end{aligned}$$

hence

$$\widehat{y}_0 \leq \widetilde{u}.$$

Therefore we have proved that

$$\widehat{y}_0 \in [0, \widetilde{u}] := \left\{u \in W_0^{1,p}(\Omega) : 0 \leq u(z) \leq \widetilde{u}(z) \text{ a.e. in } \Omega\right\}.$$

Hence (3.22) becomes

$$A_p(\widehat{y}_0) + A(\widehat{y}_0) = (\eta - \varepsilon)\widehat{y}_0 - c_\varepsilon \widehat{y}_0^{p-1},$$

hence

$$-\triangle_p \widehat{y}_0(z) - \triangle \widehat{y}_0(z) = (\eta - \varepsilon)\widehat{y}_0(z) - c_\varepsilon \widehat{y}_0(z)^{p-1} \text{ a.e. in } \Omega,\ \widehat{y}_0|_{\partial\Omega} = 0,$$

therefore (see the proof of Proposition 3.2)

$$\widehat{y}_0 \in int\ C_+ \text{ solves } (3.15).$$

It follows that $\widehat{y}_0 = u_*$ (see Proposition 3.2). Thus we have $u_* \leq \widetilde{u}$ and this proves the Claim.

We consider a chain $C \subseteq \mathcal{S}_+$ (i.e., a nonempty totally ordered subset of $\mathcal{S}_+$). From Dunford-Schwartz ([11], p.336), we know that we can find $\{u_n\}_{n\geq 1} \subseteq C$ such that

$$\inf C = \inf_{n\geq 1} u_n.$$

We have

$$A_p(u_n) + A(u_n) = N_f(u_n),\ u_* \leq u_n \leq w_+ \text{ for all } n \geq 1 \tag{3.23}$$

(see the Claim), hence

$$\{u_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega) \text{ is bounded (see } \mathbf{H}^1(f)(i)\text{)}.$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{3.24}$$

On (3.23) we act with $u_n - u \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$. Using (3.24) we obtain

$$\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle \right] = 0,$$

hence

$$\lim_{n\to\infty} \langle A_p(u_n), u_n - u\rangle = 0 \text{ (recall that } A \in \mathcal{L}\left(H_0^1(\Omega), H^{-1}(\Omega)\right)\text{)},$$

therefore

$$u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 2.4), } u_* \le u,$$

and we conclude that

$$u \in \mathcal{S}_+ \text{ and } u = \inf C.$$

Since C is an arbitrary chain in $\mathcal{S}_+$, from the Kuratowski-Zorn lemma it follows that $\mathcal{S}_+$ has a minimal element $u_+ \in \mathcal{S}_+$. The set $\mathcal{S}_+$ is downward directed (see Aizicovici-Papageorgiou-Staicu [3]), i.e., if u_1, $u_2 \in \mathcal{S}_+$, there exists $u \in \mathcal{S}_+$ such that $u \le u_1$, $u \le u_2$. From this it follows that $u_+ \in int\ C_+$ is the smallest nontrivial positive solution of (1.1).

Similarly, let $\mathcal{S}_-$ be the set of nontrivial negative solutions of (1.1) in the order interval $[w_-, 0]$. We have $\mathcal{S}_- \neq \varnothing$ and $\mathcal{S}_- \subseteq -int\ C_+$ (see Proposition 3.1). The set $\mathcal{S}_-$ is upward directed, i.e., if v_1, $v_2 \in \mathcal{S}_-$, there exists $v \in \mathcal{S}_-$ such that $v_1 \le v$, $v_2 \le v$ (see [3]). Also, we have $\widetilde{v} \le v_*$ for all $\widetilde{v} \in \mathcal{S}_-$ (see the Claim). Then as above, via the Kuratowski-Zorn lemma, we show that problem (1.1) has a biggest nontrivial negative solution $v_- \in -int\ C_+$. □

Using these extremal solutions and reasoning as in the proof of Proposition 3.1, via the mountain pass theorem (see Theorem 2.3), we produce a nontrivial solution in the order interval $[v_-, u_+]$. Evidently, this solution is nodal. So, we have:

Proposition 3.4. *If hypotheses* $\mathbf{H}^1(f)$ *hold, then problem* (1.1) *admits a nodal solution* $y_0 \in [v_-, u_+] \cap C_0^1(\Omega)$.

Closing this section, we can state the first multiplicity theorem for problem (1.1). We stress that in this theorem no growth restriction is imposed on the reaction $x \to f(z,x)$ near $\pm\infty$. In particular therefore, $f(z,.)$ may be supercritical or even exponential. Moreover, we localize the tree solutions.

Theorem 3.5. *If hypotheses* $\mathbf{H}^1(f)$ *hold, then problem* (1.1) *has at least three nontrivial solutions* $u_0 \in int\ C_+$, $v_0 \in -int\ C_+$,

$$w_-(z) < v_0(z) \le 0 \le u_0(z) < w_+(z) \text{ for all } z \in \overline{\Omega},$$

and

$$y_0 \in [v_-, u_+] \cap C_0^1(\Omega), \text{ nodal.}$$

4. Multiplicity for equations resonant at $\pm\infty$.

In this section we impose a $(p-1)$ −linearity growth condition on $f(z,.)$ near $\pm\infty$ and we produce two more nontrivial constant sign smooth solutions, for a total of five nontrivial smooth solutions, all with sign information. We stress that the hypothesis on $f(z,.)$ near $\pm\infty$ (see $\mathbf{H}^2(f)(iv)$ below) implies that we have resonance with respect to the principal eigenvalue $\widehat{\lambda}_1(p)>0$ of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$.

The hypotheses on the reaction $f(z,x)$ are the following:

$\mathbf{H}^2(f)$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $f(z,0)=0$ for a.a. $z\in\Omega$, and:

(i) there exists $a\in L^\infty(\Omega)_+$ such that

$$|f(z,x)|\le a(z)\left(1+|x|^{p-1}\right) \text{ for } a.a. z\in\Omega,\ \text{all } x\in\mathbb{R};$$

(ii) same as hypothesis $\mathbf{H}^1(f)(ii)$;

(iii) same as hypothesis $\mathbf{H}^1(f)(iii)$;

(iv) one has that:

$$\lim_{x\to\pm\infty}\frac{f(z,x)}{|x|^{p-2}x}=\widehat{\lambda}_1(p) \text{ and } \lim_{x\to\pm\infty}\frac{f(z,x)x-pF(z,x)}{x^2}=-\infty,$$

uniformly for a.a. $z\in\Omega$.

Remarks: Now $f(z,.)$ is $(p-1)$ −linear near $\pm\infty$. In hypothesis $\mathbf{H}^2(f)(iv)$, the first limit implies that at $\pm\infty$ we have resonance with respect to the principal eigenvalue $\widehat{\lambda}_1(p)$ of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$.

Example: The following function satisfies $\mathbf{H}^2(f)$ (for the sake of simplicity we drop the $z-$dependence):

$$f(x)=\begin{cases}\lambda\left(x-|x|^{p-2}x\right) & \text{if } |x|\le 1,\\ \widehat{\lambda}(p)|x|^{p-2}x-cx+\left(c-\widehat{\lambda}_1(p)\right)|x|^{\tau-2}x & \text{if } |x|>1\end{cases}$$

with $\lambda\in\left(\widehat{\lambda}_m(2),\widehat{\lambda}_{m+1}(2),\right)$, $m\ge 2$, $c>\widehat{\lambda}_1(p)$ and $\tau\in(2,p)$.

Theorem 4.1. *If hypotheses $\mathbf{H}^2(f)$ hold, then problem* (1.1) *has at least five nontrivial solutions*

$$u_0,\ \widehat{u}\in int\ C_+,\ u_0\le\widehat{u},\ u_0\ne\widehat{u},\ u_0(z)<w_+(z) \text{ for all } z\in\overline{\Omega},$$

$$v_0,\ \widehat{v}\in -int\ C_+,\ \widehat{v}\le v_0,\ v_0\ne\widehat{v},\ w_-(z)<v_0(z) \text{ for all } z\in\overline{\Omega},$$

and

$$y_0\in[v_0,u_0]\cap C_0^1\left(\overline{\Omega}\right),\ \textit{nodal.}$$

Proof. From Theorem 3.5, we already have three nontrivial solutions

$$u_0\in int\ C_+ \text{ with } u_0(z)<w_+(z) \text{ for all } z\in\overline{\Omega},$$

$$v_0\in -int\ C_+ \text{ with } w_-(z)\le v_0(z) \text{ for all } z\in\overline{\Omega},$$

and

$$y_0 \in [v_0, u_0] \cap C_0^1(\Omega), \text{ nodal.}$$

We consider the following truncation of the reaction $f(z,x)$:

$$g_+(z,x) = \begin{cases} f(z, u_0(z)) & \text{if} \quad x < u_0(z), \\ f(z,x) & \text{if} \quad u_0(z) \le x. \end{cases} \tag{4.1}$$

This is a Carathéodory function . We set $G_+(z,x) = \int_0^x g_+(z,s)\,ds$ and consider the C^1−functional $\psi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega G_+(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We show that

$$K_{\psi_+} \subseteq [u_0) := \left\{u \in W_0^{1,p}(\Omega) : u_0(z) \le u(z) \text{ a.e. in } \Omega\right\}. \tag{4.2}$$

So, let $u \in K_{\psi_+}$. Then

$$A_p(u) + A(u) = N_{g_+}(u). \tag{4.3}$$

Acting on (4.3) with $(u_0 - u)^+ \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} &\left\langle A_p(u), (u_0-u)^+\right\rangle + \left\langle A(u), (u_0-u)^+\right\rangle \\ &= \int_\Omega g_+(z,u)(u_0-u)^+\,dz \\ &= \int_\Omega f(z,u_0)(u_0-u)^+\,dz \text{ (see (4.1))} \\ &= \left\langle A_p(u_0), (u_0-u)^+\right\rangle + \left\langle A(u_0), (u_0-u)^+\right\rangle, \end{aligned}$$

hence $u_0 \le u$, which proves (4.2) .

We may assume that u_0 is the only solution of (1.1) in the order interval $[u_0, w_+]$ or otherwise we already have the second nontrivial positive solution $\widehat{u} \in int\ C_+$ of (1.1) , and $u_0 \le \widehat{u}$, $u_0 \ne \widehat{u}$.

Claim 1: $u_0 \in int\ C_+$ is a local minimizer of ψ_+.

We consider the following truncation of g_+ :

$$\widehat{g}_+(z,x) = \begin{cases} g_+(z,x) & \text{if} \quad x < w_+(z), \\ g_+(z, w_+(z)) & \text{if} \quad w_+(z) \le x. \end{cases} \tag{4.4}$$

This is a Carathéodory function . We set $\widehat{G}_+(z,x) = \int_0^x \widehat{g}_+(z,s)\,ds$ and consider the C^1−functional $\widehat{\psi}_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega \widehat{G}_+(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (4.4) it is clear that $\widehat{\psi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore, we can find $\widehat{u}_0 \in W_0^{1,p}(\Omega)$ such that

$$\widehat{\psi}_+(\widehat{u}_0) = \inf\left\{\widehat{\psi}_+(u) : u \in W_0^{1,p}(\Omega)\right\}.$$

Then

$$\widehat{\psi}'_+(\widehat{u}_0) = 0,$$

therefore

$$A_p(\widehat{u}_0) + A(\widehat{u}_0) = N_{g_+}(\widehat{u}_0). \tag{4.5}$$

On (4.5) we act with $(u_0 - \widehat{u}_0)^+ \in W_0^{1,p}(\Omega)$ and with $(\widehat{u}_0 - w_+)^+ \in W_0^{1,p}(\Omega)$, respectively, and using (4.1) and (4.4) we show that

$$\widehat{u}_0 \in [u_0, w_+].$$

So, we have

$$A_p(\widehat{u}_0) + A(\widehat{u}_0) = N_f(\widehat{u}_0)$$

(see $(4.1), (4.4)$ and (4.5)), therefore

$$\widehat{u}_0 = u_0$$

(since u_0 is the only solution of (1.1) in the order interval $[u_0, w_+]$). Since

$$\psi_+ |_{[u_0,w_+]} = \widehat{\psi}_+ |_{[u_0,w_+]},$$

it follows that $u_0 \in int\ C_+$ is a local $C_0^1(\overline{\Omega})$ $-$minimizer of ψ_+. By virtue of Proposition 2.1, u_0 is a local $W_0^{1,p}(\Omega)$ $-$minimizer of ψ_+. This proves Claim 1.

By virtue of Claim 1, we can find $\rho \in (0,1)$ small, such that

$$\psi_+(u_0) < \inf\{\psi_+(u) : \|u - u_0\| = \rho\} = \eta_\rho^+. \tag{4.6}$$

Claim 2: $\psi_+(t\widehat{u}_{1,p}) \to -\infty$ as $t \to +\infty$.

Hypothesis $\mathbf{H}^2(f)(iv)$ implies that given any $\overline{\eta} > 0$, there exists $M = M(\overline{\eta}) > \max\{1, \|u_0\|_\infty\}$ such that

$$f(z,x)x - pF(z,x) \le -\overline{\eta}x^2 \text{ for a.a. } z \in \Omega, \text{ all } x \ge M. \tag{4.7}$$

We have

$$\frac{d}{dx}\frac{F(z,x)}{x^p} = \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \le \frac{-\overline{\eta}}{x^{p-1}} \text{ for a.a. } z \in \Omega, \text{ all } x \ge M$$

(see (4.7)). Integrating this inequality, we obtain

$$\frac{F(z,x)}{x^p} - \frac{F(z,y)}{y^p} \le \frac{\overline{\eta}}{p-2}\left(\frac{1}{x^{p-2}} - \frac{1}{y^{p-2}}\right)$$
$$\text{for a.a. } z \in \Omega, \text{ all } x \ge y \ge M.$$

Passing to the limit as $x \to +\infty$ and using hypothesis $\mathbf{H}^2(f)(iv)$, we arrive at

$$\frac{\widehat{\lambda}_1(p)}{p}y^p - F(z,y) \le \frac{-\overline{\eta}}{p-2}y^2 \text{ for a.a. } z \in \Omega, \text{ all } y \ge M. \tag{4.8}$$

Note that

$$G_+(z,y) = F(z,y) - F(z,u_0(z)) + f(z,u_0(z))u_0(z)$$

$$\geq F(z,y) - c_* \text{ for a.a. } z\in\Omega, \text{ all } y \geq M \text{ and some } c_* > 0$$

(see $\mathbf{H}^2(f)(i)$). Hence, from (4.8) we have

$$\frac{\widehat{\lambda}_1(p)}{p} y^p - G_+(z,y) \leq -\frac{\overline{\eta}}{p-2} y^2 + c_* \text{ for a.a. } z\in\Omega, \text{ all } y\geq M. \tag{4.9}$$

Then by (4.9), for all $t>0$, we have

$$\begin{aligned}\psi_+(t\widehat{u}_{1,p}) &= \frac{t^p}{p}\widehat{\lambda}_1(p)\|\widehat{u}_{1,p}\|_p^p + \frac{t^2}{2}\|D\widehat{u}_{1,p}\|_2^2 - \int_\Omega G_+(z, t\widehat{u}_{1,p}(z))\,dz \\ &\leq -\frac{\overline{\eta}t^2}{p-2}\|\widehat{u}_{1,p}\|_2^2 + \frac{t^2}{2}\|D\widehat{u}_{1,p}\|_2^2 + k_{\overline{\eta}}, \\ &\text{for some positive constant } k_{\overline{\eta}} \text{ depending on } \overline{\eta}.\end{aligned}$$

Since $\overline{\eta}>0$ is arbitrary, from the above inequality we infer that $\psi_+(t\widehat{u}_{1,p}) \to -\infty$ as $t\to+\infty$. This proves Claim 2.

Claim 3: ψ_+ satisfies the $C-$condition.

Let $\{u_n\}_{n\geq1} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\psi_+(u_n)| \leq M_1 \text{ for some } M_1>0, \text{ all } n\geq 1, \tag{4.10}$$

and

$$(1+\|u_n\|)\psi_+'(u_n) \to 0 \text{ in } W_0^{1,p}(\Omega)^* \text{ as } n\to\infty. \tag{4.11}$$

From (4.11) we have that

$$\begin{aligned}\langle A_p(u_n), h\rangle + \langle A(u_n), h\rangle - \int_\Omega g_+(z,u_n)\,hdz \leq \tfrac{\varepsilon_n\|h\|}{1+\|u_n\|} \\ \text{for all } h\in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n\to 0^+.\end{aligned} \tag{4.12}$$

In (4.12) we choose $h=-u_n^-\in W_0^{1,p}(\Omega)$. Then

$$\left\|Du_n^-\right\|_p^p + \left\|Du_n^-\right\|_2^2 \leq \varepsilon_n \text{ for all } n\geq 1 \text{ (see (4.1))},$$

hence

$$u_n^- \to 0 \text{ in } W_0^{1,p}(\Omega). \tag{4.13}$$

Suppose that $\|u_n^+\|\to\infty$. We set

$$y_n = \frac{u_n^+}{\|u_n^+\|},\ n\geq 1.$$

Then $\|y_n\|=1$ for all $n\geq1$ and so, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n\to y \text{ in } L^p(\Omega) \text{ as } n\to\infty. \tag{4.14}$$

From (4.12) and (4.13) we have

$$\langle A_p(y_n),h\rangle + \frac{1}{\|u_n^+\|^{p-2}}\langle A(y_n),h\rangle - \int_\Omega \frac{g_+(z,u_n^+)}{\|u_n^+\|^{p-1}} hdz \leq \varepsilon_n'\|h\| \tag{4.15}$$

$$\text{for all } n\geq 1, \text{ with } \varepsilon_n'\to 0^+.$$

Hypothesis $\mathbf{H}^2(f)(i)$ implies that $\left\{\frac{g_+(.,u_n^+(.))}{\|u_n^+\|^{p-1}}\right\}_{n\geq 1} \subset L^{p'}(\Omega)$ is bounded $(\frac{1}{p}+\frac{1}{p'}=1)$. Hence, if in (4.15) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$, pass to the limit as $n\to\infty$ and use (4.14), then

$$\lim_{n\to\infty} \langle A_p(y_n), y_n - y\rangle = 0 \text{ (recall that } p>2),$$

and by Proposition 2.4 it follows that

$$y_n \to y \text{ in } W_0^{1,p}(\Omega), \text{ hence } \|y\| = 1,\ y \geq 0. \tag{4.16}$$

Since $\left\{\frac{g_+(.,u_n^+(.))}{\|u_n^+\|^{p-1}}\right\}_{n\geq 1} \subset L^{p'}(\Omega)$ is bounded (see $\mathbf{H}^2(f)(i)$), we may assume that

$$\frac{g_+(.,u_n^+(.))}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widetilde{g} \text{ in } L^{p'}(\Omega).$$

From hypothesis $\mathbf{H}^2(f)(iv)$ and (4.1), it follows that

$$\widetilde{g} = \widehat{\lambda}_1(p)\, y^{p-1} \tag{4.17}$$

(see [2], Proposition 14). So, if in (4.15) we pass to the limit as $n\to\infty$ and use (4.16) and (4.17), we obtain

$$\langle A_p(y), h\rangle = \widehat{\lambda}_1(p) \int_\Omega y^{p-1} h dz \text{ for all } h \in W_0^{1,p}(\Omega)$$

(recall that $p>2$), hence

$$A_p(y) = \widehat{\lambda}_1(p)\, y^{p-1},\ y\geq 0,\ y\neq 0,$$

therefore

$$-\triangle_p y(z) = \widehat{\lambda}_1(p)\, y(z)^{p-1} \text{ in } \Omega,\ y\mid_{\partial\Omega} = 0.$$

It follows that

$$y(z) > 0 \text{ for all } z\in\Omega,$$

hence

$$u_n^+(z) \to +\infty \text{ for a.a. } z\in\Omega,$$

therefore

$$\frac{pG_+(z,u_n^+(z)) - g_+(z,u_n^+(z))\, u_n^+(z)}{u_n^+(z)^2} \to +\infty \text{ for a.a. } z\in\Omega$$

(see $\mathbf{H}^2(f)(iv)$ and (4.1)), and we conclude that

$$\frac{1}{\|u_n^+\|^2}\int_\Omega \left[pG_+\left(z,u_n^+(z)\right) - g_+\left(z,u_n^+(z)\right)u_n^+(z)\right] dz \to +\infty \tag{4.18}$$

(by Fatou's Lemma). From (4.10) and (4.13), we have

(4.19)

$$-\left\|Du_n^+\right\|_p^p - \frac{p}{2}\left\|Du_n^+\right\|_2^2 + \int_\Omega pG_+\left(z,u_n^+\right) dz \leq M_2 \text{ for some } M_2>0,\ \text{all } n\geq 1.$$

Also, in (4.12) we choose $h = u_n^+ \in W_0^{1,p}(\Omega)$ and obtain

$$\left\|Du_n^+\right\|_p^p + \left\|Du_n^+\right\|_2^2 - \int_\Omega g_+\left(z, u_n^+\right) u_n^+ dz \le \varepsilon_n \text{ for all } n \ge 1. \tag{4.20}$$

Adding (4.19) and (4.20) and multiplying with $\frac{1}{\left\|u_n^+\right\|^2}$, we have

$$\begin{aligned} &\frac{1}{\left\|u_n^+\right\|^2} \int_\Omega \left[pG_+\left(z, u_n^+(z)\right) - g_+\left(z, u_n^+(z)\right) u_n^+(z)\right] dz \\ &\le \frac{M_3}{\left\|u_n^+\right\|^2} + \left(\frac{p}{2} - 1\right) \left\|Dy_n^+\right\|_2^2 \le M_4 \text{ for some } M_3, M_4 > 0, \text{ all } n \ge 1. \end{aligned} \tag{4.21}$$

Comparing (4.18) and (4.21) we reach a contradiction. This proves that $\{u_n\}_{n\ge1} \subset W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{4.22}$$

In(4.12) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (4.22). Then

$$\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle\right] = 0,$$

hence

$$\limsup_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle\right] \le 0$$

(since A is monotone), therefore

$$\limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \le 0.$$

We conclude that

$$u_n \to u \text{ in } W_0^{1,p}(\Omega)$$

(see Proposition 2.4). This proves Claim 3.

By (4.6) and Claims 2, 3, we see that we can apply Theorem 2.3 (the mountain pass theorem). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\psi_+} \text{ and } \eta_\rho^+ \le \psi_+(\widehat{u}),$$

hence $\widehat{u} \ne u_0$ (see (4.6)), $u_0 \le \widehat{u}$ (see (4.2)), $\widehat{u}$ solves (1.1)(see (4.1)) and $\widehat{u} \in int\ C_+$ (nonlinear regularity).

Similarly, starting with the Carathéodory function

$$g_-(z,x) = \begin{cases} f(z,x) & \text{if } \ x < v_0(z), \\ f(z, v_0(z)) & \text{if } \ v_0(z) \le x, \end{cases}$$

setting $G_-(z,x) = \int_0^x g_-(z,s)\,ds$ and considering the C^1−functional $\psi_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega G_-(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega),$$

we produce $\widehat{v} \le v_0$, $\widehat{v} \ne v_0$, $\widehat{v} \in -int\ C_+$, a second nontrivial solution of (1.1). □

Next, by strengthening the regularity of $f(z,.)$, we can improve the above theorem and produce a second nodal solution, for a total of six nontrivial solutions.

The new hypotheses are the following:

$\mathbf{H}^3(f)$: $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function such that for a.a. $z\in\Omega$, $f(z,0)=0$, $f(z,.)\in C^1(\mathbb{R})$ and:

(i) there exists $a\in L^\infty(\Omega)_+$ such that

$$\left|f_x'(z,x)\right|\leq a(z)\left(1+|x|^{r-2}\right) \text{ for } a.a.z\in\mathbb{R},\ p\leq r<p^*,$$

where

$$p^*=\begin{cases}\frac{Np}{N-p} & \text{if} \quad N>p\\ +\infty & \text{if} \quad N\leq p;\end{cases}$$

(ii) same as hypothesis $\mathbf{H}^1(f)(ii)$;

(iii) there exists an integer $m\geq 2$ such that

$$f_x'(z,0)\in\left[\widehat{\lambda}_m(2),\widehat{\lambda}_{m+1}(2)\right] \text{ for a.a. } z\in\Omega,$$

$$f_x'(.,0)\neq\widehat{\lambda}_m(2),\ f_x'(.,0)\neq\widehat{\lambda}_{m+1}(2), \text{ and}$$

$$f_x'(.,0)=\lim_{x\to 0}\frac{f(z,x)}{x} \text{ uniformly for a.a. } z\in\Omega;$$

(iv) same as hypothesis $\mathbf{H}^2(f)(iv)$.

Theorem 4.2. *If hypotheses $\mathbf{H}^3(f)$ hold, then problem (1.1) has at least six nontrivial solutions*

$$u_0,\ \widehat{u}\in int\ C_+,\ \widehat{u}-u_0\in int\ C_+,\ u_0(z)<w_+(z) \textit{ for all } z\in\overline{\Omega},$$

$$v_0,\ \widehat{v}\in -int\ C_+,\ v_0-\ \widehat{v}\in int\ C_+,\ w_-(z)<v_0(z) \textit{ for all } z\in\overline{\Omega},$$

and

$$y_0,\widehat{y}\in int_{C_0^1(\overline{\Omega})}[v_0,u_0]\cap C_0^1\left(\overline{\Omega}\right), \textit{ nodal.}$$

Proof. From Theorem 4.1, we already have five nontrivial solutions

$$u_0,\ \widehat{u}\in int\ C_+,\ u_0\leq\widehat{u},\ u_0\neq\widehat{u},\ u_0(z)<w_+(z) \text{ for all } z\in\overline{\Omega},$$

$$v_0,\ \widehat{v}\in -int\ C_+,\ \widehat{v}\leq v_0,\ v_0\neq\widehat{v},\ w_-(z)<v_0(z) \text{ for all } z\in\overline{\Omega},$$

and

$$y_0\in[v_0,u_0]\cap C_0^1\left(\overline{\Omega}\right), \text{ nodal.}$$

By virtue of Proposition 3.3, we may assume that u_0, v_0 are extremal nontrivial constant sign solutions of (1.1). Let $\rho=\max\{\|\widehat{v}\|_\infty,\|\widehat{u}\|_\infty\}$. Hypotheses $\mathbf{H}^3(f)(i)$ and (iii) imply that we can find $\xi_\rho^*>0$ such that for almost all $z\in\Omega$, $x\to f(z,x)+\xi_\rho^*|x|^{p-2}x$ is nondecreasing on $[-\rho,\rho]$. Then

$$\begin{aligned}&-\triangle_p u_0(z)-\triangle u_0(z)+\xi_\rho^* u_0(z)^{p-1}=f(z,u_0(z))+\xi_\rho^* u_0(z)^{p-1}\\ &\leq f(z,\widehat{u}(z))+\xi_\rho^*\widehat{u}(z)^{p-1}=-\triangle_p\widehat{u}(z)-\triangle\widehat{u}(z)+\xi_\rho^*\widehat{u}(z)^{p-1} \text{ a.e. in } \Omega.\end{aligned} \tag{4.23}$$

As in the proof of Proposition 3.1, let

$$a(y) = \|y\|^{p-2} y + y \text{ for all } y \in \mathbb{R}^N.$$

Then $a \in C^1(\mathbb{R}^N, \mathbb{R}^N)$ and

$$div\ a(Du) = \triangle_p u + \triangle u \text{ for all } u \in W_0^{1,p}(\Omega).$$

Also, we have

$$\nabla a(y) = \|y\|^{p-2} y \left(I_N + (p-2) \frac{y \otimes y}{\|y\|^2} \right) + I_N.$$

It follows that

$$(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \|\xi\|^2 \text{ for all } y, \xi \in \mathbb{R}^N.$$

So, from the tangency principle of Pucci-Serrin ([16], p. 35), we have

$$u_0(z) < \widehat{u}(z) \text{ for all } z \in \overline{\Omega}. \tag{4.24}$$

From (4.23) and (4.24) and Proposition 2.2, we have

$$\widehat{u} - u_0 \in int\ C_+.$$

A similar argument with the pairs $\{u_0, y_0\}$ and $\{y_0, v_0\}$ shows that $u_0 - y_0$, $y_0 - v_0 \in int\ C_+$. Hence, we infer that

$$y_0 \in int_{C_0^1(\overline{\Omega})}[v_0, u_0].$$

We know that y_0 is a critical point of mountain pass type for the functional φ_0 (see the proof of Proposition 3.1). Therefore

$$C_1(\varphi_0, y_0) \neq 0 \text{ (see (3.9))}. \tag{4.25}$$

Let $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z, u(z))\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Note that $\varphi \in C^2\left(W_0^{1,p}(\Omega)\right)$. Moreover, reasoning as in the proof of Theorem 4.1 (see Claim 3), we show that φ satisfies the $C-$condition.

We consider the homotopy $h(t, u)$ defined by

$$h(t, u) = t\varphi_0(u) + (1-t)\varphi(u) \text{ for all } (t, u) \in [0,1] \times W_0^{1,p}(\Omega).$$

Suppose that we can find $\{t_n\}_{n\geq 1} \subset [0,1]$ and $\{y_n\}_{n\geq 1} \subset W_0^{1,p}(\Omega)$ such that

$$t_n \to t,\ y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and } h'_u(t_n, u_n) = 0 \text{ for all } n \geq 1. \tag{4.26}$$

We have

$$A_p(y_n) + A(y_n) = t_n N_{f_0}(y_n) + (1-t_n) N_f(y_n) \text{ for all } n \geq 1$$

(see (3.5)), hence

$$-\triangle_p y_n(z) - \triangle y_n(z) = t_n f_0(z, y_n(z)) + (1-t_n) f(z, y_n(z)) \text{ a.e. in } \Omega,$$
$$y_n|_{\partial\Omega} = 0.$$

From Ladyzhenskaya-Uraltseva ([13], p.286), we know that we can find $M_5 > 0$ such that

$$\|y_n\|_\infty \leq M_5 \text{ for all } n \geq 1.$$

Then, from Lieberman ([14], p.320), it follows that there exists $\alpha \in (0,1)$ and $M_6 > 0$ such that

$$y_n \in C_0^{1,\alpha}\left(\overline{\Omega}\right) \text{ and } \|y_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq M_6 \text{ for all } n \geq 1.$$

Exploiting the compact embedding of $C_0^{1,\alpha}\left(\overline{\Omega}\right)$ into $C_0^1\left(\overline{\Omega}\right)$, we may assume that

$$y_n \to y_0 \text{ in } C_0^1\left(\overline{\Omega}\right) \text{ (see (4.26))},$$

hence

$$y_n \in int_{C_0^1(\overline{\Omega})}[v_0, u_0] \text{ for all } n \geq n_0 \text{ large enough},$$

therefore $\{y_n\}_{n \geq n_0} \subset C_0^1\left(\overline{\Omega}\right)$ are all distinct nodal solutions of (1.1) and so, we are done.

This means that we may assume that for $\rho > 0$ small, we have

$$K_{h(t,.)} \cap \overline{B}_\rho(y_0) = \{y_0\} \text{ for all } t \in [0,1],$$

where $\overline{B}_\rho(y_0) = \left\{u \in W_0^{1,p}(\Omega) : \|u - y_0\| \leq \rho\right\}$. The homotopy invariance property of critical groups implies that

$$C_k(\varphi, y_0) = C_k(\varphi_0, y_0) \text{ for all } k \geq 0,$$

hence

$$C_1(\varphi, y_0) \neq 0 \text{ (see (4.25))},$$

therefore

$$C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0$$

(see Aizicovici-Papageorgiou-Staicu [1]), and we conclude that

$$C_k(\varphi_0, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.27}$$

Recall that u_0, v_0 are local minimizers of φ_0 (see the proof of Proposition 3.1). So, we have

$$C_k(\varphi_0, u_0) = C_k(\varphi_0, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.28}$$

Also, as in Claim 2 in the proof of Proposition 3.1, we have

$$C_k(\varphi_0, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.29}$$

Finally, recall that φ_0 is coercive (see (3.5)). Hence

$$C_k(\varphi_0, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0.$$

Suppose that $K_{\varphi_0} = \{0, y_0, u_0, v_0\}$. From (4.27), (4.28), (4.29) and the Morse relation with $t = -1$ (see (2.1)), we have

$$(-1)^{d_m} + (-1)^1 + 2(-1)^0 = (-1)^0,$$

hence

$$(-1)^{d_m} = 0,$$

a contradiction. So, there exists $\widehat{y} \in K_{\varphi_0}$, $\widehat{y} \notin \{0, y_0, u_0, v_0\}$. Hence $\widehat{y} \in [v_0, u_0]$ (see the proof of Proposition 3.1). Therefore $\widehat{y}$ is a nodal solution of (1.1) (see (3.5) and recall that v_0, u_0 are extremal) and $\widehat{y} \in C_0^1\left(\overline{\Omega}\right)$ (nonlinear regularity). Moreover, as we did for y_0, using Proposition 2.2, we conclude that $\widehat{y} \in int_{C_0^1(\overline{\Omega})}\left[v_0, u_0\right]$. $\square$

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