Journal of Nonlinear and Convex Analysis
Volume 15, Number 1, 2014, 61–87

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

FIRMLY NONEXPANSIVE MAPPINGS

DAVID ARIZA-RUIZ, GENARO LÓPEZ-ACEDO, AND VICTORIA MARTÍN-MÁRQUEZ

Abstract. Firmly nonexpansive mappings play an important role in nonlinear analysis due to their correspondence with maximal monotone operators. The purpose of this work is twofold: to gather the most significant steps in the development of the theory of firmly nonexpansive mappings, and at the same time, to show the great influence of Reich's works in this direction.

1. Introduction

In the sixties the interest in the study of nonexpansive mappings experimented a boost. This was basically motivated by two facts: Browder's works on the relationship between monotone operators and nonexpansive mappings [15–20], and the seminal paper by Kirk [46] where the significance of the geometric properties of the norm for the existence of fixed points for nonexpansive mappings was highlighted. We are going to focus on the first of these facts.

Monotone operators in Hilbert spaces were introduced by Minty [65] in order to study the properties of the differential of a convex function with convex domain. These operators appear in modeling many problems in nonlinear analysis, mainly in convex analysis [11] and partial differential equations [9]; that is why the study of the class of monotone operators has became one of the main research topic in these fields. In this context, the connection between monotone operators and nonexpansive type mappings, more precisely firmly nonexpansive, has been essential. As we have previously pointed out, this relationship was exemplified by Browder in Hilbert spaces along the sixties. Browder's results were extended to the setting of Banach spaces by Bruck and Reich in the seventies; and for the last 40 years the contributions of Prof. Simeon Reich in this area, specially concerning the knowledge on the asymptotic behavior of firmly nonexpansive mappings and the extension to nonlinear metric spaces, have been extensive and very fruitful [22, 36, 37, 55, 73–87]. The purpose of this work is twofold: to gather the most significant steps in the development of the theory of firmly nonexpansive mappings, and at the same time, to show the great influence of Reich's works in this direction.

The history of firmly nonexpansive mappings goes back to the paper by Minty [66], where he implicitly used this class of mappings to study the resolvent of a monotone operator. Firmly nonexpansive mappings were first defined by Browder [17], although he named them *firmly contractive*. In this definition the interior product

---

2010 *Mathematics Subject Classification.* Primary 47H09, 47H10, 53C22; Secondary 03F10, 47H05, 90C25, 52A41.

Key words and phrases. Firmly nonexpansive mapping, resolvent, geodesic space, uniform convexity, fixed point, Δ-convergence, Picard iterates, asymptotic regularity.

This work was partially supported by DGES, Grant MTM2012-34847-C02-01 and Junta de Andalucía, Grant P08-FQM-03453.

turns out to be essential. Given C a closed convex subset of a Hilbert space H, a mapping $T : C \to H$ is *firmly nonexpansive* if for all $x, y \in C$

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle. \tag{1.1}$$

A basic example of firmly nonexpansive mappings is the metric projection

$$P_C(x) = \underset{y \in C}{\operatorname{argmin}}\{\|x - y\|\}.$$

This operator appears in many iterative methods for convex minimization problems, and its asymptotic behavior plays an essential role in the convergence of these algorithms, see [24] for a detailed treatment of this question.

Another prominent example of firmly nonexpansive mapping is the resolvent of monotone operators. Given a maximal monotone operator $A : H \to 2^H$ and $\mu > 0$, its associated *resolvent* of order μ, defined by

$$J_\mu^A := (I + \mu A)^{-1},$$

where I denotes the identity operator, is a firmly nonexpansive mapping from H to H with full domain and the set of fixed points of J_μ^A coincides with the set of zeros of A. We refer to [12] for a very nice presentation of this correspondence in Hilbert spaces.

In his study of nonexpansive projections on subsets of Banach spaces, Bruck [21] defined a *firmly nonexpansive* mapping $T : C \to E$, where C is a closed convex subset of a real Banach space E, to be a mapping such that for all $x, y \in C$ and $t \geq 0$,

$$\|Tx - Ty\| \leq \|(1 - t)(Tx - Ty) + t(x - y)\|. \tag{1.2}$$

In Hilbert spaces this definition turns into the one introduced by Browder, and also in Banach spaces a firmly nonexpansive mapping is characterized by being the resolvent of an accretive operator, see [37]. As Bruck showed, to any nonexpansive self-mapping $T : C \to C$ that has fixed points, one can associate a "large" family of firmly nonexpansive mappings having the same fixed point set as T. Hence, from the point of view of the existence of fixed points on convex closed sets, nonexpansive and firmly nonexpansive mappings exhibit a similar behavior. However, this is no longer true in non-convex domains [92].

If T is firmly nonexpansive and has fixed points, it is well known [17] that the sequence of Picard iterates $\{T^n x\}_{n \in \mathbb{N}}$ is asymptotically regular and, in more particular settings, converges weakly to a fixed point of T for any starting point x [86], while this is not true for nonexpansive mappings in general (take, for instance, $T = -I$).

A fruitful research direction is the extension of techniques and results from normed spaces to metric spaces without linear structure. For instance, minimization problems associated to convex functionals have been studied in the setting of Riemannian manifolds [32, 59], and some problems have been modeled as abstract Cauchy equations in the framework of nonpositive curvature geodesic metric spaces, see [64, 96] and references therein. Although the framework and the conceptual approach in the previous problems are seemingly quite different, it is possible, as in the case of normed spaces, to find a bridge between them through firmly nonexpansive mappings.

Other two apparently unrelated theories are the ones which deals with nonexpansive mappings and holomorphic mappings. Nevertheless, one of the links between them is the fact that holomorphic mappings are nonexpansive with respect to certain pseudo-metrics. In connection with this problem, Goebel and Reich [37] initiated the study of firmly nonexpansive mappings in a nonlinear metric setting, to be precise in the Hilbert ball with the Poincare metric. The extension of their results to hyperbolic metric spaces is due to Reich and Shafir [87]. In [60] the authors go further over the relationship between monotone operators and firmly nonexpansive mappings in the particular case of Hadamard manifolds.

To study the validity of Trotter-Kato formula [45] in the setting of gradient flows on geodesics spaces [6, 8, 95, 96], if one wants to follow the argument of the proof in Hilbert spaces, it is necessary a counterpart of the approximation of semigroups and their resolvents in these spaces. This fact motivates the study of firmly nonexpansive mappings in this setting. A first step in this direction has been already taken in [4], and related works are in progress [7, 68]. Further steps pass for a suitable definition of monotone operators in geodesic spaces, being achieved in the particular case of the Hilbert ball [55, 90] and Hadamard manifolds [60].

The remainder of the paper is organized as follows. In Section 2 we fix the notation and introduce some basic facts on geodesic metric spaces. For a comprehensive treatment of geodesic metric spaces one may check for example [14, 71]. In Section 3 we focus on the different characterizations of firmly nonexpansive mappings, which turn out to depend strongly on the framework space considered. Section 4 is devoted to the relationship between firmly nonexpansive mappings and other classes of mappings: contractions, nonexpansive mappings, strongly nonexpansive mappings, λ-firmly nonexpansive mappings, metric projections and resolvents of accretive operators. The existence and approximation of fixed points is the topic of Sections 5 and 6. In the former the existence of fixed points and periodic points is analyzed. The treatment given to the approximation problem is grounded on the asymptotic center concept.

2. Notations and preliminaries

In this section, we collect some basic definitions and facts which will be used in the following sections. Through out this paper, a Hilbert space will be denoted by H, and a normed or metric space by X.

Recall that a **geodesic space** X is a metric space satisfying that every pair of points x and y in X can be joined by a **geodesic**, that is, a map γ from a closed interval $[0, l]$ to X such that $\gamma(0) = x$, $\gamma(l) = y$, and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, l]$. The image $\gamma([0, l])$ is called a **geodesic segment** joining x and y. This geodesic segment will be denoted $[x, y]$, provided that there is no possible ambiguity. If for any points there exists a unique geodesic joining them, then X is said to be **uniquely geodesic**.

A **geodesic triangle** $\Delta(x_1, x_2, x_3)$ in a geodesic space X consists of three points x_1, x_2, x_3 in X (called **vertices** of Δ) and a geodesic segment between each pair of vertices (so-called the **edges** of Δ). A **comparison triangle** for a geodesic triangle

$\Delta(x_1, x_2, x_3)$ in X is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in $(\mathbb{R}^2, \|\cdot\|)$ such that $d(x_i, x_j) = \|\bar{x}_i - \bar{x}_j\|$ for all $i, j \in \{1, 2, 3\}$. Notice that comparison triangles always exist and are unique up to isometry. See [14, Lemma 2.14].

We are considering hyperbolic spaces which encompasses normed linear spaces [52]. In order to distinguish these spaces from Gromov hyperbolic spaces [14], or from other notions of "hyperbolic space" which can be found in the literature (we refer to [53, p.384] for a nice comparison analysis), we shall adopt the terminology first used in [58]: W-hyperbolic spaces.

Definition 2.1. A **W-hyperbolic space** (X, d, W) is a metric space (X, d) together with a convexity mapping $W : X \times X \times [0, 1] \to X$ satisfying the following four properties

(W_1) $d(z, W(x, y, \lambda)) \le (1 - \lambda)d(z, x) + \lambda d(z, y)$,
(W_2) $d(W(x, y, \lambda), W(x, y, \tilde{\lambda})) = |\lambda - \tilde{\lambda}| \cdot d(x, y)$,
(W_3) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$,
(W_4) $d(W(x, z, \lambda), W(y, w, \lambda)) \le (1 - \lambda)d(x, y) + \lambda d(z, w)$,

for all $x, y, z, w \in X$ and $\lambda, \tilde{\lambda} \in [0, 1]$.

From now on, if X is a W-hyperbolic space, given $x, y \in X$ and $\lambda \in [0, 1]$, $(1 - \lambda)\, x \oplus \lambda\, y$ stands for $W(x, y, \lambda)$.

Remark 2.2. Notice that condition (W_1) implies that all balls are convex in the following sense: a nonempty subset C of a W-hyperbolic space (X, d) is said to be **convex** if $(1 - \lambda)\, x \oplus \lambda\, y \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. The reciprocal implication, in general, is not true, see [33]. On the other hand, condition (W_2) can be interpreted as the continuity (with respect to the third variable) of the mapping W, which implies that every W-hyperbolic space is a geodesic space. Condition (W_3) yields the symmetry in the third variable, and as a consequence of the last condition (W_4), we deduce the equivalence between W-hyperbolic spaces and Busemann spaces, when the convexity mapping is unique, see [4, Proposition 2.6].

Examples of W-hyperbolic spaces are the following.

- Normed linear spaces. It is easy to check that the class of W-hyperbolic spaces includes the convex subsets of normed linear spaces, just setting $(1 - \lambda)\, x \oplus \lambda\, y = (1 - \lambda)x + \lambda y$.
- The Hilbert ball. Let $\mathbb{B}$ be the open unit ball of a complex Hilbert space H. The Hilbert ball is $\mathbb{B}$ with the hyperbolic distance, also called Kobayashi distance, defined by

$$d(x, y) := \operatorname{argtanh}\big(1 - \sigma(x, y)\big)^{1/2},$$

where

$$\sigma(x, y) := \frac{(1 - \|x\|^2)(1 - \|y\|^2)}{|1 - \langle x, y\rangle|^2}\,.$$

The **Hilbert ball** $\mathbb{B}$ is a uniquely geodesic space. Then, in this case, $(1 - \lambda)\, x \oplus \lambda\, y$ stands for the unique point $z \in \mathbb{B}$ such that

$$d(x, z) = \lambda d(x, y) \quad \text{and} \quad d(y, z) = (1 - \lambda)d(x, y),$$

being the convexity mapping which makes $\mathbb{B}$ a W-hyperbolic space. See [37] for a book treatment.

- CAT(0) spaces. A geodesic space is said to be a **CAT(0) space** if all distances between points on the sides of a geodesic triangles Δ are no larger than the distances between the corresponding points on any comparison triangle $\bar{\Delta}$ in $\mathbb{R}^2$. Every CAT(0) space is a W-hyperbolic space where the convexity mapping is defined as in the case of the Hilbert ball. For more details, see [14].
- Busseman spaces. This class of spaces was introduced by Busemann [23] to define a notion of "nonpositively curved space". Let us recall that a map $\gamma : [a,b] \to X$ is an **affinely reparametrized geodesic** if γ is a constant path or there exist an interval $[c,d]$ and a geodesic $\gamma' : [c,d] \to X$ such that $\gamma = \gamma' \circ \psi$, where $\psi : [a,b] \to [c,d]$ is the unique affine homomorphism between the intervals $[a,b]$ and $[c,d]$. A geodesic space X is a **Busemann space** if for any two affinely reparametrized geodesics $\gamma : [a,b] \to X$ and $\gamma' : [c,d] \to X$, the function $D_{\gamma,\gamma'} : [a,b] \times [c,d] \to \mathbb{R}$, defined by
$$D_{\gamma,\gamma'}(s,t) = d(\gamma(s), \gamma'(t))$$
is convex. CAT(0) spaces are Busemann spaces, see [71, Example 8.1.3] for more examples, and all Busemann spaces are W-hyperbolic spaces with the same convexity mapping as CAT(0) spaces, see [4, Proposition 2.6].

In 1936 Clarkson [27] introduced and studied a new class of Banach spaces called uniformly convex spaces. In geometrical terms the uniform convexity means that the midpoint of a variable chord of the unit sphere of the space cannot approach the surface of the sphere unless the length of the chord goes to zero. Following [37, p. 105], Leuştean [57] defined a counterpart of this concept in the setting of W-hyperbolic spaces.

Definition 2.3. A W-hyperbolic space (X,d,W) is said to be **uniformly convex** if for any $r > 0$ and any $\varepsilon \in (0,2]$ there exists $\delta \in (0,1]$ such that for all $a,x,y \in X$,
$$\left.\begin{array}{l} d(x,a) \le r \\ d(y,a) \le r \\ d(x,y) \ge \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d\big(\tfrac{1}{2}x \oplus \tfrac{1}{2}y, a\big) \le (1-\delta)\, r.$$

Remark 2.4. It should be pointed out that, unlike Banach spaces setting where there exists a natural modulus of convexity for each space which only depends on ε, in W-hyperbolic spaces we need to assume that the modulus depends on two variables: the radius of the ball and the separation condition given by ε. A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ providing the existence of such a $\delta := \eta(r,\varepsilon)$ for a given $r > 0$ and $\varepsilon \in (0,2]$ is called a **modulus of uniform convexity**. We say that η is monotone if it decreases with r (for a fixed ε). Following [58], we shall refer to uniformly convex W-hyperbolic spaces with a monotone modulus of uniform convexity η as ***UCW*-hyperbolic spaces** and denote it by (X,d,W,η). Any UCW-hyperbolic space is a Busemann space [4].

Remark 2.5. Uniformly convex Banach spaces with the natural modulus of convexity are UCW-hyperbolic, see [57]; and so are CAT(0) spaces with a modulus of uniform convexity $\eta(r,\varepsilon) = \frac{\varepsilon^2}{8}$, that does not depend on r and is quadratic in ε. For more information about the modulus of convexity of CAT(κ) spaces see [30, 31, 37].

3. Definitions and main properties

Let us start by giving the definition of a firmly nonexpansive mapping in a Banach space $(X, \|\cdot\|)$.

Definition 3.1 (In *Banach spaces*)**.** Let D be a nonempty subset of a Banach space X. We say that a mapping $T : D \to X$ is **firmly nonexpansive** if, for all $x, y \in D$, the convex function $\Phi : [0,1] \to [0,+\infty]$ defined by

$$\Phi(s) = \|(1-s)(x-y) + s(Tx - Ty)\| \tag{3.1}$$

is nonincreasing.

Recall that in any metric space (X, d), a map $T : D \subset X \to X$ is said to be **nonexpansive** if for all $x, y \in D$,

$$d(Tx, Ty) \le d(x, y).$$

Then firmly nonexpansive mappings constitute a subclass of nonexpansive mappings.

Proposition 3.2 (Equivalent definitions in *Banach spaces* [37, Lemma 11.1])**.** *Let D be a nonempty subset of a Banach space $(X, \|\cdot\|)$, T a mapping from D into X, and J the duality map of X, that is, $J(x) := \left\{ j \in X^* : \langle x, j\rangle = \|x\|^2 = \|j\|^2 \right\}$, for $x \in X$. Then the following statements are equivalent:*

(a) *T is firmly nonexpansive.*
(b) *For each $x, y \in D$, there is $j \in J(Tx - Ty)$ such that $\|Tx - Ty\|^2 \le \langle x - y, j\rangle$.*
(c) *For each $x, y \in D$, $\|Tx - Ty\| \le \|r(x-y) + (1-r)(Tx - Ty)\|$ for all $r > 0$.*
(d) *For each $x, y \in D$, $\|Tx - Ty\| \le \|(1-s)(x-y) + s(Tx - Ty)\|$ for all $0 \le s \le 1$.*

In a Hilbert space $(H, \langle\cdot,\cdot\rangle)$, since the duality mapping is the identity, the inequality in Proposition 3.2 (b) turns into the classical definition within this setting: $T : D \subset H \to H$ is firmly nonexpansive if for all $x, y \in D$

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle,$$

which yields the following equivalences.

Proposition 3.3 (Equivalent definitions in *Hilbert spaces* [35, Theorem 12.1], [11])**.** *Let D be a nonempty subset of a Hilbert space H. For a mapping $T : D \to H$ the following are equivalent.*

(a) *T is firmly nonexpansive.*
(b) *$2T - I$ is nonexpansive.*
(c) *$T = \frac{1}{2}(I + S)$ with S nonexpansive.*
(d) *$0 \le \langle Tx - Ty, (I-T)x - (I-T)y\rangle$, for all $x, y \in D$.*
(e) *$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(x - Tx) - (y - Ty)\|^2$, for all $x, y \in D$.*

Remark 3.4 (*Averaged mappings*)**.** Proposition 3.3 (b) means that every firmly nonexpansive mapping T in a Hilbert space is **averaged**, that is, $T = \alpha I + (1-\alpha)S$ for some nonexpansive mapping S and some $\alpha \in (0,1)$. In particular, we say that every firmly nonexpansive mapping is 1/2-averaged. This is no longer true when it comes to a general Banach space. Still there is a class of Banach spaces (see definition bellow) containing the Hilbert spaces where firmly nonexpansive mappings are averaged.

Definition 3.5 (*Property (S)*)**.** We say that a Banach space X has **property (S)** if there exists a constant $b > 0$ such that if $\|x + r\,y\| \geq \|x\|$ for all $r \geq 0$, then $\|x + y\| \geq \|x - b\,y\|$.

Proposition 3.6 ([22])**.** *Let X be a Banach space having property (S). If T is firmly nonexpansive, then T is averaged for $\frac{1}{1+b} \leq \alpha \leq 1$. Moreover, if T is averaged for $0 \leq \alpha \leq \frac{b}{1+b}$, then T is firmly nonexpansive.*

Remark 3.7. A Hilbert space has property (S) with $b = 1$, and thus Proposition 3.6 implies the equivalence between (a) and (c) in Proposition 3.3. There are also non-Hilbert spaces with this property, for instance, any L^p space, with $1 < p < \infty$, has property (S) (for more details, see Proposition 3.1 in [22]). Note that if X has property (S), then it is strictly convex.

The concept of firmly nonexpansive mappings was also defined in metric spaces with hyperbolic geometry; in particular, in the Hilbert ball it was called firmly nonexpansivity of the first kind, see [37, 86].

Definition 3.8 (In the *Hilbert ball*)**.** Let $\mathbb{B}$ be the Hilbert ball with its metric d. We say that a mapping $T : \mathbb{B} \to \mathbb{B}$ is **firmly nonexpansive** if, for all $x, y \in \mathbb{B}$, the convex function $\Phi : [0,1] \to [0, +\infty)$ defined by

$$\Phi(s) = d((1-s)x \oplus sTx, (1-s)y \oplus sTy) \tag{3.2}$$

is nonincreasing. Recall that the notation $(1-s)x \oplus sTx$ stands for the unique point $z \in \mathbb{B}$ such that $d(x,z) = sd(x,Tx) =$ and $d(Tx,z) = (1-s)d(x,Tx)$.

Remark 3.9. The fact that a mapping T is firmly nonexpansive if and only if is 1/2-averaged on Hilbert spaces does not carry over to this setting, see [37]. Furthermore, this definition seemingly does not enjoy the corresponding equivalences shown in Proposition 3.2 for Banach settings. However, firmly nonexpansive mappings were also studied in other metric spaces with non-linear structure: *Hadamard manifolds*, see [60]. In this case the formulation given in Definition 3.8 is equivalent to the one appearing in Proposition 3.2 (d). This fact motivates the next definition given in more general metric spaces: *W-hyperbolic spaces*, see [4].

Definition 3.10 (In *W-hyperbolic spaces*)**.** Let (X, d, W) be a W-hyperbolic space. We say that a mapping $T : D \subset X \to X$ is **firmly nonexpansive** if, for all $x, y \in D$,

$$d(Tx, Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty),$$

for all $\lambda \in (0,1)$.

Remark 3.11. It is worth mentioning that in this framework every firmly nonexpansive mapping is nonexpansive as well, see [4]. However, whether this definition would be consistent with Definition 3.8 in this setting, as happens in Hadamard manifolds, remains un-known.

Among all nonexpansive mappings, the class of firmly nonexpansive mappings is probably the closest to the class of projections (see Subsection 4.5) and contains the class of resolvents (see Subsection 4.6). It possesses, however, a serious drawback: the class of firmly nonexpansive mappings is not closed under composition as shown in the following example.

Example 3.12. Consider in $H = \mathbb{R}^2$ the mappings

$$T_1(x, y) := P_{\text{span}\{(1,0)\}}(x, y) = (x, 0)$$

and

$$T_2(x, y) := P_{\text{span}\{(1,1)\}}(x, y) = \tfrac{1}{2}(x + y, x + y).$$

Then, $T_2T_1(x, y) = \frac{1}{2}(x, x)$ and

$$\langle T_2T_1(1, -2) - T_2T_1(0, 0), (1, -2) - (0, 0)\rangle = -\frac{1}{2} \ngeq 0.$$

Thus, T_2T_1 is not firmly nonexpansive while both T_1 and T_2 are firmly nonexpansive.

An useful substitute to overcome this gap was provided by Bruck and Reich [22] who introduced the class of *strongly nonexpansive* mappings in a Banach space (see Subsection 4.3) which is closed under composition and preserves characteristic features of projections. However, in a Hilbert space the class of firmly nonexpansive mappings is closed under convex combination thanks to the fact that so is the class of nonexpansive mappings and the characterization in Proposition 3.3 (c).

Other concepts of firmly nonexpansive type mappings have been considered, see for instance [1–3] and inequality (6.6).

4. Relationship with other mappings

We have already pointed out that the class of firmly nonexpansive mappings is contained in the class of nonexpansive mappings. This section is devoted to the analysis of this class in relation with other classes of mappings, such as, *contractions, strongly nonexpansive mappings, λ-firmly nonexpansive mappings, projections* and *resolvents of accretive and monotone operators*. See the diagram in Figure 1, at the end of this section.

4.1. **Contractions.** In a metric space (X, d), a mapping $T : X \to X$ is said to be a **contraction** if there exists a constant $\alpha \in (0, 1)$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq \alpha d(x, y).$$

It is obvious that every contraction is a nonexpansive mapping. But there is no relationship between contractions and firmly nonexpansive mappings. Indeed, it is easy to see just considering the identity map that firmly nonexpansive mappings are not contractions in general. Conversely, taking the mapping $T : \mathbb{R} \to \mathbb{R}$, in $(\mathbb{R}, |\cdot|)$, defined by $Tx = -kx$ for some $k \in (0, 1)$ and all $x \in \mathbb{R}$, here is an example of contraction which is not firmly nonexpansive.

4.2. **Nonexpansive mappings.** Firmly nonexpansive mappings belong to the class of nonexpansive mappings even in *W-hyperbolic spaces*, see [4]. Furthermore, these mappings turn out to characterize nonexpansive mappings in the following sense.

Proposition 4.1 ([4, 37])**.** *Let C be a nonempty closed convex subset of either a Banach space or a complete Busemann space X. If $T : C \to C$ is a nonexpansive mapping, then we can associate a family of firmly nonexpansive mappings with the same fixed point set, defined by*

$$U_t : C \to C, \quad U_t(x) = z_t^x,$$

where z_t^x is the unique fixed point of the contraction $T_t^x = (1-t)x + tT$ (if X is a Banach space) or $T_t^x = (1-t)x \oplus tT$ (if X is a complete Busemann space).

Remark 4.2. The most recent version of Proposition 4.1, for complete Busemann spaces, generalizes the counterpart proved previously in the Hilbert ball [37] and in Hadamard manifolds [60]. A consequence of this result is that, from the point of view of fixed point theorems for the class of closed convex subsets C, firmly nonexpansive mappings $T : C \to C$ do not exhibit better behavior than nonexpansive mappings. However, this fact is no longer true if C is nonconvex. For more details, see Section 5.

4.3. **Strongly nonexpansive mappings.** In 1977 Bruck and Reich [22] introduced the concept of *strongly nonexpansive mappings* in Banach spaces so that it is a class of nonexpansive mappings properly containing the class of firmly nonexpansive mappings and with the advantage of being closed under composition opposite to firmly nonexpansive mappings which are not.

Definition 4.3. A mapping $T : D \to X$, where D is a nonempty subset of a Banach space $(X, \|\cdot\|)$, is **strongly nonexpansive** if T is nonexpansive and

$$\lim_{n\to\infty} x_n - y_n - (Tx_n - Ty_n) = 0$$

whenever $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in D such that $\{x_n - y_n\}_{n\in\mathbb{N}}$ is bounded and

$$\lim_{n\to\infty} \|x_n - y_n\| - \|Tx_n - Ty_n\| = 0.$$

Proposition 4.4 (Proposition 2.1 in [22])**.** *Let D be a nonempty subset of a uniformly convex Banach space $(X, \|\cdot\|)$. If $T : D \to X$ is firmly nonexpansive, then T is also strongly nonexpansive. And the class of strongly nonexpansive mappings is closed under composition.*

The assumption on uniform convexity is necessary since $T : C[0,1] \to C[0,1]$ defined by $(Tf)(t) = t\,f(t)$, for $0 \le t \le 1$, is firmly nonexpansive but is not strongly nonexpansive. This lager class of mappings, apart from the closedness under composition, still enjoy good properties, for example, regarding the approximation of fixed point, see [22].

Remark 4.5. Notice that in Banach spaces the class of strongly nonexpansive mappings contains the contractions. Indeed, given $T : D \subseteq X \to X$ a contraction with constant $\alpha \in [0,1)$, then if $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are sequences in D such that $\{x_n - y_n\}_{n\in\mathbb{N}}$ is bounded and

$$\lim_{n\to\infty} \|x_n - y_n\| - \|Tx_n - Ty_n\| = 0,$$

we have that

$$\|(x_n - y_n)\| - \|(Tx_n - Ty_n)\| \geq \|x_n - y_n\| - \alpha \|x_n - y_n\| = (1-\alpha) \|x_n - y_n\| .$$

Thus $\lim_{n\to\infty} \|x_n - y_n\| = \lim_{n\to\infty} \|Tx_n - Ty_n\| = 0$, and so

$$\lim_{n\to\infty} x_n - y_n - (Tx_n - Ty_n) = 0.$$

4.4. **λ-firmly nonexpansive mappings.** There exists a class of nonexpansive mappings that encompasses the firmly nonexpansive mappings and for which some fixed point results remain true. This extension has been widely analyzed in W-hyperbolic spaces, see [4]. It consists in requiring the condition in Definition 3.10 just for some $\lambda \in (0,1)$.

Definition 4.6. Let (X,d,W) be a W-hyperbolic space and $\lambda \in (0,1)$. We say that a mapping $T : D \subset X \to X$ is **λ-firmly nonexpansive** if, for all $x, y \in D$,

$$d(Tx, Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty).$$

Obviously, given $\lambda \in (0,1)$, every firmly nonexpansive mapping is λ-firmly nonexpansive. Nevertheless, the converse is not true in general because contractions with constant α are λ-firmly nonexpansive for $\lambda = \frac{1-\alpha}{1+\alpha}$. Furthermore, from the proof of Proposition 4.4 it can be deduced, in uniformly convex Banach spaces, that every λ-firmly nonexpansive mapping is strongly nonexpansive.

4.5. **Metric projection and Sunny nonexpansive retractions.** Let (X,d) be a metric space and K be a nonempty subset of X. For every $x \in K$, the distance between the point x and K is denoted by $d(x,K)$ and is defined by the following minimization problem:

$$d(x,K) := \inf_{y\in K} d(x,y).$$

The **metric projection operator**, also called the **nearest point mapping** onto the set K is the mapping $P_K : X \to 2^K$ defined by

$$P_K(x) := \{z \in K : d(x,z) = d(x,K)\} \quad \forall\, x \in X,$$

If $P_K(x)$ is singleton for every $x \in X$, then K is said to be a **Chebyshev** set.

Proposition 4.7 ([37, Proposition 3.1], [14])**.** *Every closed convex subset of either a uniformly convex Banach space or a CAT(0) space is a Chebyshev set.*

Remark 4.8. Actually it is well-known that any nonempty closed convex subset of a normed space $(X, \|\cdot\|)$ is a Chebyshev set if and only if X is reflexive and strictly convex [50].

In a Hilbert space, a classic example of firmly nonexpansive mapping is in fact the metric projection onto a closed convex set C, see [17, Proposition 2]. Furthermore, this remains true in a CAT(0) space [4].

Proposition 4.9. *The metric projection onto a nonempty closed convex set of a CAT(0) space is a firmly nonexpansive mapping.*

Out of the setting of Hilbert and CAT(0) spaces, whether the projection is firmly nonexpansive in general is an open question. However, in some particular situation, one is able to prove the extension of this fact, as we now show.

Lemma 4.10. *Let C be a Chebyshev set in a geodesic space (X,d) and $x \in X$. Then,*

$$P_C\big((1-\lambda)x \oplus \lambda P_C x\big) = P_C x \qquad \text{for all } \lambda \in [0,1]. \tag{4.1}$$

Proof. If the result was false, then there would exist $0 < \lambda < 1$ and $z \in C$ such that

$$d((1-\lambda)x \oplus \lambda P_C x, z) < d((1-\lambda)x \oplus \lambda P_C x, P_C x).$$

Hence,

$$\begin{aligned} d(x,z) &\le d(x,(1-\lambda)x \oplus \lambda P_C x) + d((1-\lambda)x \oplus \lambda P_C x, z) \\ &< d(x,(1-\lambda)x \oplus \lambda P_C x) + d((1-\lambda)x \oplus \lambda P_C x, P_C x) \\ &= \lambda\, d(x, P_C x) + (1-\lambda)\, d(x, P_C x) \\ &= d(x, C), \end{aligned}$$

which is a contradiction. □

Definition 4.11 ([72])**.** We say that a metric space (X,d) has **property (P)** if the metric projection onto any Chebyshev set is a nonexpansive mapping.

Note that any CAT(0) space has property (P). On the other hand, property (P) characterizes inner product spaces of dimension ≥ 3, see [72, Theorem 5.2], for more details.

Theorem 4.12. *Let C be a Chebyshev set in a geodesic space (X,d) with property (P). The metric projection onto C is a firmly nonexpansive mapping.*

Proof. Let $x, y \in X$ and $\lambda \in (0,1)$. Using (4.1) and the fact that X has property (P), we get

$$\begin{aligned} d(P_C x, P_C y) &= d\big(P_C((1-\lambda)x \oplus \lambda P_C x), P_C((1-\lambda)y \oplus \lambda P_C y)\big) \\ &\le d((1-\lambda)x \oplus \lambda P_C x, (1-\lambda)y \oplus \lambda P_C y). \end{aligned}$$

That is, P_C is λ-firmly nonexpansive, for all $\lambda \in (0,1)$. Therefore, P_C is a firmly nonexpansive. □

In general cases, despite not counting with the projection, there exist other tools playing a similar role which generalize the projection and characterize the firm nonexpansivity, see for instance [99]. This is the case of the *sunny nonexpansive retractions* in Banach spaces.

Given a subset K of C in a Banach space $(X, \|\cdot\|)$ and a mapping $T : C \to K$. Recall that T is a **retraction** of C onto K if $Tx = x$ for all $x \in K$. We say that T is **sunny** if for each $x \in C$ and $t \in [0,1]$, we have

$$T(tx + (1-t)Tx) = Tx,$$

whenever $tx + (1-t)Tx \in C$. Furthermore, T is a **sunny nonexpansive retraction** from C onto K if T is a retraction from C onto K which is also sunny and nonexpansive.

The following result [21, 37, 73] characterizes sunny nonexpansive retractions on a smooth Banach space.

Lemma 4.13. *Let $(X, \|\cdot\|)$ be a smooth Banach space and let $K \subseteq C$ be nonempty closed convex subsets of X. Assume that $Q : C \to K$ is a retraction from C onto K. Then the following three statements are equivalent.*

(a) *Q is sunny and nonexpansive.*
(b) *Q is firmly nonexpansive.*
(c) *$\langle x - Qx, J(y - Qx)\rangle \leq 0$ for all $x \in C$ and $y \in K$.*

Consequently, there is at most one sunny nonexpansive retraction from C onto K.

Note that when X is a Hilbert space the unique sunny nonexpansive retraction from C to K is the metric projection onto K since (c) turns into its characterization inequality.

The first result regarding the existence of sunny nonexpansive retractions on the fixed point set of a nonexpansive mapping is due to Bruck [21].

Theorem 4.14. *If $(X, \|\cdot\|)$ is strictly convex and uniformly smooth and if $T : C \to C$ is a nonexpansive mapping having a nonempty fixed point set $\mathrm{Fix}\,(T)$, then there exists a sunny nonexpansive retraction of C onto $\mathrm{Fix}\,(T)$.*

In a more general setting within the framework of smooth Banach spaces, Reich [80] and O'Hara-Pillay-Xu [69] provided constructive proof for the existence of the sunny nonexpansive retraction from C onto $\mathrm{Fix}\,(T)$.

4.6. **Resolvent of accretive and monotone operators.** The concepts of monotonicity and accretivity in Banach spaces constitute a valuable tool in studying important operators which appear in different areas.

Definition 4.15. Let $A : X \to 2^X$ be a set-valued operator with domain $\mathcal{D}(A) = \{x \in X : A(x) \neq \emptyset\}$ and range $\mathcal{R}(A) = \{x \in X : x \in A(y) \text{ for some } y \in X\}$ in a Banach space $(X, \|\cdot\|)$. The operator A is said to be

- **accretive** if for each $x, y \in \mathcal{D}(A)$ and any $u \in A(x)$, $v \in A(y)$, there exists $j(x - y) \in J(x - y)$ such that
$$\langle u - v, j(x - y)\rangle \geq 0,$$
where J is the normalized duality map.
- **m-accretive** if it is accretive and $\mathcal{R}(I + A) = X$.

Definition 4.16. Let $A : X \to 2^{X^*}$ be a set-valued operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in the dual X^* of a Banach space $(X, \|\cdot\|)$. The operator A is said to be

- **monotone** if for each $x, y \in \mathcal{D}(A)$ and any $u \in A(x)$, $v \in A(y)$,
$$\langle u - v, x - y\rangle \geq 0; \tag{4.2}$$
- **maximal monotone** if it is a monotone operator which is not proper contained in any other monotone operator on X; in other words, for any $x \in X$ and $u \in X^*$, the inequality
$$\langle u - v, x - y\rangle \geq 0, \quad \text{for all } y \in \mathcal{D}(A) \text{ and } v \in A(y), \tag{4.3}$$
implies that $u \in A(x)$;

Remark 4.17. Note that when the underlying space is a *Hilbert* space $(H, \langle\cdot,\cdot\rangle)$, the normalized duality mapping is the identity operator and then the notions of accretive and monotone operator coincide. As a consequence of the well-known Minty's Theorem (see Theorem 4.20), the notions of m-accretive and maximal monotone coincide as well, see [9, Page 100].

Definition 4.18. Let $A : X \to 2^X$ be a set-valued operator with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in a Banach space $(X, \|\cdot\|)$. Given $\lambda > 0$, the **resolvent** of order λ of A is the set-valued mapping $J_\lambda : X \to 2^X$ defined as

$$J_\lambda = (I + \lambda A)^{-1}. \tag{4.4}$$

Theorem 4.19 (Firm nonexpansivity of the resolvent [22,37])**.** *Let $A : X \to 2^X$ be an accretive operator in a Banach space $(X, \|\cdot\|)$. Then the resolvent J_λ is single-valued and firmly nonexpansive, for any $\lambda > 0$. Moreover, any firmly nonexpansive mapping is the resolvent of an accretive operator for some $\lambda > 0$.*

Recall the important Minty's Theorem in Hilbert spaces for monotone operators.

Theorem 4.20 (Minty's Theorem [65])**.** *Let $A : H \to 2^H$ be a monotone operator defined on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then A is maximal monotone if and only if $\mathcal{D}(J_\lambda) = H$, for any $\lambda > 0$.*

An example of resolvent is the proximal mapping in the sense of Moreau.

Definition 4.21 (Proximal mapping)**.** Given a lower semicontinuous convex function $f : H \to \mathbb{R}$, and $\lambda > 0$, the **proximal** or **proximity mapping** of f is the operator $\operatorname{prox}_f : H \to H$ defined by

$$\operatorname{prox}_f(x) = \operatorname*{argmin}_{y\in H}\Big\{\lambda f(y) + \frac{1}{2}\,\|x-y\|^2\Big\}. \tag{4.5}$$

This mapping is well-defined because the minimizer exists and is unique for $x \in H$. Moreover, it turns out to be the resolvent of the subdifferential of f, whose maximal monotonicity was established by Moreau [67] (in Banach spaces the fact that the subdifferential of a lower semicontinuous convex function is maximal monotone is due to Rockafellar [88]).

The concept of proximal mapping and therefore resolvent for a lower semicontinuous convex function was extended to the setting of CAT(0) spaces in [4].

Definition 4.22 (Resolvent of a lower semicontinuous convex function in CAT(0) spaces)**.** Let (X, d) be a CAT(0) space and $f : X \to \mathbb{R}$ be a lower semicontinuous convex function. The **resolvent** of order $\lambda > 0$ of f is defined by

$$\mathrm{J}_f(x) = \operatorname*{argmin}_{y\in H}\Big\{\lambda f(y) + \frac{1}{2}\,d^2(x,y)\Big\}. \tag{4.6}$$

Jost [42] proved that this operator is well-defined and nonexpansive. The firm nonexpansivity was proved later in [4]. There exists another example of resolvent of coaccretive operators in the Hilbert ball that is firmly nonexpansive due to Kopecka and Reich [55].

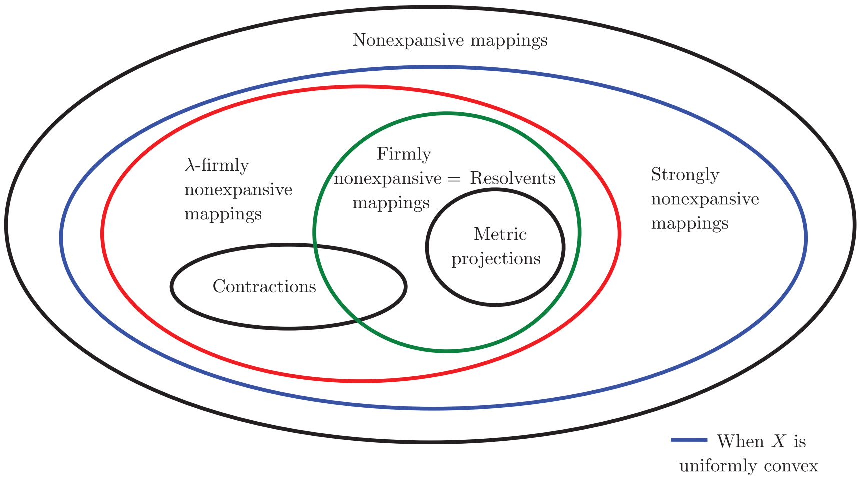

FIGURE 1. Relationship among several classes of mappings in Banach spaces

5. Existence of fixed points

By Remark 4.2 it is clear that the problem of existence of fixed point for firmly nonexpansive self-mappings and nonexpansive self-mappings is the same when the domain is convex. Recall that a Banach space X has the **fixed point property** for nonexpansive mappings (**FPP** for short) if every nonexpansive mapping $T : C \to C$ defined on a bounded closed convex subset $C \subset X$ has a fixed point. In 1965, Kirk [46] proved that all reflexive Banach spaces with normal structure (for instance, uniformly convex or uniformly smooth Banach spaces) have the FPP. Moreover it is known that there exist Banach spaces without the FPP [44]. The study of geometrical conditions on a normed space to assure the existence of fixed point has been a very active and fruitful research field in the last 50 years and many important problems regarding this question remain open. In the framework of UCW-hyperbolic spaces the more general known result so far is the following [58].

Proposition 5.1. *Let (X, d, W) be a complete UCW-hyperbolic space, C a nonempty closed convex bounded subset of X, and $T : C \to C$ a nonexpansive mapping. Then the set of fixed point of T is nonempty.*

In this section we are going to show that the situation is different if we do not assume the convexity of the domain but that the domain is the union of closed convex sets. We start by introducing the notions of orbit and periodic point that will play an essential role in our analysis.

Definition 5.2. Given a subset C of a metric space (X, d), a nonexpansive mapping $T : C \to C$ and $x \in C$, the **orbit** $\mathcal{O}(x)$ of x under T is defined by $\mathcal{O}(x) = \{T^n x : n \in \mathbb{N}\}$. As an immediate consequence of the nonexpansivity of T, if $\mathcal{O}(x)$ is bounded for some $x \in C$, then all other orbits $\mathcal{O}(y)$, for $y \in C$, are bounded. If this is the case, we say that T **has bounded orbits**. Obviously, if T has fixed points, then T has bounded orbits.

Definition 5.3. A **periodic point** of a self-mapping T with domain C is a point $x \in C$ such that there exists $m \geq 0$ with the property that $T^{m+1}x = x$.

Regarding the existence of periodic points it is known the following result [4].

Proposition 5.4. *Let (X, d, W) be a complete UCW-hyperbolic space, $C = \bigcup_{k=1}^{p} C_k$ a union of nonempty closed convex subsets C_k of X, and $T : C \to C$ a nonexpansive mapping having bounded orbits. Then T has periodic points.*

The connection between the existence of fixed point and periodic fixed point for firmly nonexpansive mappings was first studied in [4].

Proposition 5.5. *Let (X, d) be a Busemann space, $C \subset X$ nonempty and $T : C \to C$ firmly nonexpansive. Then any periodic points of T is a fixed point of T.*

As a consequence of the previous results one gets the following one.

Theorem 5.6. *Let (X, d, W) be a complete UCW-hyperbolic space, $C = \bigcup_{k=1}^{p} C_k$ be a union of nonempty closed convex subsets C_k of X, and $T : C \to C$ be firmly nonexpansive. The following two statements are equivalent:*

(a) *T has bounded orbits.*
(b) *T has fixed points.*

As an immediate consequence, we get a strengthening of Smarzewski's fixed point theorem for uniformly convex Banach spaces [92], obtained by weakening the hypothesis of C_k being bounded for all $k = 1, \ldots, p$ to T having bounded orbits. Generalizations of Smarzewski's Theorem for linear spaces have been analyzed in [26,43].

Corollary 5.7. *Let X be a uniformly convex Banach space, $C = \bigcup_{k=1}^{p} C_k$ a union of nonempty closed convex subsets C_k of X, and $T : C \to C$ firmly nonexpansive. Then T has fixed points if and only if T has bounded orbits.*

Remark 5.8. It turns out that the results in this section remain true for the larger class of λ-firmly nonexpansive mappings for some $\lambda \in (0, 1)$. And it is worth mentioning that fixed points are not guaranteed in Theorem 5.6 if T is merely nonexpansive, as the following trivial example shows. Let $x \neq y \in X$, take $C_1 = \{x\}$, $C_2 = \{y\}$, $C = C_1 \cup C_2$ and $T : C \to C$, $T(x) = y$, $T(y) = x$. Then T is fixed point free and nonexpansive. If T was λ-firmly nonexpansive for some $\lambda \in (0, 1)$, the we would get a contradiction:

$$\begin{aligned} 0 < d(x, y) &= d(Tx, Ty) \leq d((1-\lambda)x \oplus \lambda Tx, (1-\lambda)y \oplus \lambda Ty) \\ &= d((1-\lambda)x \oplus \lambda y, \lambda x \oplus (1-\lambda)y) = |2\lambda - 1| d(x, y) \quad \text{by } (W_2) \\ &< d(x, y). \end{aligned}$$

6. Approximation of fixed points

This section is devoted to a survey of diverse approximation methods for fixed points of firmly nonexpansive mappings. For the sake of simplicity we just consider Picard, Mann and Halpern iterations, however, it must be pointed out that there are other iterative methods for firmly nonexpansive mappings, such as the shrinking projection method, see for instance [1, 2]. Some results regarding the rate of asymptotic regularity are gathered.

6.1. **Picard iteration.** In general, the **Picard iterates** $\{T^n x\}_{n\in\mathbb{N}}$ of a nonexpansive mapping T with a fixed point do not converge either weakly or strongly. However, we do have positive results for firmly nonexpansive mappings. The essential core of these results are due to Reich.

Proposition 6.1 ([17], Theorem 15.1 of [37]). *Let C be a closed convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ a firmly nonexpansive mapping with a fixed point. If X and its dual X^* are uniformly convex, then for each $x \in C$, the sequence of iterates $\{T^n x\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of T.*

Remark 6.2. The above result is not true for mappings that are merely nonexpansive; consider, e.g., $T = -I$. Neither is it for all Banach spaces; for example, consider $T : C[0,1] \to C[0,1]$ defined by $(Tf)(t) = t\,f(t)$, for $0 \le t \le 1$. And the conclusion of weak convergence can not be replaced by strong convergence [34]. We do have, however, the following result [22].

Proposition 6.3. *Let T be a firmly nonexpansive self-mapping on a closed convex subset C of a uniformly convex Banach space. If $C = -C$ and T is odd, then $\{T^n x\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T.*

Using Schauder's Theorem and similar arguments that in [70] it is easy to prove the following result.

Proposition 6.4. *Let T be a firmly nonexpansive and compact self-mapping on a closed convex subset C of a uniformly convex Banach space. Then the set of fixed points of T is nonempty and $\{T^n x\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T.*

The asymptotic behavior of Picard iterates in the case of uniformly convex normed linear spaces is very well described by the following result [22, Theorem 2.4(c)].

Proposition 6.5. *Let T be a firmly nonexpansive self-mapping on a closed convex subset C of a uniformly convex Banach space. Then, T is fixed point free if and only if $\lim\limits_{n\to\infty} \|T^n x\| = \infty$ for all x in C.*

The above result is not true if T is merely nonexpansive, even in Hilbert spaces, as the following example shows [28].

Example 6.6. Consider the mapping $T : \ell^2 \to \ell^2$ defined as $T(x_1, \ldots, x_n, \ldots) = (y_1, \ldots, y_n, \ldots)$ where $y_n = e^{\frac{2\pi i}{n!}}(x_n - 1) + 1$. Then T is a fixed point free isometry and $T^{n!}(0)$ converges to 0.

The validity of Proposition 6.5 in the setting of UCW-hyperbolic spaces is an open question.

Definition 6.7. A self-mapping T on a metric space (X, d) is said to be **asymptotically regular at** $x_0 \in X$ if

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = 0.$$

If this is true for all $x_0 \in X$, we say that T is **asymptotically regular** [19].

Reich and Shafrir have studied the asymptotic regularity for nonexpansive mappings in the setting of normed linear spaces and the Hilbert ball, proving the following theorem and its corollaries, see [86].

Theorem 6.8. *Let D be a subset of a Banach space $(X, \|\cdot\|)$ and $T : D \to X$ a firmly nonexpansive mapping. If T can be iterated at $x \in D$, then for all $k \geq 1$,*

$$\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| = \lim_{n\to\infty} \frac{\left\|T^{n+k}x - T^n x\right\|}{k} = \lim_{n\to\infty} \left\|\frac{T^n x}{n}\right\|.$$

Corollary 6.9. *A firmly nonexpansive mapping which has a fixed point is asymptotically regular at each point where it can be iterated.*

The next result identifies the common limit of Theorem 6.8.

Corollary 6.10. *Let $T : D \to D$ be firmly nonexpansive, and set $d = \inf\{\|y - Ty\| : y \in D\}$. Then for each x in D,*

$$\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| = d.$$

Corollary 6.11. *Let C be a bounded closed convex subset of a Banach space $(X, \|\cdot\|)$ and $T : C \to C$ a firmly nonexpansive mapping. If X has the FPP, then T is fixed point free if and only if $\lim_{n\to\infty} \|T^n x\| = \infty$ for all x in C.*

Remark 6.12. This result improves Proposition 6.5 because each bounded closed convex subset of a uniformly convex Banach space does indeed have the FPP. It cannot, however, be obtained by the approach of [22] because in general not every firmly nonexpansive mapping is strongly nonexpansive.

Proposition 6.13 (Corollary 2.1 in [22]). *Let $(X, \|\cdot\|)$ be a Banach space. Suppose $T : X \to X$ is linear and firmly nonexpansive. Then, $\{T^n x\}_{n\in\mathbb{N}}$ converges if and only if so does $\{\frac{1}{n}\sum_{i=1}^{n} T^i x\}_{n\in\mathbb{N}}$.*

The previous results were proved for the Hilbert ball in [90]. In the case of UCW-hyperbolic spaces the asymptotic behavior for Picard iterates was considered in [4].

Theorem 6.14. *Let C be a subset of a W-hyperbolic space X and $T : C \to C$ be a firmly nonexpansive mapping. Then for all $x \in X$ and $k \geq 1$,*

$$\lim_{n\to\infty} d(T^{n+1}x, T^n x) = \frac{1}{k} \lim_{n\to\infty} d(T^{n+k}x, T^n x) = \lim_{n\to\infty} \frac{d(T^n x, x)}{n} = r_C(T),$$

where $r_C(T) := \inf\{d(x, Tx) : x \in C\}$ is the ***minimal displacement of*** *T.*

Corollary 6.15. *The following statements are equivalent:*

(a) *T is asymptotically regular at some $x \in C$.*
(b) *$r_C(T) = 0$.*
(c) *T is asymptotically regular.*

Corollary 6.16. *If T has bounded orbits, then T is asymptotically regular.*

In 1976, Lim [61] introduced a concept of convergence in the general setting of metric spaces, which is known as Δ-convergence. Jost [41] introduced a notion of weak convergence in CAT(0) spaces, which was rediscovered by Espínola and Fernández-León [30], who also proved that it is equivalent to Δ-convergence. We refer to [94] for other notions of weak convergence in geodesic spaces.

Definition 6.17 (Δ-convergence)**.** Let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence of a metric space (X,d). We say that $\{x_n\}_{n\in\mathbb{N}}$ **Δ-converges** to x if x is the unique asymptotic center of $\{u_n\}_{n\in\mathbb{N}}$ for every subsequence $\{u_n\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$. In this case we call x the Δ-limit of $\{x_n\}_{n\in\mathbb{N}}$.

The following Δ-convergence result for the Picard iteration of a firmly nonexpansive mapping is proved in [4].

Theorem 6.18. *Let (X,d,W) be a complete UCW-hyperbolic space, $C\subset X$ a nonempty closed convex set and $T:C\to C$ a firmly nonexpansive mapping. Assume that* $\mathrm{Fix}\,(T)\neq\emptyset$*. Then for all x in C, the sequence $\{T^n x\}_{n\in\mathbb{N}}$ Δ-converges to a fixed point of T.*

Remark 6.19. Theorems 6.14 and 6.18 remain true if T is λ-firmly nonexpansive for some $\lambda\in(0,1)$, see [4].

A generalization of Schauder's fixed point Theorem in Busemann spaces has been proved in [5], and, as a consequence, the following fixed point theorem for the strong convergence of Picard iterates is deduced.

Proposition 6.20. *Let T be a firmly nonexpansive and compact self-mapping on a closed convex subset C of a Bussemann space. Then the set of fixed point of T is nonempty and $\{T^n x\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T.*

We finish this section with a result about the convergence of the projection of Picard iterates on the set of fixed points, which extends Theorem 3.4 in [101] to the setting of UCW-hyperbolic spaces.

Theorem 6.21. *Let C be a nonempty closed convex subset of a complete UCW-hyperbolic space (X,d,W) and $T:X\to X$ a firmly nonexpansive mapping with* $\mathrm{Fix}\,(T)\neq\emptyset$*. Let P be the metric projection of X onto* $\mathrm{Fix}\,(T)$ *and $x_n:=T^n x_0$, for each $n\in\mathbb{N}$, the Picard iteration starting at $x_0\in C$. Then*

(i) *the sequence $\{y_n\}_{n\in\mathbb{N}}$, given by $y_n:=Px_n$ for each $n\in\mathbb{N}$, is well-defined and converges to $p\in$* $\mathrm{Fix}\,(T)$*;*

(ii) *the sequence $\{x_n\}_{n\in\mathbb{N}}$ Δ-converges to the convergence point of $\{y_n\}_{n\in\mathbb{N}}$ $p\in$* $\mathrm{Fix}\,(T)$.

Proof.

(i) By Lemma 6.2 in [4], we know that $\mathrm{Fix}\,(T)$ is closed and convex, and then $\mathrm{Fix}\,(T)$ is a Chebyshev set, see [58, Proposition 2.4]. Hence, the sequence $\{y_n\}_{n\in\mathbb{N}}$ is well defined. Next we shall prove that $\{y_n\}_{n\in\mathbb{N}}$ converges to p.

Notice that $\{d(y_n,x_n)\}_{n\in\mathbb{N}}$ is decreasing. Indeed, since $Px_n\in\mathrm{Fix}\,(T)$, we have that

$$d(y_{n+1},x_{n+1})=d(Px_{n+1},x_{n+1})\le d(y_n,x_{n+1})=d(Ty_n,Tx_n)\le d(y_n,x_n).$$

Then, there exists $d\ge 0$ such that $d(y_n,x_n)\to d$ as $n\to\infty$.

Using the nonexpansivity of T and the fact that $y_n\in\mathrm{Fix}\,(T)$ for all $n\in\mathbb{N}$, we get that

$$d(y_n,x_{n+m+1})\le d(y_n,x_n)\qquad\text{for all } n,m\in\mathbb{N}. \tag{6.1}$$

We consider two cases.

Case 1. If $d > 0$. Set

$$\ell := \limsup_{n,m\to\infty} d(y_n, y_m).$$

Suppose that $\ell > 0$. Using the fact that X has a modulus of convexity δ which is decreasing with respect to the first variable (fixed the second one) and is increasing with respect to the second variable (fixed the first one), we can assure the existence of ε in $(0,1]$ such that

$$(d+\varepsilon)\left[1-\delta\big(d+\varepsilon, \tfrac{\ell}{2(d+\varepsilon)}\big)\right] < d. \tag{6.2}$$

Since $d(y_n, x_n) \to d$ as $n \to \infty$, there exists a positive integer n_0 such that $d(y_n, x_n) < d+\varepsilon$ for all $n \geq n_0$. Then,

$$d(y_n, x_{n+m+1}) < d+\varepsilon \quad \text{and} \quad d(y_m, x_{n+m+1}) < d+\varepsilon \quad \text{for all } n, m \geq n_0.$$

On the other hand, there exists $n_1 \in \mathbb{N}$ such that

$$d(y_n, y_m) \geq \frac{\ell}{2} = \frac{\ell}{2(d+\varepsilon)} \cdot (d+\varepsilon) \qquad \text{for all } n, m \geq n_1.$$

Then, by the definition of each y_n and the uniform convexity, for all $n, m \geq \max\{n_0, n_1\}$ we get

$$\begin{aligned} d &\leq d(Px_{n+m+1}, x_{n+m+1}) \\ &\leq d(\tfrac{1}{2}y_n \oplus \tfrac{1}{2}y_m, x_{n+m+1}) \\ &\leq \Big(1-\delta(d+\varepsilon, \tfrac{\ell}{2(d+\varepsilon)})\Big) \cdot (d+\varepsilon), \end{aligned}$$

which is a contradiction with (6.2). We therefore must have that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Case 2. If $d = 0$. Using (6.1), we have

$$d(y_n, y_m) \leq d(y_n, x_{n+m+1}) + d(x_{n+m+1}, y_m) \leq d(y_n, x_n) + d(x_m, y_m).$$

Then, $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

In both cases, we obtain that $\{y_n\}_{n\in\mathbb{N}}$ converges to $p \in \mathrm{Fix}\,(T)$. Notice that using the definition of $\{y_n\}_{n\in\mathbb{N}}$ we get that

$$\lim_{n\to\infty} d(x_n, p) = \min\left\{\lim_{n\to\infty} d(x_n, z) : z \in \mathrm{Fix}\,(T)\right\}. \tag{6.3}$$

(ii) By Theorem 6.4 in [4], we know that $\{x_n\}_{n\in\mathbb{N}}$ Δ-converges to $q \in \mathrm{Fix}\,(T)$. We need to prove that $p = q$. By contradiction, suppose that $p \neq q$. Then,

$$\limsup_{n\to\infty} d(x_{n_k}, q) < \limsup_{n\to\infty} d(x_{n_k}, p),$$

for all subsequence x_{n_k} of x_n which implies that there exists a subsequence such that

$$\lim_{n_k\to\infty} d(x_{n_k}, q) < \lim_{n_k\to\infty} d(x_{n_k}, p) = \lim_{n\to\infty} d(x_n, p),$$

which is a contradiction with (6.3).

□

6.2. **Krasnosel'skiĭ-Mann iteration.** Bearing in mind that for nonexpansive mappings Picard iterates do not converge in general, an alternative iterative method, known as **Krasnosel'skiĭ-Mann iteration**, has been extensively studied; see [13, 38, 76] and references therein. For the sake of simplicity, from now on we will refer to this algorithm as **Mann iteration**.

Let C be a nonempty closed convex subset of a real Banach space $(X, \|\cdot\|)$. For a self-mapping T on C we consider the following iterative scheme: $x_0 \in C$,

$$x_{n+1} = x_n + \alpha_n (Tx_n - x_n) \quad \forall\, n \in \mathbb{N}, \tag{6.4}$$

where $\{\alpha_n\}_{n\in\mathbb{N}}$ is a real sequences in $[0, 1]$. Next result can be found in [13].

Theorem 6.22. *Let X have a uniformly convex and Fréchet differentiable norm and T be a nonexpansive mapping. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence defined by Mann iteration* (6.4)*, where the sequence of parameters satisfies the conditions $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup \alpha_n < 0$. Then either $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of T or $\{\|x_n\|\}_{n\in\mathbb{N}}$ tends to infinity.*

In the setting of Hilbert spaces, if T is firmly nonexpansive then the relaxation parameters can be in the interval $[0, 2]$, see [102].

Theorem 6.23. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and T a firmly nonexpansive mapping. Let $\{x_n\}_{n\in\mathbb{N}}$ be the sequence defined by Mann iteration* (6.4)*, where the sequence of parameters satisfies the conditions $\alpha_n \in [0, 2]$ and $\sum_{n=0}^{\infty} \alpha_n(\alpha_n - 2) = \infty$. Then either $\{x_n\}_{n\in\mathbb{N}}$ converges weakly to a fixed point of T or $\{\|x_n\|\}_{n\in\mathbb{N}}$ tends to infinity.*

Recently, Sharma and Sahu [91] have studied the convergence of Mann and Ishikawa iterations for firmly nonexpansive mappings without convex domain.

6.3. **Halpern iteration.** Let C be a nonempty closed convex subset of a real Banach space $(X, \|\cdot\|)$. For a self-mapping T on C, **Halpern iteration**, first presented in [39], generates a sequence via the recursive formula:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N} \tag{6.5}$$

where the initial guess $x_0 \in C$ and anchor $u \in C$ are arbitrary (but fixed) and the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ is a subset of the unit interval $[0, 1]$.

In contrast with Mann iteration, Halpern iteration for nonexpansive mappings provides strong convergence requiring the underlying space to be smooth enough.

Halpern initially considered the case where C is the unit closed ball and $u = 0$. He proved that $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the fixed point of T which is closest to u from $\mathrm{Fix}\,(T)$, that is, $P_{\mathrm{Fix}\,(T)}u$, essentially when $\alpha_n = n^{-a}$ with $a \in (0, 1)$. He also showed that the following two conditions

- (C1) $\lim_{n\to\infty} \alpha_n = 0$, and
- (C2) $\sum_{n=0}^{\infty} \alpha_n = \infty$

are necessary for the convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to a fixed point of the mapping T.

In [62,84,100,101] the strong convergence of Halpern iteration (6.5) was proved in the case when X is uniformly smooth Banach space and the sequence of parameters $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies (C1) and (C2) and an additional suitable condition.

Since conditions (C1) and (C2) are necessary for Halpern iteration (6.5) to converge in norm for all nonexpansive mappings T, a natural question is whether they are also sufficient. This question was answered in the negative in [98]. However, some positive answers have been given in particular cases. For averaged mappings Chidume [25] and, independently, Suzuki [97] proved that conditions (C1) and (C2) are sufficient.

Theorem 6.24. *Let C be a closed convex set of a Banach space $(X, \|\cdot\|)$ whose norm is uniformly Gâteaux differentiable and $T : C \to C$ be an averaged mapping with* $\mathrm{Fix}\,(T) \neq \emptyset$. *For arbitrary initial value $x_0 \in C$ and fixed anchor $u \in C$, define iteratively a sequence as in* (6.5), *and assume that the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies (C1) and (C2). Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T.*

Remark 6.25. Proposition 3.6 ensures that the theorem above remains true for firmly nonexpansive mappings when the space has the property (S), for example, Hilbert spaces and $L_p, 1 < p < \infty$. The question in the context of W-hyperbolic spaces is still open.

An alternative result was given in [93] for a **firmly nonexpansive type** mapping T, that is, assuming that there exists $k \in (0, \infty)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - k \|(x - Tx) - (y - Ty)\|^2 , \tag{6.6}$$

for all $x, y \in D(T)$.

Theorem 6.26. *Let X be a real reflexive Banach space with a uniformly Gâteaux differentiable norm and with the FPP. Let $C \subset X$ be nonempty closed convex and $T : C \to C$ firmly nonexpansive type with* $\mathrm{Fix}\,(T) \neq \emptyset$. *For arbitrary initial value $x_0 \in C$ and fixed anchor $u \in C$, define iteratively a sequence as in* (6.5), *and assume that the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ satisfies conditions (C1) and (C2). Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to a fixed point of T.*

Remark 6.27. It is clear that in Hilbert spaces firmly nonexpansive mappings are firmly nonexpansive type mappings ($k = 1$). An interesting question is whether this is true in the setting of Banach spaces (or UCW-hyperbolic spaces) so that Halpern iteration converges for firmly nonexpansive mappings in this framework.

6.4. **Rate of asymptotic regularity.** Asymptotic regularity is a fundamental concept in Metric Fixed Point Theory. This notion was formally introduced by Browder and Petryshyn in [19], although it was implicitly used in [28,56,89] to study the weak convergence of the iterates of an averaged mapping in uniformly convex spaces. Extension of these results to general linear spaces are due to Ishikawa [40] and Edelstein and O'Brein [29].

Recall that a mapping $T : D \to X$ is said to be **asymptotically regular** at $x \in D$ if T can be iterated at x and

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1} x_0) = 0.$$

Goebel and Kirk [35, Theorem 9.4] stated that any method which establishes the point-wise asymptotic regularity of an averaged mapping in linear spaces, actually provides estimates which are uniform over all the starting point and all averaged mapping.

Theorem 6.28. *Let X be an arbitrary Banach space and C a bounded closed convex subset of X. Let $\mathcal{F}$ denote the collection of all nonexpansive self-mappings on C. Fix $\alpha \in (0,1)$ and for any $T \in \mathcal{F}$, set $T_\alpha = (1-\alpha)I + \alpha$. Then T_α is asymptotically regular on C. Moreover, the sequence $\{\|T_\alpha^{n+1}x - T_\alpha^n x\|\}_{n\in\mathbb{N}}$ converges to 0 uniformly for $x \in C$ and $T \in \mathcal{F}$. Precisely, for any $\varepsilon > 0$ there exists an integer N depending only on ε and C such that, for $n \geq N$, $\|T_\alpha^{n+1}x - T_\alpha^n x\| \leq \varepsilon$, for any $x \in C$ and $T \in \mathcal{F}$.*

For the case of uniformly convex spaces the following quantitative version of the previous result is known, see [49].

Theorem 6.29. *Let X be an uniformly convex Banach space and C a bounded closed convex subset of X with $\operatorname{diam} C = d$. Let $\varepsilon > 0$ $(\varepsilon \leq d/2)$. If $T : C \to C$ is nonexpansive and if $S = \frac{1}{2}(I + T)$, then for any $x \in C$, $\|S^{n+1}x - S_\alpha^n x\| \leq \varepsilon$ for all n satisfying*

$$(1 - \delta(2\varepsilon/d))^n \leq \varepsilon/d.$$

Further results on quantitative rates of convergence for averaged mappings and firmly nonexpansive mappings in the setting of Banach spaces were obtained by Bruck and Baillon [10].

One of the basic tools for a quantitative study of the asymptotic regularity is the concept of rate of asymptotic regularity.

Definition 6.30. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). We say that $\gamma : (0, \infty) \to \mathbb{N}$ is a **rate of asymptotic regularity** for $\{x_n\}_{n\in\mathbb{N}}$ if for all $\varepsilon > 0$ we have that $d(x_k, x_{k+1}) \leq \varepsilon$ for each $k \in \mathbb{N}$ with $k \geq \gamma(\varepsilon)$.

By a logical analysis of the proof of the result in [13] (which generalizes Ishikawa result to unbounded sets), Kohlenbach [51] obtained for the first time explicit bounds on the asymptotic regularity. Subsequently, Kohlenbach and Leuştean [54] extended these results to the very general setting of hyperbolic spaces. Following the ideas of these works, the following results were obtained in [4] regarding the rate of asymptotic regularity for firmly nonexpansive mappings.

For $x \in C$ and $b, \varepsilon > 0$, let us denote

$$\operatorname{Fix}_\varepsilon(T, x, b) := \{y \in C : d(y, x) \leq b \text{ and } d(y, Ty) < \varepsilon\}.$$

If $\operatorname{Fix}_\varepsilon(T, x, b) \neq \emptyset$ for all $\varepsilon > 0$, we say that T **has approximate fixed points** in a b-neighborhood of x.

Theorem 6.31. *Let $b > 0$. For all UCW-hyperbolic spaces (X, d, W, η), nonempty subsets $C \subset X$, firmly nonexpansive mappings $T : C \to C$ and all $x \in C$ such that T has approximate fixed points in a b-neighborhood of x, the following holds:*

$$\forall \varepsilon > 0 \, \forall n \geq \Phi(\varepsilon, \eta, b) \left(d(T^n x, T^{n+1} x) \leq \varepsilon \right), \tag{6.7}$$

where

$$\Phi(\varepsilon, \eta, b) := \begin{cases} \left\lceil \dfrac{4(b+1)}{\varepsilon\, \eta\left(b+1, \dfrac{\varepsilon}{b+1}\right)} \right\rceil & \text{for } \varepsilon < 2b, \\ 0 & \text{otherwise.} \end{cases} \tag{6.8}$$

Remark 6.32. If, moreover, $\eta(r,\varepsilon)$ can be written as $\eta(r,\varepsilon) = \varepsilon \cdot \tilde{\eta}(r,\varepsilon)$ such that $\tilde{\eta}$ increases with ε (for a fixed r), then, for $\varepsilon < 2b$, the bound $\Phi(\varepsilon,\eta,b)$ can be replaced by

$$\tilde{\Phi}(\varepsilon,\eta,b) = \left\lceil \frac{4(b+1)}{\varepsilon\,\tilde{\eta}\left(b+1, \dfrac{\varepsilon}{b+1}\right)} \right\rceil. \tag{6.9}$$

Corollary 6.33. *Let $b > 0$. For all UCW-hyperbolic spaces (X,d,W,η), bounded subsets $C \subset X$ with diameter $d_C \le b$, firmly nonexpansive mappings $T : C \to C$ and all $x \in C$,*

$$\forall \varepsilon > 0\, \forall n \ge \Phi(\varepsilon,\eta,b)\, \big(d(T^n x, T^{n+1}x) \le \varepsilon\big),$$

where $\Phi(\varepsilon,\eta,b)$ is given by (6.8).

Thus, for bounded C, we get that T is asymptotically regular with a rate $\Phi(\varepsilon,\eta,b)$ that only depends on ε, on X via the monotone modulus of uniform convexity η, on C via an upper bound b on its diameter d_C. The rate of asymptotic regularity is uniform in the starting point $x \in C$ of the iteration and other data related with X, C and T.

CAT(0) spaces are UCW-hyperbolic spaces with a quadratic (in ε) modulus of uniform convexity $\eta(\varepsilon) = \dfrac{\varepsilon^2}{8}$, which has the form required in Remark 6.32. As an immediate consequence, we get a quadratic (in $1/\varepsilon$) rate of asymptotic regularity in the case of CAT(0) spaces.

Corollary 6.34. *Given $b > 0$, for all CAT(0) space X, bounded subset $C \subset X$ with diameter $d_C \le b$, firmly nonexpansive mapping $T : C \to C$ and $x \in C$, the following holds*

$$\forall \varepsilon > 0\, \forall n \ge \Psi(\varepsilon,b)\, \big(d(T^n x, T^{n+1}x) \le \varepsilon\big),$$

where

$$\Psi(\varepsilon,b) := \begin{cases} \left\lceil 8(b+1)\dfrac{1}{\varepsilon^2} \right\rceil & \text{for } \varepsilon < 2b, \\ 0 & \text{otherwise.} \end{cases}$$

In [57] the author presented a comparison analysis of the rate of asymptotic regularity of Mann iteration for nonexpansive mappings, obtained by diverse authors, [10, 20, 47–49, 54, 57].

References

[1] K. Aoyama and F. Kohsaka, *Existence of fixed points of firmly nonexpansive-like mappings in Banach spaces,* Fixed Point Theory and Applications, Volume 2010, Article ID 512751, 15 pages.

[2] K. Aoyama, F. Kohsaka and W. Takahashi, *Shrinking projection methods for firmly nonexpansive mappings,* Nonlinear Analysis: Theory, Methods & Applications **71** (2009), 1626–1632.

[3] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: their relations and continuity properties,* Journal of Nonlinear and Convex Analysis **10** (2009), 131–147.

[4] D. Ariza, L. Leuştean and G. López-Acedo, *Firmly nonexpansive mappings,* Trans. Amer. Math. Soc., in press.
[5] D. Ariza, C. Li and G. López-Acedo, *Schauder Theorem in geodesic spaces,* preprint.
[6] M. Bačák, *Proximal point algorithms in metric spaces,* Israel J. Math. (2012), doi: 10.1007/s11856-012-0091-3.
[7] M. Bačák, *A new proof of the Lie-Trotter-Kato formula in Hadamard spaces,* preprint.
[8] M. Bačák, *Convergence of semigroups under nonpositive curvature,* Trans. Amer. Math. Soc., in press.
[9] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces,* Springer-Verlag, New York, 2010.
[10] J.B. Baillon and R. E. Bruck, *The rate of asymptotic regularity is* $O(1/\sqrt{n})$, in Theory and applications of nonlinear operators of accretive and monotone type, Lecture Notes in Pure and Appl. Math. 178, Dekker, New York, 1996, pp. 51–81.
[11] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces,* Springer-Verlag, New York, 2011.
[12] H. H. Bauschke, S. M. Moffat and X. Wang, *Firmly nonexpansive mappings and maximally monotone operators: correspondence and duality,* Set-Valued and Variational Analysis **20** (2012), 131–153.
[13] J. Borwein, S. Reich and I. Shafrir *Krasnosel'skiĭ-Mann iteration in normed spaces,* Can. Math. Bull. **35** (1992), 21–28.
[14] M. Bridson and A. Haefliger, *Metric Spaces of Non-positive Curvature,* Grundlehren der Mathematischen Wissenschaften 319, Springer-Verlag, Berlin, 1999.
[15] F. E. Browder, *Nonlinear monotone operators and convex sets in Banach spaces,* Bull. Amer. Math. Soc. **71** (1965), 780–785
[16] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space,* Proc. Nat. Acad. Sei. **54** (1965), 1041–1044.
[17] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces,* Mathematische Zeitschrift **100** (1967), 201–225.
[18] F. E. Browder, *Convergence of approximants to fixed points of non-expansive nonlinear mappings in Banach spaces,* Arch. Rational Mech. Anal. **24** (1967), 82–90.
[19] F. E. Browder and W. V. Petryshyn, *The solution by iteration of nonlinear functional equations in Banach spaces,* Bull. Amer. Math. Soc. **72** (1966), 571–575.
[20] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces,* J. Math. Anal. Appl. **20** (1967), 197–228.
[21] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces,* Pacific J. Math. **47** (1973), 341–355.
[22] R. E. Bruck and S. Reich, *Nonexpasive projections and resolvents of accretive operators in Banach spaces,* Houston J. Math. **3** (1977), 459–470.
[23] H. Busemann, *Spaces with nonpositive curvature.* Acta Math. **80** (1948), 259–310.
[24] A. Cegielski, *Iterative Methods for Fixed Point Problems in Hilbert Spaces,* Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, 2012.
[25] C.E. Chidume and C.O. Chidume, *Iterative approximation of fixed points of nonexpansive mappings,* J. Math. Anal. Appl. **318** (2006), 288–295.
[26] W. Du, Y. Huang and C. Yen, *Fixed point theorems for nonexpansive mappings on nonconvex sets in UCED Banach spaces,* International Journal of Mathematics and Mathematical Sciences **31** (2002), 251–257.
[27] J. A. Clarkson, *Uniformly convex spaces,* Trans. Amer. Math. Soc. **40** (1936), 396–414.
[28] M. Edelstein, *On non-expansive mappings of Banach spaces,* Proc. Cambridge Philos. Soc. **60** (1964), 439–447.
[29] M. Edelstein and R. C. O'Brien, *Nonexpansive mappings, asymptotic regularity, and successive approximations,* J. London Math. Soc. **1** (1978), 547–554.
[30] R. Espínola and A. Fernández-León, *CAT(κ)-spaces, weak convergence and fixed points,* J. Math. Anal. Appl. **353** (2009), 410–427.

[31] R. Espínola and B. Piątek, *The fixed point property and unbounded sets in CAT(0) spaces*, Preprint.
[32] O. P. Ferreira, L. R. Lucambio Pérez and S. Z. Németh, *Singularities of monotone vector fields and an extragradient-type algorithm,* J. Global Optim. **31** (2005), 133–151.
[33] T. Foertsch, *Ball versus distance convexity of metric space*, Beiträge Algebra Geem. **45** (2004), 241–500.
[34] A. Genel and J. Lindenstrauss, *An example concerning fixed points,* Israel J. Math. **22** (1975), 81–86.
[35] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory,* Cambridge University Press, Cambridge, 1990.
[36] K. Goebel and S. Reich, *Iterating holomorphic self-mappings of the Hilbert ball,* Proc. Japan Acad. **58** (1982), 349–352
[37] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings,* Marcel Dekker, New York, 1984.
[38] C. W. Groetsch, *A note on segmenting Mann iterates,* J. Math. Anal. Appl. **40** (1972), 369–372.
[39] B. Halpern, *Fixed points of nonexpanding maps,* Bull. Amer. Math. Soc. **73** (1967), 591–597.
[40] S. Ishikawa, *Fixed points by a new iteration method,* Proc. Amer. Math. Soc. **44** (1974), 147–150.
[41] J. Jost, *Equilibrium maps between metric spaces,* Calc. Var. Partial Diff. Equations **2** (1994), 173–204.
[42] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature,* Comment. Math. Helv. **70** (1995), 659–673.
[43] W. Kaczor, *Fixed points of λ-firmly nonexpansive mappings on nonconvex sets,* Nonlinear Analysis **47** (2001), 2787–2792.
[44] S. Kakutani, *Topological properties of the unit sphere of a Hilbert space,* Proc. Imp. Acad. Tokyo **19** (1943), 269–271.
[45] R. Kato and K. Masuda, *Trotter's product formula for nonlinear semigroups generated by the subdiffrerential of convex functionals,* J. Math. Soc. Japan **30** (1978), 169–178.
[46] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances,* Amer. Math. Monthly **72** (1965), 1004–1006.
[47] W. A. Kirk, *Krasnosel'skiĭ iteration process in hyperbolic spaces,* Numer. Funct. Anal. and Optimiz. **4** (1982), 371–381.
[48] W. A. Kirk, *Nonexpansive mappings and asymptotic regularity,* Nonlinear Analysis, **40** (2000), 323–332.
[49] W.A. Kirk and C. Martinez-Yañez, *Approximate fixed points for nonexpansive mappings in uniformly convex spaces,* Annales Polonici Mathematici **51** (1990), 189–193.
[50] V. Klee, *Convexity or Chebyshev Sets,* Math. Annal. **142** (1961), 292–304.
[51] U. Kohlenbach, *A quantitative version of a theorem due to Borwein-Reich-Shafrir,* Numer. Funct. Anal. Optim. **22** (2001), 641–656.
[52] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis,* Trans. Amer. Math. Soc. **357** (2005), 89–128.
[53] U. Kohlenbach, *Applied Proof Theory: Proof Interpretations and Their Use in Mathematics,* Springer Monographs in Mathematics, Springer-Verlag, Berlin-Heidelberg, 2008.
[54] U. Kohlenbach and L. Leuştean, *Mann iterates of directionally nonexpansive mappings in hyperbolic spaces,* Abstr. Appl. Anal. **8** (2003), 449–477.
[55] E. Kopecka and S. Reich, *Asymptotic behavior of resolvents of coaccretive operators in the Hilber ball,* Nonlinear Anal. **70** (2009), 3187–3194.
[56] M. A. Krasnosel'skiĭ, *Two remarks on the method of successive approximation,* Usp. Math. Nauk (N.S.) **10** (1955), 123–127 (in Russian).
[57] L. Leuştean, *A quadratic rate of asymptotic regularity for CAT(0) spaces,* J. Math. Anal. Appl. **325** (2007), 386–399.
[58] L. Leuştean, *Nonexpansive iterations in uniformly convex W-hyperbolic spaces,* in Nonlinear Analysis and Optimization I: Nonlinear Analysis, Contemporary Mathematics, A. Leizarowitz,

B. S. Mordukhovich, I. Shafrir, A. Zaslavski (Editors), Vol. 513 (2010), AMS, Providence, RI, pp. 193–209.
[59] C. Li, G. López-Acedo and V. Martín-Márquez, *Monotone vector fields and the proximal point algorithm on Hadamard manifolds,* J. Lond. Math. Soc. **79** (2009), 663–683.
[60] C. Li, G. López, V. Martín-Márquez and J.-H. Wang, *Resolvents of set-valued monotone vector fields in Hadamard manifolds,* Set-Valued Var. Anal. **19** (2011), 361–383.
[61] T. C. Lim, *Remarks on some fixed point theorems,* Proc. Amer. Math. Soc. **60** (1976), 179–182.
[62] P. Lions, *Approximation des points fixes de contractions,* C.R. Acad. Sci. Ser. A-B Paris **284** (1977), 1357–1359.
[63] W. R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc. **4** (1953), 506–510.
[64] U.F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps,* Comm. Anal. Geom. **6** (1998), 199–253.
[65] G. J. Minty, *On the monoticity of the gradient of a convex function,* Pacific J. Math. **14** (1964), 243–247.
[66] G.J. Minty, *Monotone (nonlinear) operators in Hilbert space,* Duke Math. J. **29** (1962), 341–346.
[67] J.J. Moreau, *Proximité et dualité dans un espace hilbertien,* Bulletin de la Société Mathématique de France **93** (1965), 273–299.
[68] A. Nicolae, *Asymptotic behavior of averaged and firmly nonexpansive mappings in geodesic spaces,* Nonl. Anal. (In press).
[69] J.G. O'Hara, P. Pillay and H. K. Xu, *Iterative approaches to convex feasibility problems in Banach Space,* Nonlinear Analysis **64** (2006), 2002–2042.
[70] A. Pazy, *On the asymptotic behavior of iterates of non expansive mappings in Hilbert spaces,* Israel J. Math. **26** (1997), 197–204.
[71] A. Papadopoulos, *Metric Spaces, Convexity and Nonpositive Curvature,* IRMA Lectures in Mathematics and Theoretical Physics 6, European Mathematical Society, 2005.
[72] R. R. Phelps, *Convex sets and nearest points,* Proc. Amer. Math. Soc. **8** (1957), 790–797.
[73] S. Reich, *Asymptotic behavior of contractions in Banach spaces,* J. Math. Anal. Appl. **44** (1973), 57–70.
[74] S. Reich, *Asymptotic behavior of semigroups of nonlinear contractions in Banach spaces,* J. Math. Anal. Appl. **53** (1976), 271–290.
[75] S. Reich, *Extension problems for accretive sets in Banach spaces,* J. Funct. Anal. **26** (1977), 378–395.
[76] S. Reich, *Weak convergence theorems for nonexpansive mappings in Banach spaces,* J. Math. Anal. Appl. **67** (1979), 274–276.
[77] S. Reich, *Asymptotic behavior of resolvents in Banach spaces,* Atti Accad. Naz. Lincei **67** (1979), 27–30.
[78] S. Reich, *Product formulas, nonlinear semigroups, and accretive operators,* J. Funct. Anal. **36** (1980), 147–168.
[79] S. Reich, *A solution to a problem on the asymptotic behavior of nonexpansive mappings and semigroups,* Proc. Japan Acad. **56** (1980), 85–87.
[80] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces,* J. Math. Anal. Appl. **75** (1980), 287–292.
[81] S. Reich, *Convergence and approximation of nonlinear semigroups,* J. Math. Anal. Appl. **76** (1980), 77–83.
[82] S. Reich, *On the asymptotic behavior of nonlinear semigroups and the range of accretive operators,* J. Math. Anal. Appl. **79** (1981), 113–126.
[83] S. Reich, *A limit theorem for projections,* Linear and Multilinear Algebra **13** (1983), 281–290.
[84] S. Reich, *Some problems and results in fixed point theory,* Contemp. Math. **21** (1983), 179–187.
[85] S. Reich, *Nonlinear semigroups, holomorphic mappings, and integral equations,* Proc. Sympos. Pure Math. **45** (1986), 307–324.
[86] S. Reich and I. Shafrir, *The asymptotic behavior of firmly nonexpansive mappings,* Proc. Amer. Math. Soc. **101** (1987), 246–250.

[87] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces,* Nonlinear Anal. **15** (1990), 537–558.

[88] R.T. Rockafellar, *On the maximal monotonicity of subdifferential mappings,* Pacific Journal of Mathematics **33** (1970), 209–216.

[89] H. Schaefer, *Über die Methode sukzessiver Approximationen,* Jahresberichte Deutsch. Math. Verein **59** (1957), 131–140.

[90] I. Shafrir, *Coaccretive operators and firmly nonexpansive mappings in the Hilbert ball,* Nonlinear Anal. **18** (1992), 637–648.

[91] B. K. Sharma and D. R. Sahu, *Fixed point approximation for λ-firmly nonexpansive mappings,* Bull. Calcutta Math. Soc. **88** (1996), 285–290.

[92] R. Smarzewski, *On firmly nonexpansive mappings,* Proc. Amer. Math. Soc. **113** (1991), 723–725.

[93] Y. Song and X. Chai, *Halpern iteration for firmly nonexpansive type mappings,* Nonlinear Anal. **71** (2009), 4500–506.

[94] E. N. Sosov, *On analogues of weak convergence in a special metric space,* Izv. Vyssh. Uchebn. Zaved. Mat. **5** (2004), 84–89. Translation in Russian Math. (Iz.VUZ) **48** (2004), 79–83.

[95] I. Stojkovic, *Approximation for convex functionals on non-positively curved spaces and the Trotter-Kato product formula,* Adv. Calc. Var. **5** (2012), 77–126.

[96] I. Stojkovic, *Geometric approach to evolution problems in metric spaces,* PhD thesis, 2011, `http://www.math.leidenuniv.nl/scripties/PhDThesisStojkovic.pdf`.

[97] T. Suzuki, *A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings,* Proc. Amer. Math. Soc. **135** (2007), 99–106.

[98] T. Suzuki, *Reich's problem concerning Halpern's convergence,* Arch. Math. **92** (2009), 602–613.

[99] W. Takahashi, *Nonlinear Functional Analysis,* Yokohama Publishers, Yokohama 2000.

[100] R. Wittmann, *Approximation of fixed points of nonexpansive mappings,* Arch. Math. **58** (1992), 486–491.

[101] H. K. Xu, *Iterative algorithms for nonlinear operators,* Journal of London Mathematical Society **66** (2002), 240–256.

[102] J. Zhao and Q. Yang, *A note on the Krasnosel'skiĭ-Mann theorem and its generalizations,* Inverse Problems **23** (2007), 1011–1016.

Manuscript received March 5, 2013
revised July 7, 2013

David Ariza-Ruiz
Dept. Análisis Matemático, Fac. Matemáticas, Universidad de Sevilla, Apdo. 1160, 41080-Sevilla, Spain
E-mail address: `dariza@us.es`

Genaro López-Acedo
Dept. Análisis Matemático, Fac. Matemáticas, Universidad de Sevilla, Apdo. 1160, 41080-Sevilla, Spain
E-mail address: `glopez@us.es`

Victoria Martín-Márquez
Dept. Análisis Matemático, Fac. Matemáticas, Universidad de Sevilla, Apdo. 1160, 41080-Sevilla, Spain
E-mail address: `victoriam@us.es`