Journal of Nonlinear and Convex Analysis
Volume 15, Number 1, 2014, 199–209

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

THE COMMON FIXED POINT SET OF COMMUTING NONEXPANSIVE MAPPINGS IN CARTESIAN PRODUCTS OF BANACH SPACES WITH THE OPIAL PROPERTY

MONIKA BUDZYŃSKA, TADEUSZ KUCZUMOW, AND MAŁGORZATA MICHALSKA

This paper is dedicated to Professor Simeon Reich on his 65th birthday

Abstract. Using the Opial property, we establish a theorem regarding the structure of the common fixed point set of commuting nonexpansive self-mappings of sets in Cartesian products of Banach spaces.

1. Introduction

In 1967 Z. Opial [26] introduced a property which is very useful in metric fixed point theory (for the characterization of this property and its generalizations, see [1], [3], [4], [5], [6], [7], [8], [9], [12], [13], [14], [15], [19], [21], [22], [23], [24], [25], [27], [28], [30], [31]). The Opial property deals with weakly convergent sequences but in many applications we often need this property for nets especially when we study either convergence (in the Hausdorff linear topology $\mathcal{T}$) of almost orbits of various types of semigroups of self-mappings of a $\mathcal{T}$-compact subset C of X or nonexpansive retractions on fixed point sets (see, for example [16], [20], [29]). It was shown in [6] that in the case of Γ-topology the Opial property for nets is equivalent to the Opial property for sequences. For the weak topology and weak compact sets this equivalence was proved by W. Kaczor and S. Prus in [17].

In this paper we show how to construct a nonexpansive retraction on a common fixed point set of commuting nonexpansive mappings in a Cartesian product of two Banach spaces with the Opial property for nets.

2. Preliminaries

Throughout this paper all Banach spaces will be real.

Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{T}$ the Hausdorff vector topology in X. We say that a nonempty set $C \subset X$ satisfies the $\mathcal{T}$-Opial condition (or has the $\mathcal{T}$-Opial property), if whenever a bounded sequence $\{x_n\}$ of elements of C converges in the topology $\mathcal{T}$ to $x \in C$ and $y \in C\setminus\{x\}$, then we have

$$\limsup_n \|x_n - x\| < \limsup_n \|x_n - y\|.$$

We say that C satisfies the $\mathcal{T}$-Opial condition for nets, (or has $\mathcal{T}$-Opial property for nets) if whenever a bounded net $\{x_\alpha\}_{\alpha\in I}$ of elements of C converges in the topology $\mathcal{T}$ to $x \in C$ and $y \in C \setminus \{x\}$, then

$$\limsup_{\alpha\in I} \|x_\alpha - x\| < \limsup_{\alpha\in I} \|x_\alpha - y\|.$$

2010 *Mathematics Subject Classification.* 47H09.

Key words and phrases. Banach space, Cartesian product, common fixed point set, commuting nonexpansive mappings, fixed point, nonexpansive retract, Opial's property.

In many applications we consider topologies generated by some families of continuous and linear functionals. Therefore we recall the following definitions. Let $(X, \|\cdot\|)$ be a Banach space and let Γ be a nonempty subspace of its dual X^*. If

$$\sup \{f(x) : f \in \Gamma, \|f\| = 1\} = \|x\|$$

for each $x \in X$, then we say that Γ is a norming set for X. It is obvious that a norming set generates a Hausdorff linear topology $\sigma(X, \Gamma)$ on X which is weaker than the weak topology $\sigma(X, X^*)$. In the case when Γ is a norming subset of X^* and C a nonempty subset of X we say that C satisfies the Γ-Opial condition (the Γ-Opial condition for nets), if C satisfies the $\sigma(X, \Gamma)$-Opial condition (the $\sigma(X, \Gamma)$-Opial condition for nets). Now, let us observe that the following theorem is valid.

Theorem 2.1 ([6]). *Let $(X, \|\cdot\|)$ be a Banach space, Γ be a norming set for X and C a nonempty, bounded and sequentially Γ-compact subset of X. Then for such a C the Γ-Opial condition for nets is equivalent to the Γ-Opial condition.*

Remark 2.2. For the weak topology the above theorem was proved (by using a different method) by W. Kaczor and S. Prus [17].

Next, we will also use the notions of an asymptotic radius and an asymptotic center [10]. Let $(X, \|\cdot\|)$ be a Banach space and C a nonempty subset of X. For $x \in X$ and a bounded sequence $\{x_n\} \subset X$(a bounded net $\{x_\alpha\}_{\alpha \in I} \subset X$), define the asymptotic radius of $\{x_n\}$ ($\{x_\alpha\}_{\alpha\in I}$) at x as the number

$$r(x, \{x_n\}) = \limsup_n \|x_n - x\|$$

$$\left(r(x, \{x_\alpha\}_{\alpha\in I}) = \limsup_\alpha \|x_\alpha - x\| \right).$$

The asymptotic radius of $\{x_n\}$ ($\{x_\alpha\}_{\alpha\in I}$) in C is the number

$$r(C, \{x_n\}) = \inf \{r(x, \{x_n\}) : x \in C\}$$

$$(r(C, \{x_\alpha\}_{\alpha\in I}) = \inf \{r(x, \{x_\alpha\}_{\alpha\in I}) : x \in C\})$$

and the asymptotic center of $\{x_n\}$ ($\{x_\alpha\}_{\alpha\in I}$) in C is the set

$$A(C, \{x_n\}) = \{x \in C : r(x, \{x_n\}) = r(C, \{x_n\})\}$$

$$(A(C, \{x_\alpha\}_{\alpha\in I}) = \{x \in C : r(x, \{x_\alpha\}_{\alpha\in I}) = r(C, \{x_\alpha\}_{\alpha\in I})\}).$$

Finally, we recall definitions of a nonexpansive mapping and a nonexpansive retraction. Let $(X, \|\cdot\|)$ be a Banach space. If C, C_1 are nonempty subsets of X and $T : C \to C_1$ satisfies

$$\|Tx - Ty\| \le \|x - y\|$$

for each $x, y \in C$, then T is called a nonexpansive mapping.

We say that a nonempty $C \subset X$ has FPP (the fixed point property) for nonexpansive mappings if each nonexpansive $T : C \to C$ has a fixed point, i.e. there exists $x_0 \in C$ such that $Tx_0 = x_0$. Then the set $\{x \in C : Tx = x\}$ is denoted by $FixT$ and called the fixed point set of T.

If for a nonempty subset $D \subset C$ there exists a nonexpansive mapping $r : C \to D$ such that $r(x) = x$ for each $x \in D$, then D is called a nonexpansive retract of C and the mapping r is called a nonexpansive retraction.

3. Fixed point theorems in Cartesian product of sets

The following theorems are generally known.

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space with the Hausdorff vector topology $\mathcal{T}$ and let a nonempty, bounded, convex, and sequentially compact in $\mathcal{T}$ subset $C \subset X$ have the $\mathcal{T}$-Opial property. Then C has FPP for nonexpansive mappings.*

Theorem 3.2. *Let $(X, \|\cdot\|)$ be a Banach space with the Hausdorff vector topology $\mathcal{T}$ such that the norm $\|\cdot\|$ is lower semicontinuous in this topology. Let a nonempty, bounded, convex, and compact in $\mathcal{T}$ subset $C \subset X$ have the $\mathcal{T}$-Opial property for nets. Then C has FPP for nonexpansive mappings and for each nonexpansive mapping $T : C \to C$ its fixed point set $FixT$ is a nonexpansive retract of C.*

Till the end of this chapter we will always assume that if $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are Banach spaces then the norm in the Cartesian product $X_1 \times X_2$ is the max norm. We will investigate fixed point property of subsets of the Cartesian products of two sets with the Opial property (with the Opial property for nets). The two basic theorems are the following.

Theorem 3.3. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces with the Hausdorff vector topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ be lower semicontinuous with respect to the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let nonempty bounded convex sets $C_1 \subset X_1$ and $C_2 \subset X_2$ be sequentially compact in $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, and let C_1 have the $\mathcal{T}_1$-Opial property and C_2 have the $\mathcal{T}_2$-Opial property. If a nonempty and convex subset C of $C_1 \times C_2$ is closed in the topology $\mathcal{T}_1 \times \mathcal{T}_2$, then C has FPP for nonexpansive mappings.*

Proof. Let $T = (T_1, T_2) : C \to C$ be a nonexpansive mapping. Choose $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in C$. For each $n \in \mathbb{N}$ there exists a unique $x_n = (x_{1n}, x_{2n}) \in C$ such that

$$x_n = \frac{1}{n}\tilde{x} + \left(1 - \frac{1}{n}\right) Tx_n.$$

Then $\|Tx_n - x_n\| \to_n 0$. Taking a subsequence $\{x_{n_k}\}$, which tends to $y = (y_1, y_2) \in C$ in the topology $\mathcal{T}_1 \times \mathcal{T}_2$, by the Opial property we get $y \in A(C, \{x_{n_k}\})$ and next

$$\limsup_k \|x_{n_k} - Tx\| = \limsup_k \|Tx_{n_k} - Tx\| \le \limsup_k \|x_{n_k} - x\|$$

for each $x \in A(C, \{x_{n_k}\})$, i.e.

$$T\left(A(C, \{x_{n_k}\})\right) \subset A(C, \{x_{n_k}\}).$$

Without loss of generality we can assume that

$$\begin{aligned}\limsup_k \|x_{n_k} - y\| &= \max\{\limsup_k \|x_{1n_k} - y_1\|_1, \limsup_k \|x_{2n_k} - y_2\|_2\} \\ &= \limsup_k \|x_{1n_k} - y_1\|_1,\end{aligned}$$

which implies (by the Opial property)

$$A(C, \{x_{n_k}\}) = \{y_1\} \times D_2,$$

where D_2 is a nonempty, convex and norm-closed subset of C_2.

The asymptotic center $A(C,\{x_{n_k}\})$ is T-invariant. Choose $\tilde{w}=(\tilde{w}_1,\tilde{w}_2)\in A(C,\{x_{n_k}\})$. For each $n\in\mathbb{N}$ there exists a unique $w_n=(w_{1n},w_{2n})\in A(C,\{x_{n_k}\})$ such that

$$w_n=\frac{1}{n}\tilde{w}+\left(1-\frac{1}{n}\right)Tw_n.$$

Then $\|Tw_n-w_n\|\to_n 0$. Take a subsequence $\{w_{n_k}\}$ which tends to $w=(w_1,w_2)=(y_1,w_2)\in C$ in the topology $\mathcal{T}_1\times\mathcal{T}_2$. By the Opial property we obtain $w\in A(C,\{w_{n_k}\})$ and next

$$\limsup_k\|w_{n_k}-Tx\|=\limsup_k\|Tx_{n_k}-Tx\|\le\limsup_k\|x_{n_k}-x\|$$

for each $x\in A(C,\{w_{n_k}\})$. It is easy to observe that

$$A(C,\{w_{n_k}\})=(B(w_1,r(C,\{w_{n_k}\})\times\{w_2\})\cap C=B_1\times\{w_2\},$$

where $B(w_1,r(C,\{w_{n_k}\})$ is a closed ball in $(X_1,\|\cdot\|_1)$. Therefore the set $B_1\subset X_1$ is nonempty, convex, sequentially compact in $\mathcal{T}_1$, $T(\cdot,w_2)$-invariant and $T_2(B_1\times\{w_2\})=\{w_2\}$. By Theorem 3.1, the nonexpansive mapping $T_1(\cdot,w_2):B_1\to B_1$ has a fixed point $\bar{w}_1\in B_1$ and hence the point $\bar{w}=(\bar{w}_1,w_2)\in C$ is a fixed point of T. This completes the proof. □

Theorem 3.4. *Let $(X_1,\|\cdot\|_1)$ and $(X_2,\|\cdot\|_2)$ be Banach spaces with the Hausdorff vector topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ be lower semicontinuous with respect to the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, and let nonempty bounded convex sets $C_1\subset X_1$ and $C_2\subset X_2$ be compact in $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let C_1 have the $\mathcal{T}_1$-Opial property for nets and C_2 have the $\mathcal{T}_2$-Opial property for nets. If a nonempty and convex subset C of $C_1\times C_2$ is closed in the topology $\mathcal{T}_1\times\mathcal{T}_2$, then C has FPP for nonexpansive mappings and for each nonexpansive mapping $T:C\to C$ a fixed point set $FixT$ is a nonexpansive retract of T.*

Proof. Let $T=(T_1,T_2):C\to C$ be a nonexpansive mapping. For each $x=(x_1,x_2)\in C$ and for each $n\in\mathbb{N}$ there exists a unique $F(x,n)=(F_1(x,n),F_2(x,n))\in C$ such that

$$F(x,n)=\frac{1}{n}x+\left(1-\frac{1}{n}\right)T(F(x,n)).$$

Each mapping $F(\cdot,n):C\to C$ is nonexpansive. The set $\mathbb{N}$ of all natural numbers can be treated as a sequence $\{n\}_{n\in\mathbb{N}}$. Hence it has a subnet $\{n_\alpha\}_{\alpha\in I}$ which is an ultranet (see [11] and [18] for properties of ultranets). Then for each $x\in C$ we get convergence of $\{F(x,n_\alpha)\}_{\alpha\in I}$ to $y(x)=(y_1(x),y_2(x))\in C$ (in the topology $\mathcal{T}_1\times\mathcal{T}_2$). By lower semicontinuity of norms $\|\cdot\|_1$ and $\|\cdot\|_1$ in the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, the mapping $y(\cdot):C\to C$ is nonexpansive. Next, by the Opial property for nets, we get $y(x)\in A(C,\{F(x,n_\alpha)\}_{\alpha\in I})$ and

$$\lim_\alpha\|T(F(x,n_\alpha))-F(x,n_\alpha)\|=0,$$

$$\|F(x,n_\alpha)-F(\tilde{x},n_\alpha)\|\le\|x-\tilde{x}\|$$

for each $x,\tilde{x}\in C$, and

$$\limsup_\alpha\|F(x,n_\alpha)-T(z)\|=\limsup_k\|T(F(x,n_\alpha))-T(z))\|\le\limsup_k\|F(x,n_\alpha)-z\|$$

for each $z \in A(C, \{F(x, n_\alpha\}_{\alpha\in I})$, i.e., $T\,(A(C, \{F(x, n_\alpha)\}_{\alpha\in I})) \subset A(C, \{F(x, n_\alpha)\}_{\alpha\in I})$ for each $x \in C$. Let us observe that now we have two cases: either

$$\begin{aligned} &\limsup_\alpha \|F(x, n_\alpha) - y(x)\| \\ &= \max\{\limsup_\alpha \|F_1(x, n_\alpha) - y_1(x)\|_1, \limsup_\alpha \|F_2(x, n_\alpha) - y_2(x)\|_2\} \\ &= \limsup_\alpha \|F_1(x, n_\alpha) - y_1(x)\|_1 \end{aligned}$$

or

$$\begin{aligned} &\limsup_\alpha \|F(x, n_\alpha) - y(x)\| \\ &= \max\{\limsup_\alpha \|F_1(x, n_\alpha) - y_1(x)\|_1, \limsup_\alpha \|F_2(x, n_\alpha) - y_2(x)\|_2\} \\ &= \limsup_\alpha \|F_2(x, n_\alpha) - y_2(x)\|_2. \end{aligned}$$

Then (by the Opial property for nets) we get

$$A(C, \{F(x, n_\alpha)\}_\alpha) = \{y_1(x)\} \times D_2$$

in the first case and

$$(A(C, \{F(x, n_\alpha)\}_\alpha) = D_1 \times \{y_1(x)\})$$

in the second one, where D_2 (D_1) is a nonempty, convex and norm-closed subset of C_2 (C_1). The asymptotic center $A(C, \{F(x, n_\alpha)\}_\alpha)$ is T-invariant and therefore for each $n \in \mathbb{N}$ there exists a unique $G(x, n) = (G_1(x, n), G_2(x, n)) \in A(C, \{F(x, n_\alpha)\}_\alpha)$ such that

$$\begin{aligned} G(x, n) = F(y(x), n) &= \frac{1}{n} y(x) + \left(1 - \frac{1}{n}\right) TF(y(x), n) \\ &= \frac{1}{n} y(x) + \left(1 - \frac{1}{n}\right) TG(x, n). \end{aligned}$$

Then $\|TG(x, n) - G(x, n)\| \to_n 0$. Taking an ultranet $\{G(x, n_\alpha)\}_{\alpha\in I}$, which tends (in the topology $\mathcal{T}_1 \times \mathcal{T}_2$) to

$$w(x) = (w_1(x), w_2(x)) = (y_1(x), w_2(x)) \in C$$

in the first case and to

$$w(x) = (w_1(x), w_2(x)) = (w_1(x), y_2(x)) \in C$$

in the second case, we obtain $w(x) \in A(C, \{G(x, n_\alpha)\}_{\alpha\in I})$ (by the Opial property for nets). Next, we have

$$\limsup_\alpha \|G(x, n_\alpha) - Tz\| = \limsup_\alpha \|T(G(x, n_\alpha) - Tz\| \leq \limsup_\alpha \|G(x, n_\alpha) - z\|$$

for each $z \in A(C, \{G(x, n_\alpha)\}_{\alpha\in I})$. As in the first part of this proof we can show that each mapping $G(\cdot, n) : C \to C$ and the mapping $w(\cdot) : C \to C$ are nonexpansive. It is easy to observe that

$$\begin{aligned} A(C, \{G(x, n_\alpha)\}_{\alpha\in I}) &= (B(w_1, r(C, \{G(x, n_\alpha)\}_{\alpha\in I})) \times \{w_2(x)\}) \cap C \\ &= B_1(x) \times \{w_2(x)\} \end{aligned}$$

in the first case and

$$\begin{aligned} A(C, \{G(x, n_\alpha)\}_{\alpha\in I}) &= (\{w_1(x)\} \times (B(w_2, r(C, \{G(x, n_\alpha)\}_{\alpha\in I})) \cap C \\ &= \{w_1(x)\} \times B_2(x) \end{aligned}$$

in the second one. Therefore the set $B_1(x) \subset X_1$ ($B_2(x) \subset X_2$) is nonempty, convex, compact in $\mathcal{T}_1$ ($\mathcal{T}_2$) and $T_1(\cdot, w_2(x))$-invariant ($T_2(w_1(x), \cdot)$-invariant). Additionally, we have $T_2(B_1(x) \times \{w_2(x)\}) = \{w_2(x)\}$ ($T_1(\{w_1(x)\} \times B_2(x)) = \{w_1(x)\}$). Once more we take a sequence $\{H(x, n)\}$ such that

$$H(x, n) = (H_1(x, n), H_2(x, n)) \in A(C, \{G(x, n_\alpha)\}_\alpha) = B_1(x) \times \{w_2(x)\}$$

or, respectively,

$$H(x, n) = (H_1(x, n), H_2(x, n)) \in A(C, \{G(x, n_\alpha)\}_\alpha) = \{w_1(x)\} \times B_2(x)$$

and

$$\begin{aligned} H(x, n) = F(w(x), n) &= \frac{1}{n} w(x) + \left(1 - \frac{1}{n}\right) TF(w(x), n) \\ &= \frac{1}{n} w(x) + \left(1 - \frac{1}{n}\right) TH(x, n) \end{aligned}$$

for $n \in \mathbb{N}$. Then the ultranet $\{H(x, n_\alpha)\}_{\alpha\in I}$ tends to

$$r(x) = (r_1(x), r_2(x)) = (r_1(x), w_2(x)) \in A(C, \{H(x, n_\alpha)\}_\alpha) = B_1(x) \times \{w_2(x)\}$$

or, respectively, to

$$r(x) = (r_1(x), r_2(x)) = (w_1(x), r_2(x)) \in A(C, \{H(x, n_\alpha)\}_\alpha) = \{w_2(x)\} \times B_2(x)$$

in the topology $\mathcal{T}_1 \times \mathcal{T}_2$. By the Opial property for nets we obtain

$$\begin{aligned} \{r(x)\} &= A(A(C, \{G(x, n_\alpha)\}_\alpha), \{H(x, n_\alpha)\}_{\alpha\in I}) \\ &= A(B_1(x) \times \{w_2(x)\}, \{H(x, n_\alpha)\}_{\alpha\in I}) \end{aligned}$$

in the first case and

$$\begin{aligned} \{r(x)\} &= A(A(C, \{G(x, n_\alpha)\}_\alpha), \{H(x, n_\alpha)\}_{\alpha\in I}) \\ &= A(\{w_1(x)\} \times B_2(x), \{H(x, n_\alpha)\}_{\alpha\in I}) \end{aligned}$$

in the second one. Since

$$A(A(C, \{G(x, n_\alpha)\}_\alpha), \{H(x, n_\alpha)\}_{\alpha\in I})$$

is T-invariant, we have $T(r(x)) = r(x)$. It is obvious that if $x \in FixT$, then $T(r(x)) = x$. As in the previous step we can prove that the mappings $H(\cdot, n)$ and $r(\cdot) : C \to C$ are nonexpansive. This completes the proof. □

4. The case of families of mappings

We begin this part of the paper with a result about a finite family of nonexpansive retracts in a more general setting. The idea of this theorem and its proof are due to R. E. Bruck [2].

Theorem 4.1. *Let C be a subset of a Banach space X with the FPP and for each nonexpansive $T : C \to C$ let the fixed point set $FixT$ be a nonexpansive retract of C. Then every finite family $\{T_1, \dots, T_m\}$ of commuting nonexpansive mappings from C to C has a nonempty common fixed point set $FixT_1 \cap \dots \cap FixT_n$, which is a nonexpansive retract of C.*

Our first result in this section is the following.

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a Banach space with the Hausdorff vector topology $\mathcal{T}$ such that $(X, \|\cdot\|)$ has the $\mathcal{T}$-Opial property for nets. Let the norm $\|\cdot\|$ be lower semicontinuous with respect to the topology $\mathcal{T}$. If a nonempty bounded convex set $C \subset X$ is compact in $\mathcal{T}$, then for any infinite family $\mathcal{M} = \{T_\alpha\}_{\alpha \in I}$ of commuting nonexpansive self-mappings of C the set $Fix(\mathcal{M}) = \bigcap_{\alpha \in I} Fix(T_\alpha)$ of common fixed points of $\mathcal{M}$ is a nonempty nonexpansive retract of C.*

Proof. Let J be the set of all finite subsets of I. To each $j \in J$ there corresponds a subset $\{\alpha_1, ..., \alpha_m\}$, and by Theorems 3.4 and 4.1, there exists a nonexpansive retraction $r_j : C \to \bigcap_{i=1}^m Fix(T_{\alpha_i})$. The set J is directed by inclusion and can be considered as a net itself. Let $(j_\beta)_{\beta \in J'}$ be an ultranet in J. For each $x \in C$ consider the ultranet $(r_{j_\beta}(x))_{\beta \in J'}$. By the $\mathcal{T}$-compactness of C this ultranet tends in $\mathcal{T}$ to $r(x) \in C$. By the lower semicontinuity of the norm $\|\cdot\|$ with respect to the topology $\mathcal{T}$ the mapping $r : C \to C$ is nonexpansive. Moreover, for each $x \in C$ and each $\alpha \in I$ we obtain

$$\begin{aligned}\limsup_{\beta \in J'} \|T_\alpha(r(x)) - r_{j_\beta}(x)\| &= \limsup_{\beta \in J'} \|T_\alpha(r(x)) - T_\alpha(r_{j_\beta}(x))\| \\ &\le \limsup_{\beta \in J'} \|r(x) - r_{j_\beta}(x)\|,\end{aligned}$$

which combined with the Opial property for nets gives

$$T_\alpha(r(x)) = r(x).$$

It then follows that the set of common fixed points $\bigcap_{\alpha \in I} Fix(T_\alpha)$ is nonempty. Next, we have $r(x) = x$ for each $x \in Fix(\mathcal{T}) = \bigcap_{\alpha \in I} Fix(T_\alpha)$. Hence

$$r(C) = \bigcap_{\alpha \in I} Fix(T_\alpha)$$

and r is the desired retraction. □

Now we consider the Cartesian product of two Banach spaces.

Theorem 4.3. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces with the Hausdorff vector topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ be lower semicontinuous with respect to the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, and let nonempty bounded convex sets $C_1 \subset X_1$ and $C_2 \subset X_2$ be compact in $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. Let C_1 have the $\mathcal{T}_1$-Opial property for nets and C_2 have the $\mathcal{T}_2$-Opial property for nets. If a nonempty and convex subset C of $C_1 \times C_2$ is closed in the topology $\mathcal{T}_1 \times \mathcal{T}_2$, then for any infinite family $\mathcal{M} = \{T_\alpha\}_{\alpha \in I}$ of commuting nonexpansive self-mappings of C the set $Fix(\mathcal{M}) = \bigcap_{\alpha \in I} Fix(T_\alpha)$ of common fixed points of $\mathcal{M}$ is a nonempty nonexpansive retract of C.*

Proof. As in the previous proof we consider the set J of all finite subsets of I. To each $j \in J$ there corresponds a subset $\{\alpha_1, ..., \alpha_m\}$, and by Theorems 4.1 and 4.2, there exists a nonexpansive retraction $r_j = (r_{1j}, r_{2j}) : C \to \bigcap_{i=1}^{m} Fix(T_{\alpha_i})$. Once more the set J is directed by inclusion and can be considered as a net itself. Let $(j_\beta)_{\beta \in J'}$ be an ultranet in J. Choose $x \in C$. For this x consider the ultranet $\{r_{j_\beta}(x)\}_{\beta \in J'}$. By the $\mathcal{T}_1 \times \mathcal{T}_2$-compactness of C this ultranet tends in $\mathcal{T}_1 \times \mathcal{T}_2$ to $y(x) = (y_1(x), y_2(x)) \in C$. Now we use the asymptotic center method. The asymptotic center of $\{r_{j_\beta}(x)\}_{\beta \in J'}$ with respect to C is a nonempty, closed in norm and convex subset of C. Next, it is easy to note that if $j \in J, \beta \in J'$ and $j \leq j_\beta$, then

$$r_j\left(r_{j_\beta}(x)\right) = r_{j_\beta}(x)$$

and hence the inequality

$$\begin{aligned} r\left(r_j(z), \{r_{j_\beta}(x)\}_{\beta \in J'}\right) &= \limsup_{\beta} \|r_j(z) - r_{j_\beta}(x)\| \\ &\leq \limsup_{\beta} \|z - r_{j_\beta}(x)\| \\ &= r\left(z, \{r_{j_\beta}(x)\}_{\beta \in J'}\right) \end{aligned}$$

is valid for each $j \in J$ and each $z \in C$. This implies that the asymptotic center $A\left(C, \{r_{j_\beta}(x)\}_{\beta \in J'}\right)$ is r_j-invariant for each $j \in J$. Moreover, by the Opial property for nets, the asymptotic center $A\left(C, \{r_{j_\beta}(x)\}_{\beta \in J'}\right)$ is equal either to the Cartesian product

$$\{y_1(x)\} \times D_2(x)$$

or to the Cartesian product

$$D_1 \times \{y_2(x)\},$$

where D_1 and D_2 are nonempty, convex and norm- closed subsets of C. This procedure defines the mapping $y(\cdot) : C \to C$ which is nonexpansive (by lower semicontinuity of the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ in the topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively). Now, we repeat the above reasoning. Namely, we consider the ultranet $\{r_{j_\beta}(y(x))\}_{\beta \in J'}$ and asymptotic center $A(C, \{r_{j_\beta}(y(x))\}_{\beta \in J'})$ of this net with respect to the set C. The ultranet $\{r_{j_\beta}(y(x))\}_{\beta \in J'}$ tends to $w(x) = (w_1(x), w_2(x)) = (y_1(x), w_2(x)) \in C$ ($w(x) = (w_1(x), w_2(x)) = (w_1(x), y_2(x)) \in C$ in the second case) in the topology $\mathcal{T}_1 \times \mathcal{T}_2$. Again, by the Opial property for nets, we obtain $w(x) \in A(C, \{r_{j_\beta}(y(x))\}_{\beta \in J'})$ and next

$$\begin{aligned} r\left(r_j(z), \{r_{j_\beta}(y(x))\}_{\beta \in J'}\right) &= \limsup_{\beta} \|r_j(z) - r_{j_\beta}(y(x))\| \\ &\leq \limsup_{\beta} \|z - r_{j_\beta}(y(x))\| \\ &= r\left(z, \{r_{j_\beta}(y(x))\}_{\beta \in J'}\right) \end{aligned}$$

is valid for each $j \in J$ and each $z \in C$. This implies that the asymptotic center $A\left(C, \{r_{j_\beta}(y(x))\}_{\beta \in J'}\right)$ is r_j-invariant for each $j \in J$. Likewise, we can prove that the mapping $w(\cdot) : C \to C$ is nonexpansive. Now, it is easy to observe that we have

$$\begin{aligned} A\left(C, \{r_{j_\beta}(y(x))\}_{\beta \in J'}\right) &= (B(w_1(x), r(C, \{G(x, n_\alpha)\}_{\alpha \in I})) \times \{w_2(x)\}) \cap C \\ &= B_1(x) \times \{w_2(x)\} \end{aligned}$$

in the first case and

$$\begin{aligned} A\left(C,\{r_{j_\beta}(y(x))\}_{\beta\in J'}\right) &= \left(\{w_1(x)\}\times(B(w_2(x),r(C,\{r_{j_\beta}(y(x))\}_{\beta\in J'})\right)\cap C \\ &= \{w_1(x)\}\times B_2(x) \end{aligned}$$

in the second case. The set $B_1(x)\subset X_1$ ($B_2(x)\subset X_2$) is nonempty, convex, compact in $\mathcal{T}_1$ ($\mathcal{T}_2$) and $r_{1j}(\cdot,w_2(x))$-invariant ($r_{2j}(w_1(x),\cdot)$-invariant) for each $j\in J$. Additionally, we have $r_{2j}(B_1(x)\times\{w_2(x)\})=\{w_2(x)\}$ ($r_{1j}(\{w_1(x)\}\times B_2(x))=\{w_1(x)\}$) for each $j\in J$. Once more we take the ultranet $\{r_{j_\beta}(w(x))\}_{\beta\in J'}$ which tends (in the topology $\mathcal{T}_1\times\mathcal{T}_2$) to

$$\begin{aligned} R(x) &= (R_1(x),R_2(x)) = (R_1(x),w_2(x)) \in A(C,\{r_{j_\beta}(w(x))\}_{\beta\in J'}) \\ &= B_1(x)\times\{w_2(x)\} \end{aligned}$$

in the first case and to

$$\begin{aligned} R(x) &= (R_1(x),R_2(x)) = (w_1(x),r_2(x)) \in A(C,\{r_{j_\beta}(w(x))\}_{\beta\in J'}) \\ &= \{w_2(x)\}\times B_2(x) \end{aligned}$$

in the second case. By the Opial property for nets we obtain, respectively,

$$\begin{aligned} \{R(x)\} &= A(A(C,\{r_{j_\beta}(y(x))\}_{\beta\in J'}),\{r_{j_\beta}(w(x))\}_{\beta\in J'}) \\ &= A(B_1(x)\times\{w_2(x)\},\{r_{j_\beta}(y(x))\}_{\beta\in J'}) \end{aligned}$$

and

$$\begin{aligned} \{R(x)\} &= A(A(C,\{r_{j_\beta}(y(x))\}_{\beta\in J'}),\{r_{j_\beta}(w(x))\}_{\beta\in J'}) \\ &= A(\{w_1(x)\}\times B_2(x),\{r_{j_\beta}(y(x))\}_{\beta\in J'}). \end{aligned}$$

Now, since $A(A(C,\{r_{j_\beta}(y(x))\}_{\beta\in J'}),\{r_{j_\beta}(y(x))\}_{\beta\in J'})$ is r_j-invariant for each $j\in J$, we have $r_j(R(x))=R(x)$. It is obvious that if $x\in Fix(\mathcal{M})=\bigcap_{\alpha\in I}Fix(T_\alpha)$, then $r_j(R(x))=x$. As in the previous step we can prove that the mapping $R(\cdot):C\to C$ is nonexpansive. This completes the proof. □

References

[1] H. Brezis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[2] R. E. Bruck, *Properties of fixed point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.

[3] R. E. Bruck, T. Kuczumow and S. Reich, *Convergence of iterates of asymptotically nonexpansive mappings in Banach spaces with the uniform Opial property*, Colloq. Math. **65** (1993), 169–179.

[4] M. Budzyńska, W. Kaczor and M. Koter-Mórgowska, *Asymptotic normal structure, the semi-Opial property and fixed points*, Ann. Univ. Mariae Curie Skłodowska **50** (1996), 33–41.

[5] M. Budzyńska, W. Kaczor, M. Koter-Mórgowska and T. Kuczumow, *Asymptotic normal structure and the semi-Opial property*, Nonlinear Analysis **30** (1997), 3505–3515.

[6] M. Budzyńska, T. Kuczumow and M. Michalska, Γ-*Opial property*, Bull. Austr. Math. Soc. **73** (2006), 473–476.

[7] M. Budzyńska, T. Kuczumow and S. Reich, *Uniform asymptotic normal structure, the uniform semi-Opial property and fixed points of asymptotically regular uniformly Lipschitzian semigroups. Part I*, Abstr. Appl. Anal. **3** (1998), 133–151.

[8] T. Dalby and B. Sims, *Duality map characterisations for Opial conditions*, Bull. Austral. Math. Soc. **53** (1996), 413–417.

[9] J. M. Dye, T. Kuczumow, P.-K. Lin and S. Reich, *Convergence of unrestricted products of nonexpansive mappings in spaces with the Opial property*, Nonlinear Analysis **26** (1996), 767–773.

[10] M. Edelstein, *The construction of an asymptotic center with a fixed point property*, Bull. Amer. Math. Soc. **78** (1972), 206–208.

[11] R. Engelking, *Outline of General Topology*, Elsevier, Amsterdam, 1968.

[12] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[13] K. Goebel and T. Kuczumow, *Irregular convex sets with fixed-point property for nonexpansive mappings*, Colloq. Math. **40** (1978/79), 259–264.

[14] K. Goebel, T. Sękowski and A. Stachura, *Uniform convexity of the hyperbolic metric and fixed points of holomorphic mappings in the Hilbert ball*, Nonlinear Analysis **4** (1980), 1011–1021.

[15] J.-P. Gossez and E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. **40** (1972), 565–573.

[16] W. Kaczor, T. Kuczumow and M. Michalska, *Convergence of ergodic means of orbits of semigroups of nonexpansive mappings in sets with* Γ*-Opial property*, Nonlinear Analysis **67** (2007), 2122–2130.

[17] W. Kaczor and S. Prus, *Asymptotical smoothness and its applications*, Bull. Austral. Math. Soc. **66** (2002), 405–418.

[18] J. L. Kelley, *General Topology*, Springer, New York, 1975.

[19] M. A. Khamsi, *On uniform Opial condition and uniform Kadec-Klee property in Banach and metric spaces*, Nonlinear Analysis **26** (1996), 1733–1748.

[20] M. A. Khamsi and W. A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory, Pure and Applied Mathematics (New York)*, John Willey & Sons, Inc., New York, 2001.

[21] T. Kuczumow, *Opial's modulus and fixed points of semigroups of mappings*, Proc. Amer. Math Soc. **127** (1999), 2671–2678.

[22] T. Kuczumow and S. Reich, *Opial's property and James' quasi-reflexive space*, Comment. Math. Univ. Carolinae **35** (1994), 283–289.

[23] T. Kuczumow and S. Reich, *An application of Opial's modulus to the fixed point theory of semigroups of Lipschitzian mappings*, Ann. Univ. Mariae Curie Skłodowska **51** (1997), 185–192.

[24] T. Kuczumow, S. Reich and M. Schmidt, *A fixed point property of* l_1*-product spaces*, Proc. Amer. Math. Soc. **119** (1993), 457–463.

[25] T. Kuczumow, S. Reich and D. Shoikhet, *Fixed points of holomorphic mappings: a metric approach*, Handbook of Metric Fixed Point Theory (Eds. W. A. Kirk and B. Sims), Kluwer Academic Publishers, Dordrecht, 2001, pp. 437-515.

[26] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[27] S. Prus, *Geometrical background of metric fixed point theory*, Handbook of Metric Fixed Point Theory (Eds. W. A. Kirk and B. Sims), Kluwer Acad. Publ., Dordrecht, 2001, 93-132.

[28] B. Sims, *A support map characterization of the Opial conditions*, Miniconference on Linear Analysis and Function Spaces (Canberra, 1984), Proc. Centre Math. Anal. Austral. Nat. Univ., 9, Austral. Nat. Univ., Canberra, 1985, pp. 259–264.

[29] T. Suzuki, *Some remarks on the set of common fixed points of one-parameter semigroups of nonexpansive mappings in Banach spaces with the Opial property*, Nonlinear Anal. **58** (2004), 441–458.

[30] D. Van Dulst, *Equivalent norms and the fixed point property for nonexpansive mappings*, J. London Math. Soc. 25 (1982), 139–144.

[31] A. Wiśnicki, *On the structure of fixed-point sets of asymptotically regular semigroups*, J. Math. Anal. Appl. **393** (2012), 177–184.

Manuscript received April 12, 2013
revised July 29, 2013

Monika Budzyńska
Instytut Matematyki UMCS, 20-031 Lublin, Poland
E-mail address: monikab1@hektor.umcs.lublin.pl

Tadeusz Kuczumow
Instytut Matematyki UMCS, 20-031 Lublin, Poland
E-mail address: tadek@hektor.umcs.lublin.pl

Małgorzata Michalska
Instytut Matematyki UMCS, 20-031 Lublin, Poland
E-mail address: daisy@hektor.umcs.lublin.pl