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REMARKS ON CONVEX COMBINATIONS IN GEODESIC SPACES

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Dedicated to Simeon Reich on the occasion of his 65th birthday.

Abstract. Let X be a Busemann space and $F \subset X$. The convex hull of F, denoted $conv\,(F)$, is typically to be $\bigcup F_n$ where $F_0 = F$, and for $n \geq 1$, F_n is the set of all geodesics with endpoints in F_{n-1}. In this paper it is shown that if X is also complete there is another approach to define the convex combination of a finite set of points in F. As in the case of normed spaces, this opens the possibility of defining the convex hull of F to be the union of all convex combinations of all finite subsets of F. The objective is to attempt to find a more analytic approach to the study of convex combinations in geodesic spaces. Although many questions remain unclear, this approach might open the door to some new applications.

1. Introduction

We follow the terminology of [16]. Let (X, ρ) be a metric space and $x, y \in X$. A geodesic path from x to y is a mapping $c : [0, \ell] \to X$ with $c(0) = x$, $c(\ell) = y$, and $\rho(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, \ell]$. The image of $(c[0, \ell])$ in X is a *geodesic* (*or metric*) *segment* which joins x and y. When no confusion arises, and in particular when this segment is unique, we shall denote it $[x, y]$. In this section we deal with a class of uniquely geodesic spaces called Busemann spaces (cf. also [3]).

A metric space (X, ρ) is said to be a *Busemann space* if X is a geodesic space and for any two affinely reparametrized geodesics $\gamma : [a, b] \to X$ and $\gamma' : [a', b'] \to X$, the map $D_{\gamma,\gamma'} : [a, b] \times [a', b'] \to \mathbb{R}$ defined by

$$D_{\gamma,\gamma'}\left(t, t'\right) = \left|\gamma(t) - \gamma'\left(t'\right)\right|$$

is convex.

Suppose (X, ρ) is a uniquely geodesic space, and let $x, y \in X$. Then a point $u \in [x, y]$ if and only if there exists $t \in [0, 1]$ such that $\rho(u, x) = t\rho(x, y)$ and $\rho(u, y) = (1 - t)\rho(x, y)$. For simplicity we will write $u = (1 - t)x \oplus ty$. If X is a Busemann space the metric ρ on X is *convex*. This means that for any $z \in X$,

$$\rho(z, (1 - t)x \oplus ty) \leq (1 - t)\rho(z, x) + t\rho(z, y) \text{ for all } t \in [0, 1]. \tag{1.1}$$

First observe that a point u is in $[x, y]$ if and only if there exist $\lambda_1, \lambda_2 \in [0, 1]$ with $\lambda_1 + \lambda_2 = 1$ such that $\rho(u, x) = \lambda_1\rho(x, y)$ and $\rho(u, y) = \lambda_2\rho(x, y)$. We shall

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use the notation $u = \lambda_1 x \oplus \lambda_2 y$ to denote this fact. With this notation condition (1.1) implies that for any $z \in X$

$$\rho(z, \lambda_1 x \oplus \lambda_2 y) \leq \lambda_1 \rho(z, x) + \lambda_2 \rho(z, y). \tag{1.2}$$

This condition can be shown (see [16]) to be equivalent to the following: Let $p, x, y \in X$. If m_1 is the midpoint of $[p, x]$ and m_2 is the midpoint of $[p, y]$, then

$$\rho(m_1, m_2) \leq \frac{1}{2}\rho(x, y). \tag{1.3}$$

Busemann spaces include all strictly convex normed linear spaces and all CAT(0) spaces. An important class of Busemann spaces are the CAT(0) spaces of Gromov (see [1]). These spaces are characterized by the so-called CN inequality of Bruhat and Tits [2] (see [1, p. 163]). A Busemann space (X, ρ) is a CAT(0) space if and only if the following holds. For all $p, q, r \in X$ and all $m \in X$ with $\rho(q, m) = \rho(r, m) = \rho(q, r)/2$, one has:

$$\rho(p, q)^2 + \rho(p, r)^2 \geq 2\rho(m, p)^2 + \frac{1}{2}\rho(q, r)^2. \tag{CN}$$

Therefore these spaces include, among others, the complex Hilbert ball with a hyperbolic metric (see Goebel and Reich [6]; also see inequality (4.2) of Reich and Shafrir [17] and subsequent comments).

In this paper we propose a way to analytically define what we call the *convex combination* of a finite set of points in a Busemann space. Our approach reduces to the classical one for subsets of a strictly convex Banach space. However its full implications are as yet unclear.

2. Preliminaries

We begin by describing the approach of [12]. For $x_1, x_2 \in X$ and $a_1, a_2 \in [0, 1]$ with $a_1 + a_2 = 1$, let $a_1 x_1 \oplus a_2 x_2$ be the point of $[x_1, x_2]$ for which

$$\rho(x_1, a_1 x_1 \oplus a_2 x_2) = a_2 \rho(x_1, x_2) \text{ and } \rho(x_2, a_1 x_1 \oplus a_2 x_2) = a_1 \rho(x_1, x_2).$$

Having defined the *ordered* convex combination $a_1 x_1 \oplus \cdots \oplus a_{n-1} x_{n-1}$ for $a_1, \ldots, a_{n-1} \in [0, 1]$ with $\sum_{i=1}^{n-1} a_i = 1$ $(n > 1)$,

$$(x_1, \ldots, x_{n-1}) \in \prod_{i=1}^{n-1} X,$$

let $a_1, \ldots, a_n \in [0, 1]$ with $\sum_{i=1}^{n} a_i = 1$, and let $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X$. If $a_n = 1$, set

$$a_1 x_1 \oplus \cdots \oplus a_n x_n = x_n.$$

Otherwise, set

$$a_1 x_1 \oplus \cdots \oplus a_n x_n = a_n x_n \oplus (1 - a_n)\left[\frac{a_1}{1 - a_n} x_1 \oplus \cdots \oplus \frac{a_{n-1}}{1 - a_n} x_{n-1}\right].$$

We now adopt the notation

$$a_1 x_1 \oplus \cdots \oplus a_n x_n = \sum_{i=1}^{n} [a_i x_i].$$

With this definition it follows immediately from (1.2) that if $x, x_1, \ldots, x_n \in X$, then

$$\rho\left(x, \sum_{i=1}^{n} [a_i x_i]\right) \leq \sum_{i=1}^{n} a_i \rho(x, x_i).$$

It was shown in [12] that this approach has applications in the study of approximate fixed points for mappings that are 'approximately' continuous. However this definition has the defect that the convex combination depends on the order of the n-tuple $\{x_1, \ldots, x_n\}$. For example in general, $a_1x_1 \oplus a_2x_2 \oplus a_3x_3$ may not be the same as $a_2x_2 \oplus a_1x_1 \oplus a_3x_3$. In the next section we describe a related method for defining $a_1x_1 \oplus \cdots \oplus a_nx_n$ in a *complete* Busemann space which does not depend on the order in which the sum is written. Our approach in the next section is inspired by a technique of Ivanshin [8] (based on an approach of [5]) for defining the 'mean point' of a finite set of points in such spaces.

3. Convex combinations

Throughout this section X denotes a complete Busemann space. We take as our point of departure the approach of the previous section, but with the goal of defining the convex combination of a finite set of points of X that is independent of the order in which they are chosen. This procedure suggests two new ways to define the convex hull of a subset of X. We discuss this in more detail at the end of the section. Our motivation is to try to find a more analytic approach the study of convex hulls of subsets of Busemann spaces.

We describe the general procedure in this section. To help make the general case clear, the case $k = 3$ is discussed in detail in Section 5.

We proceed by induction. Having defined $a_1x_1 \oplus a_2x_2$ for $\{x_1, x_2\} \subset X$ and $\{a_1, a_2\} \subset [0,1]$ with $a_1 + a_2 = 1$, we now proceed by induction. Suppose $k > 2$ and suppose $a_1x_1 \oplus \cdots \oplus a_{k-1}x_{k-1}$ has been defined, regardless of order, for all sets of $k-1$ points of X and all $\{a_1, \ldots, a_{k-1}\} \subset [0,1]$ satisfying $\sum_{i=1}^{k-1} a_i = 1$. Now consider a k-tuple: $\{x_1, x_2, \ldots, x_k\} \subset X$ and suppose $\{a_1, \ldots, a_k\} \subset [0,1]$ satisfies $\sum_{i=1}^{k} a_i = 1$. By the inductive assumption we may further assume that $\{a_1, \ldots, a_k\} \subset (0,1)$. Now set

$$\begin{aligned}
x_1^1 &= a_1x_1 \oplus (1-a_1)\left(\frac{a_2}{1-a_1}x_2 \oplus \frac{a_3}{1-a_1}x_3 \oplus \cdots \oplus \frac{a_k}{1-a_1}x_k\right)\\
x_2^1 &= a_2x_2 \oplus (1-a_2)\left(\frac{a_1}{1-a_2}x_1 \oplus \frac{a_3}{1-a_2}x_3 \oplus \cdots \oplus \frac{a_k}{1-a_2}x_k\right)\\
x_3^1 &= a_3x_3 \oplus (1-a_3)\left(\frac{a_1}{1-a_3}x_1 \oplus \frac{a_2}{1-a_3}x_2 \oplus \cdots \oplus \frac{a_k}{1-a_3}x_k\right)\\
&\vdots\\
x_k^1 &= a_kx_k \oplus (1-a_k)\left(\frac{a_1}{1-a_k}x_1 \oplus \frac{a_2}{1-a_k}x_2 \oplus \cdots \oplus \frac{a_{k-1}}{1-a_k}x_{k-1}\right).
\end{aligned}$$

In general, let

$$\begin{aligned}
x_1^n &= a_1 x_1^{n-1} \oplus (1-a_1)\left(\frac{a_2}{1-a_1}x_2^{n-1} \oplus \frac{a_3}{1-a_1}x_3^{n-1} \oplus \cdots \oplus \frac{a_k}{1-a_1}x_k^{n-1}\right) \\
x_2^n &= a_2 x_2^{n-1} \oplus (1-a_2)\left(\frac{a_1}{1-a_2}x_1^{n-1} \oplus \frac{a_3}{1-a_2}x_3^{n-1} \oplus \cdots \oplus \frac{a_k}{1-a_2}x_k^{n-1}\right) \\
x_3^n &= a_3 x_3^{n-1} \oplus (1-a_3)\left(\frac{a_1}{1-a_3}x_1^{n-1} \oplus \frac{a_2}{1-a_3}x_2^{n-1} \oplus \cdots \oplus \frac{a_k}{1-a_3}x_k^{n-1}\right) \\
&\vdots \\
x_k^n &= a_k x_k^{n-1} \oplus (1-a_k)\left(\frac{a_1}{1-a_k}x_1^{n-1} \oplus \frac{a_2}{1-a_k}x_2^{n-1} \oplus \cdots \oplus \frac{a_{k-1}}{1-a_k}x_{k-1}^{n-1}\right).
\end{aligned}$$

We now estimate $\rho\left(x_i^n, x_j^n\right)$ for $i, j \in \{1, \ldots, k\}$, $i < j$. By iterated use of (1.2) we obtain

$$\begin{aligned}
\rho\left(x_i^n, x_j^n\right) &\leq \sum_{i=1}^{k} a_i \rho\left(x_i^{n-1}, x_j^n\right) \\
&\leq \sum_{i=1}^{k} a_i \sum_{j=1}^{k} a_j \rho\left(x_i^{n-1}, x_j^{n-1}\right) \\
&= \sum_{i,j=1}^{k} a_i a_j \rho\left(x_i^{n-1}, x_j^{n-1}\right) \\
&\leq 2\left[\sum_{i,j=1(i<j)}^{k} a_i a_j\right] diam\left(\left\{x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, \ldots, x_k^{n-1}\right\}\right).
\end{aligned}$$

Thus

$$diam\left(\{x_1^n, x_2^n, x_3^n, \ldots, x_k^n\}\right) \leq 2\left[\sum_{i,j=1(i<j)}^{k} a_i a_j\right] diam\left(\left\{x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, \ldots, x_k^{n-1}\right\}\right).$$

Next observe that if $\{a_1, a_2, \ldots, a_n\} \subset (0,1)$ and $\sum_{i=1}^{k} a_i = 1$, then

$$2\sum_{i,j=1(i<j)}^{k} a_i a_j < 1.$$

Indeed

$$\begin{aligned}
2\sum_{i,j=1(i<j)}^{k} a_i a_j &= a_1\left(\sum_{j=2}^{k} a_j\right) + a_2\left(\sum_{j=1, j\neq 2}^{k} a_j\right) + \cdots + a_k\left(\sum_{j=1}^{k-1} a_j\right) \\
&= a_1(1-a_1) + a_2(1-a_2) + \cdots + a_k(1-a_k) \\
&= \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} a_i^2 = 1 - \sum_{i=1}^{k} a_i^2 < 1.
\end{aligned}$$

Letting

$$\delta = 2 \sum_{i,j=1 (i<j)}^{k} a_i a_j,$$

we now have

$$diam\left(\{x_1^n, x_2^n, x_3^n, \ldots, x_k^n\}\right) \leq \delta diam\left(\left\{x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, \ldots, x_k^{n-1}\right\}\right)$$

with $\delta < 1$. It follows that

$$diam\left(\{x_1^n, x_2^n, x_3^n, \ldots, x_k^n\}\right) \leq \delta^n diam\left(\{x_1, x_2, x_3, \ldots, x_k\}\right). \tag{3.1}$$

Now let $\overline{conv}\,(F)$ denote the closed convex hull of a subset $F \subset X$ in the usual sense. Thus $\overline{conv}\,(F)$ denotes the closure of the set

$$conv\,(F) = \bigcup_{n=0}^{\infty} F_n, \tag{3.2}$$

where $F_0 = F$, and for $n \geq 1$ the set F_n consists of all points in the space which lie on geodesics which have endpoints in F_{n-1}. With this definition it is clear via (1.2) that $diam\,(F) = diam\,(F_1) = diam\,(F_2) = \cdots = diam\,(\mathrm{conv}\,(F))$.

By construction, the set $\{x_1^n, x_2^n, x_3^n, \ldots, x_k^n\}$ lies in the convex hull of the set $\left\{x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, \ldots, x_k^{n-1}\right\}$; thus

$$conv\,\{x_1^n, x_2^n, x_3^n, \ldots, x_k^n\} \subset conv\left\{x_1^{n-1}, x_2^{n-1}, x_3^{n-1}, \ldots, x_k^{n-1}\right\}.$$

Now, from inequality (3.1), we conclude that

$$diam\left(\overline{conv}\,\{x_1^n, x_2^n, x_3^n, \cdots, x_k^n\}\right) \leq \delta^n diam\left(\overline{conv}\,\{x_1, x_2, x_3, \ldots, x_k\}\right).$$

We can now apply Cantor's intersection theorem to the descending sequence of closed sets

$$\{\overline{conv}\,\{x_1^n, x_2^n, x_3^n, \cdots, x_k^n\}\}_{n=1}^{\infty}$$

and conclude that for $1 \leq j \leq k$, each of the sequences $\left\{x_j^n\right\}_{n=1}^{\infty}$ is a Cauchy sequence, and each converges to a common limit, which we denote $a_1x_1 \oplus \cdots \oplus a_kx_k$.

As in the approach of [12], with this definition we have the following estimate: If $x, x_1, \ldots, x_n \in X$, then

$$\rho\left(x, a_1x_1 \oplus \cdots \oplus a_kx_k\right) \leq \sum_{i=1}^{k} a_i \rho\left(x, x_i\right).$$

If $a_i \equiv \dfrac{1}{k}$ then we have another definition of the mean point (or 'barycenter') $\dfrac{x_1 \oplus \cdots \oplus x_k}{k}$ analogous to the one given in [8]. In this case,

$$2 \sum_{i,j=1 (i<j)}^{k} a_i a_j = \frac{k-1}{k}$$

and for each $x \in X$,

$$\rho\left(x, \frac{x_1 \oplus \cdots \oplus x_k}{k}\right) \leq \frac{1}{k} \sum_{i=1}^{k} \rho(x, x_i).$$

Remark 3.1. If X is a closed subset of a strictly convex Banach space then the iterative process described above for defining the convex combination terminates at the first step. It is also known that X is isometric to a convex subset of a normed space if and only if affine functions on X separate points (see Theorem 1.1 in [7]).

Remark 3.2. Let $co\,(F)$ denote the collection of all convex combinations of finite subsets of F as defined above, and let $\overline{co}\,(F)$ denote its closure. All that is clear at this point is that $\overline{co}\,(F) \subset \overline{conv}\,(F)$. It is probably asking too much to expect the two sets to coincide. A third approach might be to set $F_0 = F$ and for $n \geq 1$, set $F_n = co\,(F_{n-1})$. It is now possible to define a new concept of 'convex hull' of F by taking the union of the sets $co\,(F_n)$. In general this 'convex hull' would appear to lie between $\overline{co}\,(F)$ and $\overline{conv}\,(F)$.

4. Possible applications

4.1. **A KKM principle.** Niculescu and Rovenţa have shown in [14] that the classical KKM lemma due to Knaster, Kuratowski, and Mazurkiewicz extends to complete CAT(0) spaces. We suggest another approach here via the following definition.

Definition 4.1. *Let (X, ρ) be a Busemann space and $K \subset X$. We call a multivalued mapping $G : K \to 2^X$ a* KKM-map *if for any finite set of points $x_1, \ldots, x_n \in K$,*

$$co(\{x_1, \ldots, x_n\}) \subset \bigcup_{1 \leq i \leq n} G(x_i),$$

where $co\,(\{x_1, \ldots, x_n\})$ denotes the collection of all convex combinations (as defined in Section 3) of the set $\{x_1, \ldots, x_n\}$. Let $\overline{co}\,(\{x_1, \ldots, x_n\})$ denote the closure of the set $co\,(\{x_1, \ldots, x_n\})$.

A question left open is whether the convex combination $a_1 x_1 \oplus \cdots \oplus a_n x_n$ of a fixed finite set of points $\{x_1, \ldots, x_n\}$ in a complete Busemann space is always a continuous function of the parameters $\{a_1, \ldots, a_n\}$. *The lemma below only applies in situations where this is true.*

Lemma 4.2. *Suppose X is a complete Busemann space and suppose*

$$\overline{co}\,(\{x_1, \ldots, x_n\})$$

has the fixed point property for continuous maps for every finite subset $\{x_1, \ldots, x_n\}$ of X. Let $G : H \to 2^X$ be a KKM-map for which each set $G(x)$ is closed. Then the family $\{G(x) : x \in H\}$ has the finite intersection property. Thus if $G(x_0)$ is compact for some $x_0 \in H$, $\bigcap_{x \in H} G(x) \neq \emptyset$.

Proof. (This mimics the proof of Theorem 3 of [10].) Suppose there exists $F := \{x_1, \ldots, x_n\} \subset H$ such that $\bigcap_{i=1}^{n} G(x_i) = \emptyset$. Set $C = \overline{co}\,(F)$. Then for every

$c \in C$ there exists $i_0 \in \{1, \ldots, n\}$ such that $c \notin G(x_{i_0})$. Since $G(x_{i_0})$ is closed, $dist(c, G(x_{i_0})) > 0$. Therefore,

$$\alpha(c) = \sum_{i=1}^{n} dist(c, G(x_i)) > 0.$$

Define$f : C \to C$ by setting

$$f(c) = \frac{dist(c, G(x_1))}{\alpha(c)} x_1 \oplus \cdots \oplus \frac{dist(c, G(x_n))}{\alpha(c)} x_n.$$

Then f is continuous, and $f(c) \in co(C)$ for each $c \in C$. By assumption, there exists $c_0 \in C$ such that $f(c_0) = c_0$. Set

$$I := \{i \in \{1, \ldots, n\} : dist(c_0, G(x_i)) > 0\}.$$

Then

$$c_0 = \frac{1}{\alpha(c_0)} \sum_{i \in I} dist(c, G(x_i))\, x_i.$$

Thus $c_0 \in co\{x_i : i \in I\}$. However $c_0 \notin \bigcup_{i \in I} G(x_i)$, contradicting the assumption

$$co\{x_i : i \in I\} \subset \bigcup_{i \in I} G(x_i).$$

□

It is known that in a complete CAT(0) space X, every compact convex set K has the fixed point property for continuous maps ([15], Theorem 1.5). Therefore Lemma 4.2 holds in any such space for which $\overline{co}F$ is compact and convex for any finite subset F of X (under the additional continuity assumption).

4.2. **Kannan maps.** In this section we illustrate the *potential* usefulness of an analytic approach to convex combinations. We begin with a classical definition from Banach space geometry.

Definition 4.3. *A convex set K in a Banach space X is said to have quasi-normal structure if every bounded convex subset H of K which contains more than one point contains a point x_0* such that $\|x_0 - y\| < diam(H)$ for each $x \in H$.

The concept of quasi-normal structure is a very mild condition in Banach spaces (see e.g., [20], [18]). However it has a rather striking fixed point implication. A mapping $T : K \to K$ is called a *Kannan mapping* if

$$\|T(x) - T(y)\| \leq \frac{1}{2}[\|x - T(x)\| + \|y - T(y)\|]$$

for each $x, y \in K$. Kannan proved in [9] that if K is a weakly compact convex subset of a Banach space and if for any T-invariant subset H of K with more than one point, $\sup\{\|y - T(y)\| : y \in H\} < diam(H)$, then T has a fixed point. Subsequently Wong proved in [19] that a weakly compact convex subset of a Banach space has the fixed point property for Kannan maps if and only if it has quasi-normal structure.

It is known that if K is a bounded closed convex subset of complete CAT(0) space then

$$rad\,(K) \leq \frac{\sqrt{2}}{2} diam\,(K)\,,$$

where rad(K) denotes the Chebyshev radius of K. This is a particular case of Theorem B of [13] (also see [11], [4]). Thus if $diam\,(K) > 0$ then K must have nondiametral points. This raises the question of whether every Kannan map $T : K \to K$ has a fixed point. We show that the answer is affirmative if $\overline{co}\,(F)$ is convex for every $F \subset K$. It remains open whether this is true in general for a complete CAT(0) space.

Theorem 4.4. *Let K be a bounded closed convex subset of a complete CAT(0) space (X, ρ), and let $T : K \to K$ be a Kannan map. Suppose also that $\overline{co}\,(F)$ is convex for every $F \subset K$. Then there exists $x \in K$ such that*

$$\rho\,(x, T\,(x)) = \inf \{\rho\,(y, T\,(y)) : y \in K\}\,.$$

Proof. This theorem can be proved following the corresponding proof of Wong [19], simply by replacing the norm $\|\cdot\|$ with the metric $\rho\,(\cdot)$. Specifically, let $r_0 = \inf \{\rho\,(y, T\,(y)) : y \in K\}$, and for $r > r_0$, let

$$K_r = \{y \in K : \rho\,(y, T\,(y)) \leq r\}\,.$$

As in [19], $T : K_r \to K_r$. Now let $H_r = \overline{co}\,(T\,(K_r))$. Since the family $\{T\,(K_r)\}_{r>r_0}$ is a descending chain, the family $\{H_r\}_{r>r_0}$ has the finite intersection property. It is known that in a CAT(0) space any descending sequence of bounded closed convex sets has nonempty intersection. Thus

$$\bigcap_{r>r_0} H_r \neq \emptyset.$$

Next it is necessary to show that for each $r > r_0$, $H_r \subseteq K_r$. (This is the only step where the definition of Section 3 is needed.) For such r, let $y \in H_r$ and let $\varepsilon > 0$. Then there exist $\{t_1, t_2, \ldots, t_n\} \subset \mathbb{R}^+$ and $\{y_1, y_2, \ldots, y_n\} \subset K_r$ such that $\sum_{i=1}^n t_i = 1$ and for which

$$\rho\,(y, t_1 T\,(y_1) \oplus \cdots \oplus t_n T\,(y_n)) < \varepsilon.$$

Thus

$$\begin{aligned}
\rho\,(y, T\,(y)) &\leq \rho\,(y, t_1 T\,(y_1) \oplus \cdots \oplus t_n T\,(y_n)) + \rho\,(t_1 T\,(y_1) \oplus \cdots \oplus t_n T\,(y_n)\,, T\,(y)) \\
&< \varepsilon + \rho\,(t_1 T\,(y_1) \oplus \cdots \oplus t_n T\,(y_n)\,, T\,(y)) \\
&\leq \varepsilon + \sum_{i=1}^n t_i \rho\,(T\,(y_i)\,, T\,(y)) \\
&\leq \varepsilon + \sum_{i=1}^n t_i \frac{1}{2} \left[\rho\,(y_i, T\,(y_i)) + \rho\,(y, T\,(y))\right] \\
&\leq \varepsilon + \sum_{i=1}^n t_i \frac{1}{2} \left[r + \rho\,(y, T\,(y))\right] \\
&= \varepsilon + \frac{1}{2} \left[r + \rho\,(y, T\,(y))\right].
\end{aligned}$$

From this we conclude that $\rho(y, T(y)) < \varepsilon + r$, and since $\varepsilon > 0$ is arbitrary, $y \in K_r$. Thus $\bigcap_{r > r_0} K_r \neq \emptyset$, and the conclusion follows. $\square$

Theorem 4.5. *Let K be as in Theorem 4.4. Then every Kannan map $T : K \to K$ has a fixed point.*

Proof. Let K_0 be a minimal nonempty closed convex subset of K which is invariant under T. (Such a minimal set is known to exist; see [1]; also [11]). Upon replacing $\|\cdot\|$ with $\rho(\cdot)$ in part (a) of Theorem 3 of [19], it is possible to show that for each $x \in K_0$, $\rho(x, T(x)) = diam(K_0)$. However, as noted above, if $diam(K) > 0$ then K has nondiametral points. It follows that $diam(K_0) = 0$ and the conclusion follows. $\square$

5. Appendix

To help make the general case more transparent, we describe the procedure of Section 3 in detail for three points, i.e., for defining $a_2x_2 \oplus a_1x_1 \oplus a_3x_3$ for $\{x_1, x_2, x_3\} \subset X$, and $\{a_1, a_2, a_3\} \subset [0, 1]$ with $a_1 + a_2 + a_3 = 1$.

If $a_n = 1$ for some $n \in \{1, 2, 3\}$, set

$$a_1x_1 \oplus a_2x_2 \oplus a_3x_3 = x_n.$$

Since the convex combination of two points has already been defined, we may assume $\{a_1, a_2, a_3\} \subset (0, 1)$ and set

$$\begin{aligned}
x_1^1 &= a_1x_1 \oplus (1 - a_1)\left(\frac{a_2}{1 - a_1}x_2 \oplus \frac{a_3}{1 - a_1}x_3\right)\\
x_2^1 &= a_2x_2 \oplus (1 - a_2)\left(\frac{a_1}{1 - a_2}x_1 \oplus \frac{a_3}{1 - a_2}x_3\right)\\
x_3^1 &= a_3x_3 \oplus (1 - a_3)\left(\frac{a_1}{1 - a_3}x_1 \oplus \frac{a_2}{1 - a_3}x_2\right);
\end{aligned}$$

$$\begin{aligned}
x_1^2 &= a_1x_1^1 \oplus (1 - a_1)\left(\frac{a_2}{1 - a_1}x_2^1 \oplus \frac{a_3}{1 - a_1}x_3^1\right)\\
x_2^2 &= a_2x_2^1 \oplus (1 - a_2)\left(\frac{a_1}{1 - a_2}x_1^1 \oplus \frac{a_3}{1 - a_2}x_3^1\right)\\
x_3^2 &= a_3x_3^1 \oplus (1 - a_3)\left(\frac{a_1}{1 - a_3}x_1^1 \oplus \frac{a_2}{1 - a_3}x_2^1\right).
\end{aligned}$$

Having defined $x_1^{n-1}, x_2^{n-1}, x_3^{n-1}$ set

$$\begin{aligned}
x_1^n &= a_1x_1^{n-1} \oplus (1 - a_1)\left(\frac{a_2}{1 - a_1}x_2^{n-1} \oplus \frac{a_3}{1 - a_1}x_3^{n-1}\right)\\
x_2^n &= a_2x_2^{n-1} \oplus (1 - a_2)\left(\frac{a_1}{1 - a_2}x_1^{n-1} \oplus \frac{a_3}{1 - a_2}x_3^{n-1}\right)\\
x_3^n &= a_3x_3^{n-1} \oplus (1 - a_3)\left(\frac{a_1}{1 - a_3}x_1^{n-1} \oplus \frac{a_2}{1 - a_3}x_2^{n-1}\right).
\end{aligned}$$

We assert that the sequences $\{x_k^n\}_{n=1}^{\infty}$, $k = 1,2,3$, are Cauchy and all converge to a common point. Indeed two applications of the inequality (1.2) yields

$$\begin{aligned}
\rho\left(x_1^n, x_2^n\right) &\leq a_1 a_2 \rho\left(x_1^{n-1}, x_2^{n-1}\right) + a_1 a_3 \rho\left(x_1^{n-1}, x_3^{n-1}\right) + a_2 a_1 \rho\left(x_2^{n-1}, x_1^{n-1}\right) \\
&\quad + a_2 a_3 \rho\left(x_2^{n-1}, x_3^{n-1}\right) + a_3 a_1 \rho\left(x_3^{n-1}, x_1^{n-1}\right) + a_3 a_2 \rho\left(x_3^{n-1}, x_2^{n-1}\right) \\
&= 2a_1 a_2 \rho\left(x_1^{n-1}, x_2^{n-1}\right) + 2a_1 a_3 \rho\left(x_1^{n-1}, x_3^{n-1}\right) \\
&\quad + 2a_2 a_3 \rho\left(x_3^{n-1}, x_2^{n-1}\right) \\
&\leq 2\left[a_1 a_2 + a_1 a_3 + a_2 a_3\right] diam\left(\left\{x_1^{n-1}, x_2^{n-1}, x_3^{n-1}\right\}\right).
\end{aligned}$$

In general for $i, j \in \{1,2,3\}$,

$$\rho\left(x_i^n, x_j^n\right) \leq 2\left[a_1 a_2 + a_1 a_3 + a_2 a_3\right] diam\left(\left\{x_1^{n-1}, x_2^{n-1}, x_3^{n-1}\right\}\right).$$

It follows that

$$diam\left(\{x_1^n, x_2^n, x_3^n\}\right) \leq \delta^n diam\left(\{x_1, x_2, x_3\}\right)$$

where $\delta = 2\left[a_1 a_2 + a_1 a_3 + a_2 a_3\right]$. Also, $\delta < 1$ because

$$\begin{aligned}
2\left[a_1 a_2 + a_1 a_3 + a_2 a_3\right] &= a_1\left(a_2 + a_3\right) + a_2\left(a_1 + a_3\right) + a_3\left(a_2 + a_1\right) \\
&= a_1\left(1 - a_1\right) + a_2\left(1 - a_2\right) + a_3\left(1 - a_3\right) \\
&= 1 - \left(a_1^2 + a_2^2 + a_3^2\right).
\end{aligned}$$

Since $\overline{conv}(\{x_1^n, x_2^n, x_3^n\}) \subset \overline{conv}(\{x_1^{n-1}, x_2^{n-1}, x_3^{n-1}\})$, by Cantor's intersection theorem the three sequences $\left\{x_j^n\right\}_{n=1}^{\infty}$, $j = 1,2,3$, are Cauchy with a common limit which we denote $a_1 x_1 \oplus a_2 x_2 \oplus a_3 x_3$.

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