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CONDITIONS FOR ZERO DUALITY GAP IN CONVEX PROGRAMMING

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Abstract. We introduce and study a new dual condition which characterizes zero duality gap in nonsmooth convex optimization. We prove that our condition is less restrictive than all existing constraint qualifications, including the closed epigraph condition. Our dual condition was inspired by, and is less restrictive than, the so-called Bertsekas' condition for monotropic programming problems. We give several corollaries of our result and special cases as applications. We pay special attention to the polyhedral and sublinear cases, and their implications in convex optimization.

1. Introduction

Duality theory establishes an interplay between an optimization problem, called the *primal*, and another optimization problem, called the *dual*. A main target of this approach is the establishment of the so-called *zero duality gap*, which means that the optimal values of primal and dual problems coincide. Not all convex problems enjoy the zero duality gap property, and this has motivated the quest for assumptions on the primal problem which ensure zero duality gap (see [29] and references therein).

Recently Bertsekas considered such an assumption for a specific convex optimization problem, called the *extended monotropic programming problem*, the origin of which goes back to Rockafellar (see [24, 25]). Following Boţ and Csetnek [6], we study this problem in the following setting. Let $\{X_i\}_{i=1}^m$ be separated locally convex spaces and let $f_i : X_i \to \,]-\infty, +\infty]$ be proper lower semicontinuous and convex for every $i \in \{1, 2, \ldots, m\}$. Consider the minimization problem

$$(P) \qquad p := \inf \Big(\sum_{i=1}^{m} f_i(x_i) \Big) \quad \text{subject to}\, (x_1, \ldots, x_m) \in S,$$

where $S \subseteq X_1 \times X_2 \times \cdots X_m$ is a linear closed subspace. The dual problem is given as follows:

$$(D) \qquad d := \sup \Big(\sum_{i=1}^{m} -f_i^*(x_i^*) \Big) \quad \text{subject to}\, (x_1^*, \ldots, x_m^*) \in S^{\perp}.$$

We note that formulation (P) includes any general convex optimization problem. Indeed, for X a separated locally convex space, and $f : X \to \,]-\infty, +\infty]$ a proper

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lower semicontinuous and convex function, consider the problem

$$(CP) \qquad \inf f(x) \quad \text{subject to}\, x \in C,$$

where C is a closed and convex set. Problem (CP) can be reformulated as

$$\inf \big\{ f(x_1) + \iota_C(x_2) \big\} \quad \text{subject to}\, (x_1, x_2) \in S = \{(y_1, y_2) \in X \times X \,:\, y_1 = y_2\},$$

where ι_C is the indicator function of C.

Denote by $v(P)$ and $v(D)$ the optimal values of (P) and (D), respectively. In the finite dimensional setting, Bertsekas proved in [3, Proposition 4.1] that a *zero duality gap* holds for problems (P) and (D) (i.e., $p = v(P) = v(D) = d$) under the following condition:

$$N_S(x) + \big(\partial_\varepsilon f_1(x_1), \ldots, \partial_\varepsilon f_m(x_m)\big) \quad \text{is closed}$$
$$\text{for every } \varepsilon > 0,\ (x_1, \ldots, x_m) \in S \text{ and } x_i \in \operatorname{dom} f_i,\ \forall i \in \{1, 2, \ldots, m\},$$

where the sets $\partial_\varepsilon f_i(x_i)$ are the epsilon-subdifferentials of the f_i at x_i (see (2.2) for the definition). In [6, Theorem 3.2], Boţ and Csetnek extended this result to the setting of separated locally convex spaces.

Burachik and Majeed [14] presented a zero duality gap property for a monotropic programming problem in which the subspace constraint S in (P) is replaced by a closed cone C, and the orthogonal subspace $S^\perp$ in (D) is replaced by the dual cone $C^* := \{x^* \mid \inf_{c \in C} \langle x^*, C\rangle \geq 0\}$. Defining $g_i : X_1 \times X_2 \times \cdots \times X_m \to \,]-\infty, +\infty]$ by $g_i(x_1, \ldots, x_m) := f_i(x_i)$, we have

$$\begin{aligned}
(P) \qquad p &= \inf \Big(\sum_{i=1}^m f_i(x_i) \Big) \quad \text{subject to}\, (x_1, \ldots, x_m) \in C \\
&= \inf \Big(\iota_C(x) + \sum_{i=1}^m g_i(x) \Big) \\
(D) \qquad d &= \sup_{(x_1^*, \ldots, x_m^*) \in C^*} \sum_{i=1}^m -f_i^*(x_i^*),
\end{aligned}$$

where $C \subseteq X_1 \times X_2 \times \cdots \times X_m$ is a closed convex cone. In [14, Theorem 3.6], Burachik and Majeed proved that

$$\text{(1.1)} \qquad \text{if } \partial_\epsilon \iota_C(x) + \sum_{i=1}^m \partial_\varepsilon g_i(x) \text{ is weak}^* \text{ closed for every } x \in C \cap \Big(\bigcap_{i=1}^m \operatorname{dom} g_i \Big),$$

then $v(p) = v(D)$. Note that $\partial_\epsilon \iota_C(x) + \sum_{i=1}^m \partial_\varepsilon g_i(x) = \partial_\epsilon \iota_C(x) + \big(\partial_\varepsilon f_1(x_1), \ldots, \partial_\varepsilon f_m(x_m)\big)$. Thence, Burachik and Majeed's result extends Boţ and Csetnek's result and Bertsekas' result to the case of cone constraints. From now on, we focus on a more general form of condition (1.1), namely

$$\text{(1.2)} \qquad \sum_{i=1}^m \partial_\varepsilon f_i(x) \text{ is weak}^* \text{ closed},$$

where $f_i : X \to \,]-\infty, +\infty]$ is a proper lower semicontinuous and convex function for all $i = 1, \ldots, m$. We will refer to (1.2) as *the Bertsekas Constraint Qualification.*

In none of these results, however, is there a direct link between (1.2) and the zero duality gap property. One of the aims of this paper is to establish such a link precisely.

Another constraint qualification is the so-called *closed epigraph condition*, which was first introduced by Burachik and Jeyakumar in [11, Theorem 1] (see also [9, 20]). This condition is stated as

$$\text{(1.3)} \qquad \operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^* \text{ is weak}^* \text{ closed in the topology } \omega(X^*, X) \times \mathbb{R}.$$

Condition (1.3) does not imply (1.2). This was recently shown in [14, Example 3.1], in which (1.2) (and hence zero duality gap) holds, while (1.3) does not.

We recall from [19, Proposition 6.7.3] the following characterization of the zero duality gap property for (P) and (D), which uses the *infimal convolution* (see (2.3) for its definition) of the conjugate functions f_i^*.

$$(P) \qquad p = \inf\Big(\sum_{i=1}^m f_i(x)\Big) = -\Big(\sum_{i=1}^m f_i\Big)^*(0)$$

$$(D) \qquad d = -\,(f_1^* \square \cdots \square f_m^*)\,(0).$$

Hence, zero duality gap is tantamount to the equality

$$\Big(\sum_{i=1}^m f_i\Big)^*(0) = \big(f_1^* \square \cdots \square f_m^*\big)(0).$$

In our main result (Theorem 3.2 below), we introduce a new closedness property, stated as follows. There exists $K > 0$ such for every $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$ and every $\varepsilon > 0$,

$$\text{(1.4)} \qquad \overline{\Big[\sum_{i=1}^m \partial_\varepsilon f_i(x)\Big]}^{\mathrm{w}^*} \subseteq \sum_{i=1}^m \partial_{K\varepsilon} f_i(x).$$

Theorem 3.2 below proves that this property is equivalent to

$$\text{(1.5)} \qquad \Big(\sum_{i=1}^m f_i\Big)^*(x^*) = \big(f_1^* \square \cdots \square f_m^*\big)(x^*), \text{ for all } x^* \in X^*.$$

Condition (1.4) is easily implied by (1.1), since the latter implies that (1.4) is true for the choice $K = 1$. Hence, Theorem 3.2 shows exactly how and why (1.1) implies a zero duality gap. Moreover, in view of [11, Theorem 1], we see that our new condition (1.4) is strictly less restrictive than the closed epigraph condition. Indeed, the latter implies not only (1.5) but also exactness of the infimal convolution everywhere. Condition(1.5) with exactness is equivalent to (1.3). Condition (1.3), in turn, is less restrictive than the interiority-type conditions.

In the present paper, we focus on the following kind of interiority condition:

$$\text{(1.6)} \qquad \operatorname{dom} f_1 \cap \Big(\bigcap_{i=2}^m \operatorname{int}\operatorname{dom} f_i\Big) \neq \varnothing.$$

In summary, we have

$$\begin{array}{ccccc} & & \textit{Closed Epigraph Condition}\,(1.3) & & \\ & \nearrow & & \not\nwarrow \quad \searrow & \\ \textit{Interiority Condition}\,(1.6) & & \Downarrow ? \quad \not\Uparrow & & (1.4) \Longleftrightarrow (1.5) \\ & \searrow & & \nearrow & \\ & & \textit{Bertsekas Constraint Qualification}\,(1.2) & & \end{array}$$

Example 3.1 in [14] allows us to assert that the Bertsekas Constraint Qualification is not more restrictive than the Closed Epigraph Condition. This example also shows that our condition (1.4) does not imply the closed epigraph condition. It is still an open question whether a more precise relationship can be established between the closed epigraph condition and Bertsekas Constraint Qualification. The arrow linking (1.6) to (1.3) has been established by Zălinescu in [30, 31]. All other arrows are, as far as we know, new, and are established by us in this paper. Some clarification is in order regarding the arrow from (1.6) to the *Bertsekas Constraint Qualification*(1.2). It is clear that for every $x_0 \in \operatorname{dom} f_1 \cap \left(\bigcap_{i=2}^m \operatorname{int} \operatorname{dom} f_i\right)$, the set $\sum_{i=1}^m \partial_\varepsilon f_i(x_0)$ is weak* closed. Indeed, this is true because the latter set is the sum of a weak* compact set and a weak* closed set. Our Lemma 4.2 establishes that, under assumption (1.6), the set $\sum_{i=1}^m \partial_\varepsilon f_i(x)$ is weak* closed *for every point* $x \in \left(\bigcap_{i=1}^m \operatorname{dom} f_i\right)$.

A well-known result, which is not easily found in the literature, is the equivalence between (1.3) and the equality (1.5) with exactness of the infimal convolution everywehere in X^*. For convenience and possible future use, we have included the proof of this equivalence in the present paper (see Proposition 3.11).

The layout of our paper is as follows. The next section contains the necessary preliminary material. Section 3 contains our main result, and gives its relation with the Bertsekas Constraint Qualification (1.2), with the closed epigraph condition (1.3), and with the interiority conditions (1.6). Still in this section we establish stronger results for the important special case in which all f_is are sublinear. We finish this section by showing that our closedness condition allows for a simplification of the well-known Hiriart-Urruty and Phelps formula for the subdifferential of the sum of convex functions. In Section 4 we show that (generalized) interiority conditions imply (1.2), as well as (1.3). We also provide some additional consequences of Corollary 4.3, including various forms of Rockafellar's Fenchel duality result. At the end of Section 4 we establish stronger results for the case involving polyhedral functions. We end the paper with some conclusions and open questions.

2. Preliminaries

Let I be a directed set with a partial order $\preceq$. A subset J of I is said to be *terminal* if there exists $j_0 \in I$ such that every successor $k \succeq j_0$ verifies $k \in J$. We say that a net $\{s_\alpha\}_{\alpha \in I} \subseteq \mathbb{R}$ is *eventually bounded* if there exists a terminal set J and $R > 0$ such that $|s_\alpha| \leq R$ for every $\alpha \in J$.

We assume throughout that X is a separated (i.e., Hausdorff) locally convex topological vector space and X^* is its continuous dual endowed with the weak* topology $\omega(X^*, X)$. Given a subset C of X, $\operatorname{int} C$ is the *interior* of C. We next recall standard notions from convex analysis, which can be found, e.g., in [2, 5, 10, 21, 23, 26, 31]. For the set $D \subseteq X^*$, $\overline{D}^{\mathrm{w}^*}$ is the weak* closure of D. The *indicator function* of C, written as ι_C, is defined at $x \in X$ by

$$\iota_C(x) := \begin{cases} 0, & \text{if } x \in C; \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

The *normal cone* operator of C at x is defined by $N_C(x) := \big\{x^* \in X^* \mid \sup_{c \in C}\langle c - x, x^*\rangle \leq 0\big\}$, if $x \in C$; and $N_C(x) := \varnothing$, if $x \notin C$. If $S \subseteq X$ is a subspace, we define $S^\perp$ by $S^\perp := \{z^* \in X^* \mid \langle z^*, s\rangle = 0, \quad \forall s \in S\}$. Let $f\colon X \to [-\infty, +\infty]$. Then $\operatorname{dom} f := f^{-1}[-\infty, +\infty[$ is the *domain* (or *effective domain*) of f, and $f^*\colon X^* \to [-\infty, +\infty] : x^* \mapsto \sup_{x \in X}\{\langle x, x^*\rangle - f(x)\}$ is the *Fenchel conjugate* of f. The *epigraph* of f is $\operatorname{epi} f := \big\{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\big\}$.

The *lower semicontinuous hull* of f is denoted by $\overline{f}$. We say f is proper if $\operatorname{dom} f \neq \varnothing$ and $f > -\infty$. Given a function f, the *subdifferential* of f is the point-to-set mapping $\partial f\colon X \rightrightarrows X^*$ defined by

$$\partial f(x) := \begin{cases} \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \leq f(y)\} & \text{if } f(x) \in \mathbb{R}; \\ \varnothing & \text{otherwise.} \end{cases}$$

Given $\varepsilon \geq 0$, the $\varepsilon-$*subdifferential* of f is the point-to-set mapping $\partial_\varepsilon f\colon X \rightrightarrows X^*$ defined by

$$\partial_\varepsilon f(x) := \begin{cases} \{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \leq f(y) + \varepsilon\} & \text{if } f(x) \in \mathbb{R}; \\ \varnothing & \text{otherwise.} \end{cases} \tag{2.2}$$

Thus, if f is not proper, then $\partial_\varepsilon f(x) = \varnothing$ for every $\varepsilon \geq 0$ and $x \in X$. Note also that if f is convex and there exists $x_0 \in X$ such that $\overline{f}(x_0) = -\infty$, then $\overline{f}(x) = -\infty, \forall x \in \operatorname{dom} \overline{f}$ (see [13, Proposition 2.4] or [16, page 867]).

Let $f : X \to\ ]-\infty, +\infty]$. We say f is a *sublinear* function if $f(x+y) \leq f(x)+f(y)$, $f(0) = 0$, and $f(tx) = tf(x)$ for every $x, y \in \operatorname{dom} f$ and $t \geq 0$.

Let Z be a separated locally convex space and let $m \in \mathbb{N}$. For a family of functions $\psi_1, \ldots, \psi_m$ such that $\psi_i : Z \to [-\infty, +\infty]$ for all $i = 1, \ldots, m$, we define its *infimal convolution* as the function $(\psi_1 \square \cdots \square \psi_m) : Z \to [-\infty, +\infty]$ as

$$\big(\psi_1 \square \cdots \square \psi_m\big)z = \inf_{\sum_{i=1}^m z_i = z} \big\{\psi(z_1) + \cdots + \psi_m(z_m)\big\}. \tag{2.3}$$

We denote by $\rightharpoonup_{\mathrm{w}^*}$ the weak* convergence of nets in X^*.

3. Our main results

The following formula will be important in the proof of our main result.

Fact 3.1. (See [31, Corollary 2.6.7] or [6, Theorem 3.1].) *Let $f, g : X \to\ ]-\infty, +\infty]$ be proper lower semicontinuous and convex. Then for every $x \in X$ and $\varepsilon \geq 0$,*

$$\partial_\varepsilon(f + g)(x) = \bigcap_{\eta > 0} \overline{\Big[\bigcup_{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta} \big(\partial_{\varepsilon_1} f(x) + \partial_{\varepsilon_2} g(x)\big)\Big]}^{\mathrm{w}^*}.$$

We now come to our main result. The proof in part follows that of [6, Theorem 3.2].

Theorem 3.2. *Let $m \in \mathbb{N}$, and $f_i : X \to]-\infty, +\infty]$ be proper convex with $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$, where $i \in \{1, 2, \ldots, m\}$. Suppose that $\overline{f_i} = f_i$ on $\bigcap_{i=1}^m \operatorname{dom} \overline{f_i}$. Then the following four conditions are equivalent.*

(i) *There exists $K > 0$ such that for every $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$, and every $\varepsilon > 0$,*

$$\overline{\left[\sum_{i=1}^m \partial_\varepsilon f_i(x)\right]}^{\mathrm{w}^*} \subseteq \sum_{i=1}^m \partial_{K\varepsilon} f_i(x).$$

(ii) $\left(\sum_{i=1}^m f_i\right)^* = f_1^* \square \cdots \square f_m^*$ *in X^*.*

(iii) $f_1^* \square \cdots \square f_m^*$ *is weak* lower semicontinuous.*

(iv) *For every $x \in X$ and $\varepsilon \geq 0$,*

$$\partial_\varepsilon (f_1 + \cdots + f_m)(x) = \bigcap_{\eta > 0} \Big[\bigcup_{\varepsilon_i \geq 0, \sum_{i=1}^m \varepsilon_i = \varepsilon + \eta} \big(\partial_{\varepsilon_1} f_1(x) + \cdots + \partial_{\varepsilon_m} f_m(x)\big)\Big].$$

Proof. First we show that our basic assumptions imply that $\overline{f_i}$ is proper for every $i \in \{1, 2, \ldots, m\}$. Let $i \in \{1, 2, \ldots, m\}$.

Since $\varnothing \neq \left(\bigcap_{j=1}^m \operatorname{dom} f_j\right) \subseteq \left(\bigcap_{j=1}^m \operatorname{dom} \overline{f_j}\right)$, then $\bigcap_{j=1}^m \operatorname{dom} \overline{f_j} \neq \varnothing$. Let $x_0 \in \bigcap_{i=j}^m \operatorname{dom} \overline{f_j}$. Suppose to the contrary that $\overline{f_i}$ is not proper and thus there exists $y_0 \in X$ such that $\overline{f_i}(y_0) = -\infty$. Then by [13, Proposition 2.4], $\overline{f_i}(x_0) = -\infty$. By the assumption, $\overline{f_i}(x_0) = f_i(x_0) > -\infty$, which is a contradiction. Hence $\overline{f_i}$ is proper.

(i)⇒(ii): Let $x^* \in X^*$. Clearly, we have $\left(f_1^* \square \cdots \square f_m^*\right)(x^*) \geq \left(\sum_{i=1}^m f_i\right)^*(x^*)$. It suffices to show that

$$\left(\sum_{i=1}^m f_i\right)^*(x^*) \geq \left(f_1^* \square \cdots \square f_m^*\right)(x^*). \tag{3.1}$$

First we show that

$$\sum_{i=1}^m \overline{f_i}(y) \geq \sum_{i=1}^m f_i(y), \quad \forall y \in X. \tag{3.2}$$

Indeed, let $y \in X$. If $y \notin \bigcap_{i=1}^m \operatorname{dom} \overline{f_i}$. Clearly, (3.2) holds. Now assume that $y \in \bigcap_{i=1}^m \operatorname{dom} \overline{f_i}$. By our assumption $\overline{f_i}(y) = f_i(y)$, we conclude that (3.2) holds. Combining both cases, we conclude that (3.2) holds everywhere.

Since $\sum_{i=1}^m \overline{f_i} \leq \sum_{i=1}^m f_i$, (3.2) implies that

$$\sum_{i=1}^m \overline{f_i} = \sum_{i=1}^m f_i. \tag{3.3}$$

Taking the lower semicontinuous hull in the equality above, we have

$$\sum_{i=1}^m \overline{f_i} = \sum_{i=1}^m f_i = \overline{f_1 + \cdots + f_m}. \tag{3.4}$$

Clearly, if $(\sum_{i=1}^m f_i)^*(x^*) = +\infty$, then (3.1) holds. Now assume that $(\sum_{i=1}^m f_i)^*(x^*) < +\infty$. Then we have $(\sum_{i=1}^m f_i)^*(x^*) \in \mathbb{R}$ and thus $x^* \in$

$\operatorname{dom}(\sum_{i=1}^m f_i)^*$. Since $(\sum_{i=1}^m f_i)^*$ is lower semicontinuous, given $\varepsilon > 0$, there exists $x \in X$ such that $x \in \partial_\varepsilon(\sum_{i=1}^m f_i)^*(x^*)$. Then

$$\Big(\sum_{i=1}^m f_i\Big)^*(x^*) + \overline{\Big(\sum_{i=1}^m f_i\Big)}(x) = \Big(\sum_{i=1}^m f_i\Big)^*(x^*) + \Big(\sum_{i=1}^m f_i\Big)^{**}(x) \leq \langle x, x^*\rangle + \varepsilon.$$

By (3.4), we have

$$\Big(\sum_{i=1}^m f_i\Big)^*(x^*) + \Big(\sum_{i=1}^m f_i\Big)(x) = \Big(\sum_{i=1}^m \overline{f_i}\Big)^*(x^*) + \Big(\sum_{i=1}^m \overline{f_i}\Big)(x) \leq \langle x, x^*\rangle + \varepsilon.$$

Hence

$$x^* \in \partial_\varepsilon\Big(\sum_{i=1}^m f_i\Big)(x) \quad \text{and} \quad x^* \in \partial_\varepsilon\Big(\sum_{i=1}^m \overline{f_i}\Big)(x). \tag{3.5}$$

Next, we claim that there exists $K > 0$ such that

$$x^* \in \sum_{i=1}^m \partial_{Km\varepsilon} f_i(x). \tag{3.6}$$

Set $f := \overline{f_1}$, $g := (\sum_{i=2}^m \overline{f_i})$, and $\eta = \varepsilon$ in Fact 3.1, and use (3.5) to write

$$x^* \in \partial_\varepsilon(\sum_{i=1}^m \overline{f_i})(x) \Rightarrow x^* \in \overline{\Big[\partial_{2\varepsilon}\overline{f_1}(x) + \partial_{2\varepsilon}\Big(\sum_{i=2}^m \overline{f_i}\Big)(x)\Big]}^{w^*}.$$

We repeat the same idea with $f := \overline{f_2}$, $g := (\sum_{i=3}^m \overline{f_i})$ in Fact 3.1, and continue iteratively to obtain

$$\Rightarrow x^* \in \overline{\Big[\partial_{2\varepsilon}\overline{f_1}(x) + \overline{\partial_{3\varepsilon}\overline{f_2}(x) + \partial_{3\varepsilon}\Big(\sum_{i=3}^m \overline{f_i}\Big)(x)}^{w^*}\Big]}^{w^*}$$

$$\Rightarrow x^* \in \overline{\Big[\partial_{2\varepsilon}\overline{f_1}(x) + \partial_{3\varepsilon}\overline{f_2}(x) + \partial_{3\varepsilon}\Big(\sum_{i=3}^m \overline{f_i}\Big)(x)\Big]}^{w^*}$$

$\dots$

$$\Rightarrow x^* \in \overline{\Big[\partial_{2\varepsilon}\overline{f_1}(x) + \partial_{3\varepsilon}\overline{f_2}(x) + \cdots + \partial_{m\varepsilon}\overline{f_m}(x)\Big]}^{w^*}$$

$$\Rightarrow x^* \in \overline{\Big[\partial_{2\varepsilon}f_1(x) + \partial_{3\varepsilon}f_2(x) + \cdots + \partial_{m\varepsilon}f_m(x)\Big]}^{w^*} \quad \text{(by (3.5) and } \overline{f_i}(x) = f_i(x), \forall i)$$

$$\Rightarrow x^* \in \overline{\Big[\partial_{m\varepsilon}f_1(x) + \partial_{m\varepsilon}f_2(x) + \cdots + \partial_{m\varepsilon}f_m(x)\Big]}^{w^*}.$$

By assumption (i), the last inclusion implies that there exists $K > 0$ such that

$$x^* \in \partial_{Km\varepsilon}f_1(x) + \partial_{Km\varepsilon}f_2(x) + \cdots + \partial_{Km\varepsilon}f_m(x)$$

Hence (3.6) holds. Thus, there exists $y_i^* \in \partial_{K\,m\varepsilon}f_i(x)$ such that $x^* = \sum_{i=1}^m y_i^*$ and

$$f_i^*(y_i^*) + f_i(x) \leq \langle x, y_i^*\rangle + Km\varepsilon, \quad \forall i \in \{1, 2, \dots, m\}.$$

Thus,

$$(f_1^*\Box\cdots\Box f_m^*)(x^*) \leq \sum_{i=1}^m f_i^*(y_i^*) \leq -\sum_{i=1}^m f_i(x) + \langle x, x^*\rangle + Km^2\varepsilon$$
$$\leq \Big(\sum_{i=1}^m f_i\Big)^*(x^*) + Km^2\varepsilon.$$

Letting $\varepsilon \longrightarrow 0$ in the above inequality, we have

$$\Big(f_1^*\Box\cdots\Box f_m^*\Big)(x^*) \leq \Big(\sum_{i=1}^m f_i\Big)^*(x^*).$$

Hence (3.1) holds and so

$$\Big(\sum_{i=1}^m f_i\Big)^* = f_1^*\Box\cdots\Box f_m^*. \tag{3.7}$$

(ii)⇒(iii): This clearly follows from the lower semicontinuity of $(\sum_{i=1}^m f_i)^*$.

(iii)⇒(i): Let $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$ and $\varepsilon > 0$, and $x^* \in \overline{[\sum_{i=1}^m \partial_\varepsilon f_i(x)]}^{\mathrm{w}^*}$. Then for each $i = 1, \ldots, m$ there exists a net $(x_{i,\alpha}^*)_{\alpha\in I}$ in $\partial_\varepsilon f_i(x)$ such that

$$\sum_{i=1}^m x_{i,\alpha}^* \rightharpoonup_{\mathrm{w}^*} x^*. \tag{3.8}$$

We have

$$f_i(x) + f_i^*(x_{i,\alpha}^*) \leq \langle x, x_{i,\alpha}^*\rangle + \varepsilon, \quad \forall i \in \{1, 2, \ldots, m\} \quad \forall \alpha \in I. \tag{3.9}$$

Thus

(3.10)
$$\sum_{i=1}^m f_i(x) + (f_1^*\Box\cdots\Box f_m^*)\Big(\sum_{i=1}^m x_{i,\alpha}^*\Big) \leq \sum_{i=1}^m f_i(x) + \sum_{i=1}^m f_i^*(x_{i,\alpha}^*) \leq \Big\langle x, \sum_{i=1}^m x_{i,\alpha}^*\Big\rangle + m\varepsilon,$$
$$\forall \alpha \in I.$$

Since $f_1^*\Box\cdots\Box f_m^*$ is weak* lower semicontinuous, it follows from (3.10) and (3.8) that

$$\sum_{i=1}^m f_i(x) + (f_1^*\Box\cdots\Box f_m^*)(x^*) \leq \langle x, x^*\rangle + m\varepsilon. \tag{3.11}$$

There exists $y_i^* \in X^*$ such that $\sum_{i=1}^m y_i^* = x^*$ and $\sum_{i=1}^m f_i^*(y_i^*) \leq (f_1^*\Box\cdots\Box f_m^*)(x^*) + \varepsilon$. Then by (3.11),

$$\sum_{i=1}^m f_i(x) + \sum_{i=1}^m f_i^*(y_i^*) \leq \langle x, x^*\rangle + (m+1)\varepsilon.$$

Thus, we have

$$y_i^* \in \partial_{(m+1)\varepsilon} f_i(x), \quad \forall i \in \{1, 2, \ldots, m\}.$$

Hence

$$x^* = \sum_{i=1}^m y_i^* \in \sum_{i=1}^m \partial_{(m+1)\varepsilon} f_i(x),$$

and the statement in (i) holds for $K := (m+1)$.

(ii)⇒(iv): Let $x \in X$ and $\varepsilon \geq 0$.

We have

$$\begin{aligned}
&\bigcap_{\eta>0} \Big[\bigcup_{\varepsilon_i \geq 0,\, \sum_{i=1}^m \varepsilon_i = \varepsilon+\eta} \big(\partial_{\varepsilon_1} f_1(x) + \cdots + \partial_{\varepsilon_m} f_m(x)\big) \Big] \\
&\quad \subseteq \bigcap_{\eta>0} \Big[\bigcup_{\varepsilon_i \geq 0,\, \sum_{i=1}^m \varepsilon_i = \varepsilon+\eta} \partial_{\sum_i^m \varepsilon_i} (f_1 + \cdots + f_m)(x) \Big] \\
&\quad = \bigcap_{\eta>0} \partial_{\varepsilon+\eta}(f_1 + \cdots + f_m)(x) \\
&\quad = \partial_\varepsilon (f_1 + \cdots + f_m)(x).
\end{aligned}$$

Now we show the other inclusion:

(3.12)
$$\partial_\varepsilon (f_1 + \cdots + f_m)(x) \subseteq \Big(\bigcap_{\eta>0} \Big[\bigcup_{\varepsilon_i \geq 0,\, \sum_{i=1}^m \varepsilon_i = \varepsilon+\eta} \big(\partial_{\varepsilon_1} f_1(x) + \cdots + \partial_{\varepsilon_m} f_m(x)\big) \Big] \Big).$$

Let $x^* \in \partial_\varepsilon (f_1 + \cdots + f_m)(x)$. Then we have $\sum_{i=1}^m f_i(x) + (\sum f_i)^*(x^*) \leq \langle x, x^* \rangle + \varepsilon$. By (ii), we have

$$\sum_{i=1}^m f_i(x) + \big(f_1^* \square \cdots \square f_m^*\big)(x^*) \leq \langle x, x^* \rangle + \varepsilon. \tag{3.13}$$

Let $\eta > 0$. Then there exists $y_i^* \in X^*$ such that $\sum_{i=1}^m y_i^* = x^*$ and $\sum_{i=1}^m f_i^*(y_i^*) \leq \big(f_1^* \square \cdots \square f_m^*\big)(x^*) + \eta$. Then by (3.13),

$$\sum_{i=1}^m f_i(x) + \sum_{i=1}^m f_i^*(y_i^*) \leq \langle x, x^* \rangle + \varepsilon + \eta. \tag{3.14}$$

Set $\gamma_i := f_i(x) + f_i^*(y_i^*) - \langle x, y_i^* \rangle$. Then $\gamma_i \geq 0$ and $y_i^* \in \partial_{\gamma_i} f_i(x)$. By (3.14),

$$\langle x, x^* \rangle + \sum_{i=1}^m \gamma_i = \sum_{i=1}^m \big[\langle x, y_i^* \rangle + \gamma_i\big] \leq \langle x, x^* \rangle + \varepsilon + \eta. \tag{3.15}$$

Hence $\sum_{i=1}^m \gamma_i \leq \varepsilon + \eta$. Set $\varepsilon_1 := \varepsilon + \eta - \sum_{i=2}^m \gamma_i$ and $\varepsilon_i := \gamma_i$ for every $i = \{2, 3, \ldots, m\}$. Then $\varepsilon_1 \geq \gamma_1$ and we have

$$x^* = \sum_{i=1}^m y_i^* \in \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x).$$

Hence $x^* \in \bigcup_{\varepsilon_i \geq 0,\, \sum_{i=1}^m \varepsilon_i = \varepsilon+\eta} \big(\partial_{\varepsilon_i} f_i(x) + \cdots + \partial_{\varepsilon_m} f_m(x)\big)$ and therefore (3.12) holds.

(iv)⇒(i): Let $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$, $\varepsilon > 0$, and $x^* \in \overline{[\sum_{i=1}^m \partial_\varepsilon f_i(x)]}^{\mathrm{w}^*}$. Then for each $i = 1, \ldots, m$ there exists a net $(x^*_{i,\alpha})_{\alpha \in I}$ in $\partial_\varepsilon f_i(x)$ such that

$$\sum_{i=1}^m x^*_{i,\alpha} \rightharpoonup_{\mathrm{w}^*} x^*, \tag{3.16}$$

and this implies that

$$\sum_{i=1}^m x^*_{i,\alpha} \in \Big(\bigcap_{\eta>0} \Big[\bigcup_{\varepsilon_i \geq 0, \sum_{i=1}^m \varepsilon_i = m\varepsilon+\eta} \Big(\partial_{\varepsilon_1} f_1(x) + \cdots + \partial_{\varepsilon_m} f_m(x) \Big) \Big] \Big). \tag{3.17}$$

Assumption (iv) yields $\sum_{i=1}^m x^*_{i,\alpha} \in \partial_{m\varepsilon}(f_1 + \cdots + f_m)(x)$. Since $\partial_{m\varepsilon}(f_1 + \cdots + f_m)(x)$ is weak* closed, (3.16) shows that $x^* \in \partial_{m\varepsilon}(f_1 + \cdots + f_m)(x)$. Using (iv) again for $\eta = \varepsilon$, we conclude that $x^* \in \big(\partial_{(m+1)\varepsilon} f_i(x) + \cdots + \partial_{(m+1)\varepsilon} f_m(x)\big)$.

Therefore, statement (i) holds for $K := m + 1$. □

Remark 3.3. (a) We point out that the proof of Theorem 3.2(i) actually shows that $K = m+1$, and this constant is *independent of the functions* $f_1, \ldots, f_m$.

(b) Part (i) implies (ii) of Theorem 3.2 generalizes [3, Proposition 4.1], [6, Theorem 3.2] by Boţ and Csetnek, and [14, Theorem 3.6] by Burachik and Majeed.

(c) A result similar to Theorem 3.2(iii)⇔(iv) has been established in [7, Corollary 3.9] by Boţ and Grad.

An immediate corollary follows:

Corollary 3.4. *Let* $f, g : X \to\,]-\infty, +\infty]$ *be proper convex with* $\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$. *Suppose that* $\overline{f} = f$ *and* $\overline{g} = g$ *on* $\operatorname{dom} \overline{f} \cap \operatorname{dom} \overline{g}$. *Suppose also that for every* $x \in \operatorname{dom} f \cap \operatorname{dom} g$ *and* $\varepsilon > 0$,

$$\partial_\varepsilon f(x) + \partial_\varepsilon g(x) \quad \textit{is weak}^* \textit{ closed.}$$

Then $(f+g)^* = f^* \Box g^*$ *in* X^*. *Consequently,* $\inf(f+g) = \sup_{x^* \in X^*}\{-f^*(x^*) - g^*(-x^*)\}$.

Note that, for a linear subspace $S \subseteq X$, we have $\partial_\varepsilon \iota_S = S^\perp$. Taking this into account we derive the Bertsekas Constraint Qualification result from Theorem 3.2.

Corollary 3.5 (Bertsekas)**.** (See [3, Proposition 4.1].) *Let* $m \in \mathbb{N}$ *and suppose that* X_i *is a finite dimensional space, and let* $f_i : X_i \to\,]-\infty, +\infty]$ *be proper lower semicontinuous and convex, where* $i \in \{1, 2, \ldots, m\}$. *Let* S *be a linear subspace of* $X_1 \times X_2 \times \cdots \times X_m$ *with* $S \cap \big(\bigcap_{i=1}^m \operatorname{dom} f_i\big) \neq \varnothing$. *Define* $g_i : X_1 \times X_2 \times \cdots \times X_m \to\,]-\infty, +\infty]$ *by* $g_i(x_1, \ldots, x_m) := f_i(x_i)$. *Assume that for every* $x \in S \cap \big(\bigcap_{i=1}^m \operatorname{dom} f_i\big)$ *and for every* $\varepsilon > 0$ *we have that*

$$S^\perp + \sum_{i=1}^m \partial_\varepsilon g_i(x) \quad \textit{is closed.}$$

Then $v(P) = \inf_{x \in S}\{\sum_{i=1}^m f_i(x)\} = \sup_{x^* \in S^\perp}\{-\sum_{i=1}^m f_i^*(x^*)\} = v(D)$.

The following example which is due to [14, Example 3.1] and [9, Example, page 2798], shows that the infimal convolution in Corollary 3.4 is not always achieved (exact).

Example 3.6. Let $X = \mathbb{R}^2$, and $f := \iota_C, g := \iota_D$, where $C := \{(x, y) \in \mathbb{R}^2 \mid 2x + y^2 \leq 0\}$ and $D := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$. Then f and g are proper lower semicontinuous and convex with $\operatorname{dom} f \cap \operatorname{dom} g = \{(0, 0)\}$. For every $\varepsilon > 0$, $\partial_\varepsilon f(0, 0) + \partial_\varepsilon g(0, 0)$ is closed. Hence $(f + g)^* = f^*\Box g^*$. But $f^*\Box g^*$ is not exact everywhere and $\partial(f + g)(0) \neq \partial f(0) + \partial g(0)$. Consequently, $\operatorname{epi} f^* + \operatorname{epi} g^*$ is not closed in the topology $\omega(X^*, X) \times \mathbb{R}$.

Proof. Clearly, f and g are proper lower semicontinuous convex. Let $\varepsilon > 0$. Then by [14, Example 3.1]

$$\partial_\varepsilon f(0,0) = \bigcup_{u\geq 0} \left(u \times \left[-\sqrt{2\varepsilon u}, \sqrt{2\varepsilon u}\right]\right) \quad \text{and} \quad \partial_\varepsilon g(0,0) = \left]-\infty, 0\right] \times \{0\}. \tag{3.18}$$

Thus, $\partial_\varepsilon f(0,0)+\partial_\varepsilon g(0,0) = \mathbb{R}^2$ and then $\partial_\varepsilon f(0,0)+\partial_\varepsilon g(0,0)$ is closed. Corollary 3.4 implies that $(f + g)^* = f^*\Box g^*$. [9, Example, page 2798] shows that $(f^*\Box g^*)$ is not exact at $(1, 1)$ and $\partial(f + g)(0) \neq \partial f(0) + \partial g(0)$. By [11, 9], $\operatorname{epi} f^* + \operatorname{epi} g^*$ is not closed in the topology $\omega(X^*, X) \times \mathbb{R}$. □

The following result is classical, we state and prove it here for more convenient and clear future use.

Lemma 3.7 (Hiriart-Urruty). *Let $m \in \mathbb{N}$, and $f_i : X \to \left]-\infty, +\infty\right]$ be proper convex with $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$, where $i \in \{1, 2, \ldots, m\}$. Assume that $(\sum_{i=1}^m f_i)^* = f_1^*\Box \cdots \Box f_m^*$ in X^* and the infimal convolution is exact (attained) everywhere. Then*

$$\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m.$$

Proof. Let $x \in X$. We always have $\partial(f_1+f_2+\cdots+f_m)(x) \supseteq \partial f_1(x)+\cdots+\partial f_m(x)$. So it suffices to show that

$$\partial(f_1 + f_2 + \cdots + f_m)(x) \subseteq \partial f_1(x) + \cdots + \partial f_m(x). \tag{3.19}$$

Let $w^* \in \partial(f_1 + f_2 + \cdots + f_m)(x)$. Then

$$(f_1 + f_2 + \cdots + f_m)(x) + (f_1 + f_2 + \cdots + f_m)^*(w^*) = \langle x, w^*\rangle.$$

By the assumption, there exists $w_i^* \in X^*$ such that $\sum_{i=1}^m w_i^* = w^*$ and

$$f_1(x) + f_2(x) + \cdots + f_m(x) + f_1^*(w_1^*) + \cdots + f_m^*(w_m^*) = \langle x, w_1^* + \cdots + w_m^*\rangle.$$

Hence

$$w_i^* \in \partial f_i(w_i), \quad \forall i \in \{1, 2, \ldots, m\}.$$

Thus

$$w^* = \sum_{i=1}^m w_i^* \in \sum_{i=1}^m \partial f_i(w_i),$$

and (3.19) holds. □

A less immediate corollary is:

Corollary 3.8. (See [8, Theorem 3.5.8].) *Let $m \in \mathbb{N}$, and $f_i : X \to \,]-\infty,+\infty]$ be proper convex with $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$, where $i \in \{1,2,\ldots,m\}$. Suppose that $\overline{f_i} = f_i$ on $\bigcap_{i=1}^m \operatorname{dom} \overline{f_i}$. Assume that* $\operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$ *is closed in the topology $\omega(X^*,X)\times\mathbb{R}$.*

Then $(\sum_{i=1}^m f_i)^* = f_1^* \Box \cdots \Box f_m^*$ in X^* and the infimal convolution is exact (attained) everywhere. In consequence, we also have

$$\partial(f_1+f_2+\cdots+f_m) = \partial f_1 + \cdots + \partial f_m.$$

Proof. Let $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$, $x^* \in \overline{[\sum_{i=1}^m \partial_\varepsilon f_i(x)]}^{\mathrm{w}^*}$ and $\varepsilon > 0$. We will show that

$$x^* \in \sum_{i=1}^m \partial_{m\varepsilon} f_i(x). \tag{3.20}$$

The assumption on x^* implies that for each $i = 1,\ldots,m$ there exists $(x^*_{i,\alpha})_{\alpha\in I}$ in $\partial_\varepsilon f_i(x)$ such that

$$\sum_{i=1}^m x^*_{i,\alpha} \rightharpoonup_{\mathrm{w}^*} x^*. \tag{3.21}$$

We have

$$f_i^*(x^*_{i,\alpha}) \le -f_i(x) + \langle x, x^*_{i,\alpha}\rangle + \varepsilon, \quad \forall i \in \{1,2,\ldots,m\} \quad \forall \alpha \in I. \tag{3.22}$$

Thus $(x^*_{i,\alpha}, -f_i(x) + \langle x, x^*_{i,\alpha}\rangle + \varepsilon) \in \operatorname{epi} f_i^*, \forall i$ and hence

$$\Big(\sum_{i=1}^m x^*_{i,\alpha}, -\sum_{i=1}^m f_i(x) + \langle x, \sum_{i=1}^m x^*_{i,\alpha}\rangle + m\varepsilon\Big) \in \operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*. \tag{3.23}$$

Now $\operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$ is closed in the topology $\omega(X^*,X)\times\mathbb{R}$. Thus, by (3.21) and (3.23), we have

$$\Big(x^*, -\sum_{i=1}^m f_i(x) + \langle x, x^*\rangle + m\varepsilon\Big) \in \operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*. \tag{3.24}$$

Consequently, there exists $y_i^* \in X^*$ and $t_i \ge 0$ such that

$$x^* = \sum_{i=1}^m y_i^* \tag{3.25}$$

$$-\sum_{i=1}^m f_i(x) + \langle x, x^*\rangle + m\varepsilon = \sum_{i=1}^m (f^*(y_i^*) + t_i).$$

Hence

$$-\sum_{i=1}^m f_i(x) + \langle x, x^*\rangle + m\varepsilon \ge \sum_{i=1}^m f^*(y_i^*). \tag{3.26}$$

Then we have

$$y_i^* \in \partial_{m\varepsilon} f_i(x), \quad \forall i \in \{1,2,\ldots,m\}.$$

Thus by (3.25),

$$x^* \in \sum_{i=1}^{m} \partial_{m\varepsilon} f_i(x).$$

Hence (3.20) holds. Applying Theorem 3.2, part (i) implies (ii), we have

$$(\sum_{i=1}^{m} f_i)^* = f_1^* \Box \cdots \Box f_m^*. \tag{3.27}$$

Let $z^* \in X^*$. Next we will show that $(f_1^* \Box \cdots \Box f_m^*)(z^*)$ is achieved. If $z^* \notin \operatorname{dom}(\sum_{i=1}^{m} f_i)^*$, then $(f_1^* \Box \cdots \Box f_m^*)(x^*) = +\infty$ by (3.27) and hence $(f_1^* \Box \cdots \Box f_m^*)(z^*)$ is achieved.

Now suppose that $z^* \in \operatorname{dom}(\sum_{i=1}^{m} f_i)^*$ and then $(\sum_{i=1}^{m} f_i)^*(z^*) \in \mathbb{R}$. By (3.27), there exists $(z_{i,n}^*)_{n\in\mathbb{N}}$ such that $\sum_{i=1}^{m} z_{i,n}^* = z^*$ and

$$\Big(\sum_{i=1}^{m} f_i\Big)^*(z^*) \leq f_1^*(z_{1,n}^*) + f_2^*(z_{2,n}^*) + \cdots + f_m^*(z_{m,n}^*) \leq \Big(\sum_{i=1}^{m} f_i\Big)^*(z^*) + \frac{1}{n}.$$

Then we have

$$f_1^*(z_{1,n}^*) + f_2^*(z_{2,n}^*) + \cdots + f_m^*(z_{m,n}^*) \longrightarrow \Big(\sum_{i=1}^{m} f_i\Big)^*(z^*). \tag{3.28}$$

Since $\big(z^*, \sum_{i=1}^{m} f_i^*(z_{i,n}^*)\big) = \big(\sum_{i=1}^{m} z_{i,n}^*, \sum_{i=1}^{m} f_i^*(z_{i,n}^*)\big) \in \operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$ and $\operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$ is closed in the topology $\omega(X^*, X) \times \mathbb{R}$, (3.28) implies that

$$\Big(z^*, \Big(\sum_{i=1}^{m} f_i\Big)^*(z^*)\Big) \in \operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*.$$

Thus, there exists $v_i^* \in X^*$ such that $\sum_{i=1}^{m} v_i^* = z^*$ and

$$\Big(\sum_{i=1}^{m} f_i\Big)^*(z^*) \geq \sum_{i=1}^{m} f_i^*(v_i^*) \geq (f_1^* \Box \cdots \Box f_m^*)(z^*). \tag{3.29}$$

Since $(\sum_{i=1}^{m} f_i)^*(z^*) = (f_1^* \Box \cdots \Box f_m^*)(z^*)$ by (3.27), it follows from (3.29) that $(\sum_{i=1}^{m} f_i)^*(z^*) = \sum_{i=1}^{m} f_i^*(v_i^*)$. Hence $(f_1^* \Box \cdots \Box f_m^*)(z^*)$ is achieved.

The applying Lemma 3.7, we have $\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m$. □

When there are precisely two functions this reduces to:

Corollary 3.9 (Boţ and Wanka)**.** (See [9, Theorem 3.2].) *Let $f, g : X \to]-\infty, +\infty]$ be proper lower semicontinuous and convex with $\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$. Assume that $\operatorname{epi} f^* + \operatorname{epi} g^*$ is closed in the topology $\omega(X^*, X) \times \mathbb{R}$. Then $(f+g)^* = f^* \Box g^*$ in X^* and the infimal convolution is exact everywhere. In consequence, $\partial(f+g) = \partial f + \partial g$.*

Proof. Directly apply Corollary 3.8. □

Remark 3.10. In the setting of Banach space, Corollary 3.9 was first established by Burachik and Jeyakumar [11]. Example 3.6 shows that the equality $(f+g)^* = f^* \Box g^*$ is not a sufficient condition for $\operatorname{epi} f^* + \operatorname{epi} g^*$ to be closed.

The following result, stating the equivalence between the closed epigraph condition and condition (ii) in Theorem 3.2 *with exactness*, is well known but hard to track down.

Proposition 3.11. *Let $m \in \mathbb{N}$, and $f_i : X \to]-\infty, +\infty]$ be proper lower semicontinuous and convex with $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$, where $i \in \{1, 2, \ldots, m\}$. Then* $\operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$ *is closed in the topology $\omega(X^*, X) \times \mathbb{R}$ if and only if $(\sum_{i=1}^m f_i)^* = f_1^* \Box \cdots \Box f_m^*$ in X^* and the infimal convolution is exact.*

Proof. $\Rightarrow$: This follows directly from Corollary 3.10.

$\Leftarrow$: Assume now that $(\sum_{i=1}^m f_i)^* = f_1^* \Box \cdots \Box f_m^*$ in X^* and the infimal convolution is always exact. Note that this assumption implies that the function $f_1^* \Box \cdots \Box f_m^*$ is lower semicontinuous in X^*. Let $(w^*, r) \in X^* \times \mathbb{R}$ be in the closure of $\operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$ in the topology $\omega(X^*, X) \times \mathbb{R}$. We will show that $(w^*, r) \in \operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$. The assumption on (w^*, r) implies that there exist $(x_{i,\alpha}^*)_{\alpha \in I}$ in $\operatorname{dom} f_i^*$ and $(r_{i,\alpha})_{\alpha \in I}$ in $\mathbb{R}$ such that

$$w_\alpha^* := \sum_{i=1}^m x_{i,\alpha}^* \rightharpoonup_{\mathrm{w}^*} w^*, \quad f_i^*(x_{i,\alpha}^*) \leq r_{i,\alpha}, \forall\, i,\, \alpha \quad \text{and} \quad \sum_{i=1}^m r_{i,\alpha} \longrightarrow r. \tag{3.30}$$

Then

$$\big(f_1^* \Box \cdots \Box f_m^*\big)(w_\alpha^*) \leq \sum_{i=1}^m f_i^*(x_{i,\alpha}^*) \leq \sum_{i=1}^m r_{i,\alpha}. \tag{3.31}$$

Our assumption implies that $f_1^* \Box \cdots \Box f_m^*$ is lower semicontinuous, hence by taking limits in (3.31) and using (3.30) we obtain

$$\big(f_1^* \Box \cdots \Box f_m^*\big)(w^*) \leq r. \tag{3.32}$$

By assumption, $\big(f_1^* \Box \cdots \Box f_m^*\big)(w^*)$ is exact. Therefore there exists w_i^* such that $w^* = \sum_{i=1}^m w_i^*$ and $\big(f_1^* \Box \cdots \Box f_m^*\big)(w^*) = \sum_{i=1}^m f_i^*(w_i^*)$. The latter fact and (3.32) show that $(w^*, r) \in \operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$. ☐

We next dualize Corollary 3.8.

Corollary 3.12 (Dual conjugacy)**.** *Suppose that X is a reflexive Banach space. Let $m \in \mathbb{N}$, and $f_i : X \to]-\infty, +\infty]$ be proper lower semicontinuous and convex with $\bigcap_{i=1}^m \operatorname{dom} f_i^* \neq \varnothing$, where $i \in \{1, 2, \ldots, m\}$. Assume that* $\operatorname{epi} f_i + \cdots + \operatorname{epi} f_m$ *is closed in the weak topology $\omega(X, X^*) \times \mathbb{R}$.*

Then $(\sum_{i=1}^m f_i^*)^* = f_1 \Box \cdots \Box f_m$ in X and the infimal convolution is exact (attained) everywhere. In consequence, we also have

$$\partial(f_1^* + f_2^* + \cdots + f_m^*) = \partial f_1^* + \cdots + \partial f_m^*.$$

Proof. Apply Corollary 3.8 to the functions f_i^*. ☐

In a Banach space we can add a general interiority condition for closure.

Remark 3.13 (Transversality)**.** Suppose that X is a Banach space, and let f, g be defined as in Corollary 3.9. If $\bigcup_{\lambda > 0} \lambda\, [\operatorname{dom} f - \operatorname{dom} g]$ is a closed subspace, then the Attouch-Brezis theorem implies that $\operatorname{epi} f^* + \operatorname{epi} g^*$ is closed in the topology $\omega(X^*, X) \times \mathbb{R}$ [1, 27, 9, 11]. This result works also in a locally convex Fréchet space [4].

The following result shows that sublinearity rules out the pathology of Example 3.6 in Theorem 3.2(i).

Corollary 3.14 (Sublinear functions)**.** *Let $m \in \mathbb{N}$, and $f_i : X \to]-\infty, +\infty]$ be proper sublinear, where $i \in \{1, 2, \dots, m\}$. Suppose that $\overline{f_i} = f_i$ on $\bigcap_{i=1}^m \operatorname{dom} \overline{f_i}$. Then the following eight conditions are equivalent.*

(i) *There exists $K > 0$ such that for every $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$, and every $\varepsilon > 0$,*

$$\overline{\Big[\sum_{i=1}^m \partial_\varepsilon f_i(x)\Big]}^{\mathrm{w}^*} \subseteq \sum_{i=1}^m \partial_{K\varepsilon} f_i(x).$$

(ii) $\sum_{i=1}^m \partial f_i(0)$ *is weak* closed.*

(iii) $\Big(\sum_{i=1}^m f_i\Big)^* = f_1^* \Box \cdots \Box f_m^*$ *in X^*.*

(iv) $f_1^* \Box \cdots \Box f_m^*$ *is weak* lower semicontinuous.*

(v) *For every $x \in X$ and $\varepsilon \geq 0$,*

$$\partial_\varepsilon (f_1 + \cdots + f_m)(x) = \bigcap_{\eta > 0} \Big[\bigcup_{\varepsilon_i \geq 0,\, \sum_{i=1}^m \varepsilon_i = \varepsilon + \eta} \big(\partial_{\varepsilon_1} f_1(x) + \cdots + \partial_{\varepsilon_m} f_m(x)\big)\Big].$$

(vi) $\operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^*$ *is closed in the topology $\omega(X^*, X) \times \mathbb{R}$.*

(vii) $\Big(\sum_{i=1}^m f_i\Big)^* = f_1^* \Box \cdots \Box f_m^*$ *in X^* and the infimal convolution is exact (attained) everywhere it is finite.*

(viii)

$$\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m.$$

Proof. We first show that (i)⇔(ii)⇔(iii)⇔(iv)⇔(v). By Theorem 3.2, it suffices to show that (i)⇔(ii).

(i)⇒(ii): Let $x^* \in \overline{[\sum_{i=1}^m \partial f_i(0)]}^{\mathrm{w}^*}$. Then $x^* \in \overline{[\sum_{i=1}^m \partial_1 f_i(0)]}^{\mathrm{w}^*}$. By (i), there exists $K > 0$ such that $x^* \in \sum_{i=1}^m \partial_K f_i(0)$. [31, Theorem 2.4.14(iii)] shows that $x^* \in \sum_{i=1}^m \partial f_i(0)$. Hence $\sum_{i=1}^m \partial f_i(0)$ is weak* closed.

(ii)⇒(i): Let $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$ and $\varepsilon > 0$, and $x^* \in \overline{[\sum_{i=1}^m \partial_\varepsilon f_i(x)]}^{\mathrm{w}^*}$. Then there exists a net $(x^*_{i,\alpha})_{\alpha \in I}$ in $\partial_\varepsilon f_i(x)$ such that

$$\sum_{i=1}^m x^*_{i,\alpha} \rightharpoonup_{\mathrm{w}^*} x^*. \tag{3.33}$$

Then by [31, Theorem 2.4.14(iii)], we have

$$x^*_{i,\alpha} \in \partial f_i(0) \quad \text{and} \quad f_i(x) \leq \langle x, x^*_{i,\alpha}\rangle + \varepsilon, \quad \forall i \in \{1, 2, \dots, m\} \quad \forall \alpha \in I. \tag{3.34}$$

Hence

$$\sum_{i=1}^m x^*_{i,\alpha} \in \sum_{i=1}^m \partial f_i(0) \quad \text{and} \quad \sum_{i=1}^m f_i(x) \leq \Big\langle x, \sum_{i=1}^m x^*_{i,\alpha}\Big\rangle + m\varepsilon, \quad \forall \alpha \in I. \tag{3.35}$$

Thus, by (3.33) and (3.35),

$$x^* \in \overline{\Big[\sum_{i=1}^m \partial f_i(0)\Big]}^{\mathrm{w}^*} \quad \text{and} \quad \sum_{i=1}^m f_i(x) \le \langle x, x^*\rangle + m\varepsilon. \tag{3.36}$$

Since $\sum_{i=1}^m \partial f_i(0)$ is weak* closed, by (3.36), $x^* \in \sum_{i=1}^m \partial f_i(0)$. Then there exists $y_i^* \in \partial f_i(0)$ such that

$$x^* = \sum_{i=1}^m y_i^*. \tag{3.37}$$

By (3.36) and [31, Theorem 2.4.14(i)], we have

$$\sum_{i=1}^m \big(f_i(x) + f_i^*(y_i^*)\big) = \sum_{i=1}^m \big(f_i(x) + \iota_{\partial f_i(0)}(y_i^*)\big) \le \langle x, x^*\rangle + m\varepsilon$$

Hence

$$y_i^* \in \partial_{m\varepsilon} f_i(x), \quad \forall i \in \{1, 2, \ldots, m\}.$$

Then by (3.37), $x^* \in \sum_{i=1}^m \partial_{m\varepsilon} f_i(x)$. Setting $K := m$, we obtain (i).

Hence (i)⇔(ii)⇔(iii) ⇔(iv)⇔(v).

(ii)⇔(vi): By [31, Theorem 2.4.14(i)], we have

$$\operatorname{epi} f_1^* + \cdots + \operatorname{epi} f_m^* = \big(\partial f_1(0) + \cdots + \partial f_m(0)\big) \times \{r \mid r \ge 0\}.$$

The rest is now clear.

(vi)⇒(vii): Apply Corollary 3.8.

(vii)⇒(viii): Apply Lemma 3.7 directly.

(viii)⇒(ii): Since $\sum_{i=1}^m \partial f_i(0) = \partial(f_1 + f_2 + \cdots + f_m)(0)$, we conclude that $\sum_{i=1}^m \partial f_i(0)$ is weak* closed □

Remark 3.15. By applying Corollary 3.14 to a single sublinear function, we conclude that $f = \overline{f}$ and is lower semicontinuous everywhere (see (3.3)). By [31, Theorem 2.4.14], this implies existence of subdifferentials at 0 (as indeed can also be deduced from Corollary 3.14).

Corollary 3.16 (Burachik, Jeyakumar and Wu)**.** (See [12, Corollary 3.3].) *Suppose that X is a Banach space. Let $f, g : X \to\ ]-\infty, +\infty]$ be proper lower semicontinuous and sublinear. Then the following are equivalent.*

(i) $\operatorname{epi} f^* + \operatorname{epi} g^*$ *is closed in the topology* $\omega(X^*, X) \times \mathbb{R}$.
(ii) $(f+g)^* = f^* \Box g^*$ *in* X^* *and the infimal convolution is exact (attained) everywhere.*
(iii) $\partial(f+g) = \partial f + \partial g$.

Proof. Apply Corollary 3.14 directly. □

We end this section with a corollary of our main result involving the subdifferential of the sum of convex functions. We recall that a formula known to hold in general, without any constraint qualification, has been given by Hiriart-Urruty and

Phelps in [18, Theorem 2.1] (see also [15, Corollary 5.1] and [17, Theorem 3.1]) and is as follows.

$$\partial(f_1 + \cdots + f_m)(x) = \bigcap_{\eta>0} \overline{[\partial_\eta f_1(x) + \cdots + \partial_\eta f_m(x)]}^{\mathrm{w}^*}. \tag{3.38}$$

Several constraint qualifications have been given in the literature to obtain simpler expressions for the right hand side in (3.38). As we mentioned before, the closed epigraph condition allows one to conclude the subdifferential sum formula, so both the intersection symbol and the closure operator become superfluous under this constraint qualification. Hence it is valid to ask whether our closedness condition in Theorem 3.2(i) allows us to simplify the right hand side in (3.38). The following corollary shows that this is indeed the case, and we are able to remove the weak* closure from (3.38).

Corollary 3.17. *Let $m \in \mathbb{N}$, and $f_i : X \to \,]-\infty, +\infty]$ be proper convex with $\bigcap_{i=1}^m \operatorname{dom} f_i \neq \varnothing$, where $i \in \{1, 2, \ldots, m\}$. Suppose that $\overline{f_i} = f_i$ on $\bigcap_{i=1}^m \operatorname{dom} \overline{f_i}$. Assuming any of the assumptions (i)-(iv) in Theorem 3.2, the following equality holds for every $x \in X$,*

$$\partial(f_1 + \cdots + f_m)(x) = \bigcap_{\eta>0} [\partial_\eta f_1(x) + \cdots + \partial_\eta f_m(x)].$$

Proof. By Theorem 3.2(iv), we have

$$\begin{aligned}\partial(f_1 + \cdots + f_m)(x) &= \bigcap_{\eta>0} \Big[\bigcup_{\varepsilon_i \geq 0,\, \sum_{i=1}^m \varepsilon_i = \eta} \big(\partial_{\varepsilon_1} f_1(x) + \cdots + \partial_{\varepsilon_m} f_m(x)\big) \Big] \\ &\subseteq \bigcap_{\eta>0} \Big(\sum_{i=1}^m \partial_\eta f_i(x) \Big) \subseteq \bigcap_{\eta>0} \Big(\partial_{m\eta} \Big(\sum_{i=1}^m f_i \Big)(x) \Big) = \partial \Big(\sum_{i=1}^m f_i \Big)(x).\end{aligned}$$

Hence $\partial(f_1 + \cdots + f_m)(x) = \bigcap_{\eta>0} [\partial_\eta f_1(x) + \cdots + \partial_\eta f_m(x)]$. □

Without the constraint qualification in Theorem 3.2, Corollary 3.17 need not hold, as shown in the following example. We denote by $\overline{\operatorname{span}\{C\}}$ the closed linear subspace spanned by a set C.

Example 3.18. Let $\mathbb{N} := \{0, 1, 2, \ldots\}$. Suppose that H is an infinite-dimensional Hilbert space and let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal sequence in H. Set

$$C := \overline{\operatorname{span}\{e_{2n}\}_{n\in\mathbb{N}}} \quad \text{and} \quad D := \overline{\operatorname{span}\{\cos(\theta_n) e_{2n} + \sin(\theta_n) e_{2n+1}\}_{n\in\mathbb{N}}},$$

where $(\theta_n)_{n\in\mathbb{N}}$ is a sequence in $\left]0, \frac{\pi}{2}\right]$ such that $\sum_{n\in\mathbb{N}} \sin^2(\theta_n) < +\infty$. Define $f, g : H \to \,]-\infty, +\infty]$ by

$$f := \iota_{C^\perp} \quad \text{and} \quad g := \iota_{D^\perp}. \tag{3.39}$$

Then f and g are proper lower semicontinuous and convex, and constraint qualifications in Theorem 3.2 fail. Moreover,

$$\partial(f + g)(x) \neq \bigcap_{\eta>0} [\partial_\eta f(x) + \partial_\eta g(x)], \quad \forall x \in \operatorname{dom} f \cap \operatorname{dom} g.$$

Proof. Since C, D are closed linear subspaces, f and g are proper lower semicontinuous and convex. Let $x \in \operatorname{dom} f \cap \operatorname{dom} g$ and $\eta > 0$. Then we have $\partial_\eta f(x) = C^{\perp\perp} = C$ and $\partial_\eta g(x) = D^{\perp\perp} = D$ and thus $\partial_\eta f(x) + \partial_\eta g(x) = C + D$. Hence

$$\bigcap_{\eta>0} [\partial_\eta f(x) + \partial_\eta g(x)] = C + D. \tag{3.40}$$

Then by [2, Example 3.34], $\bigcap_{\eta>0} [\partial_\eta f(x) + \partial_\eta g(x)]$ is not norm closed and hence $\bigcap_{\eta>0} [\partial_\eta f(x) + \partial_\eta g(x)]$ is not weak* closed by [2, Theorem 3.32]. However, $\partial(f + g)(x)$ is weak* closed. Hence $\partial(f + g)(x) \neq \bigcap_{\eta>0} [\partial_\eta f(x) + \partial_\eta g(x)]$.

Note that $\overline{\partial_\eta f(x) + \partial_\eta g(x)}^{\mathrm{w}^*} = \overline{C + D}^{\mathrm{w}^*} \not\subseteq C + D = \partial_\varepsilon f(x) + \partial_\varepsilon g(x)$, $\forall \varepsilon > 0$. Hence the constraint qualification in Theorem 3.2(i) fails. ☐

4. Further consequences of our main result

In this section, we will recapture various forms of Rockafellar's Fenchel duality theorem.

Lemma 4.1 (Interiority). *Let $m \in \mathbb{N}$, and $\varepsilon_i \geq 0$ and let $f_i : X \to]-\infty, +\infty]$ be proper convex, where $i \in \{1, 2, \ldots, m\}$. Assume that there exists $x_0 \in \left(\bigcap_{i=1}^m \operatorname{dom} f_i\right)$ such that f_i is continuous at x_0 for every $i \in \{2, 3, \ldots, m\}$. Then for every $x \in \left(\bigcap_{i=1}^m \operatorname{dom} f_i\right)$, the set $\sum_{i=1}^m \partial_{\varepsilon_i} f_i(x)$ is weak* closed. Moreover, for every $z \in \left(\bigcap_{i=1}^m \operatorname{dom} \overline{f_i}\right)$, the set $\sum_{i=1}^m \partial_{\varepsilon_i} \overline{f_i}(z)$ is weak* closed.*

Proof. We can and do suppose that $x_0 = 0$. Then there exist a neighbourhood V of 0 and $K > \max\{0, f_1(0)\}$ such that $V = -V$ (see [28, Theorem 1.14(a)]) and

$$V \subseteq \operatorname{dom} f_i \quad \text{and} \quad \sup_{y\in V} \overline{f_i}(y) \leq \sup_{y\in V} f_i(y) \leq K, \quad \forall i \in \{2, 3, \ldots, m\}. \tag{4.1}$$

Let $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$, $x^* \in \overline{[\sum_{i=1}^m \partial_{\varepsilon_i} f_i(x)]}^{\mathrm{w}^*}$. We will show that

$$x^* \in \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x). \tag{4.2}$$

Our assumption on x^* implies that for every $i = 1, \ldots, m$ there exists a net $(x^*_{i,\alpha})_{\alpha\in I}$ in $\partial_{\varepsilon_i} f_i(x)$ such that

$$\sum_{i=1}^m x^*_{i,\alpha} \rightharpoondown_{\mathrm{w}^*} x^*. \tag{4.3}$$

We have

$$f_i^*(x^*_{i,\alpha}) \leq -f_i(x) + \langle x, x^*_{i,\alpha}\rangle + \varepsilon_i, \quad \forall i \in \{1, 2, \ldots, m\} \quad \forall \alpha \in I \tag{4.4}$$

Now we claim that

$$\Big\{\sum_{i=2}^m \sup |\langle x^*_{i,\alpha}, V\rangle|\Big\}_{\alpha\in I} = \Big\{\sum_{i=2}^m \sup \langle x^*_{i,\alpha}, V\rangle\Big\}_{\alpha\in I} \quad \text{is eventually bounded.} \tag{4.5}$$

In other words, we will find a terminal set $J \subseteq I$ and $R > 0$ such that

$\sum_{i=2}^m \sup\langle x_{i,\alpha}^*, V\rangle \leq R$ for all $\alpha \in J$. Fix $i \in \{2, \ldots, m\}$. By (4.4), we have

$$-f_i(x) + \langle x, x_{i,\alpha}^*\rangle + \varepsilon_i \geq \sup_{y\in V}\{\langle x_{i,\alpha}^*, y\rangle - f_i(y)\} \geq \sup_{y\in V}\{\langle x_{i,\alpha}^*, y\rangle - K\} \quad \text{(by (4.1))}$$
$$= \sup\langle x_{i,\alpha}^*, V\rangle - K. \tag{4.6}$$

Then we have

$$-\sum_{i=2}^m f_i(x) + \Big\langle x, \sum_{i=2}^m x_{i,\alpha}^*\Big\rangle + \sum_{i=2}^m \varepsilon_i \geq \sum_{i=2}^m \sup\langle x_{i,\alpha}^*, V\rangle - (m-1)K, \quad \forall \alpha \in I. \tag{4.7}$$

Since $0 \in \operatorname{dom} f_1$ and , $f_1^*(x_{1,\alpha}^*) \geq -f_1(0) \geq -K$. Then by (4.4),

$$-f_1(x) + \langle x, x_{1,\alpha}^*\rangle + \varepsilon_1 \geq -K, \quad \forall \alpha \in I. \tag{4.8}$$

Combining (4.7) and (4.8)

$$-\sum_{i=1}^m f_i(x) + \Big\langle x, \sum_{i=1}^m x_{i,\alpha}^*\Big\rangle + \sum_{i=1}^m \varepsilon_i \geq \sum_{i=2}^m \sup\langle x_{i,\alpha}^*, V\rangle - mK, \quad \forall \alpha \in I.$$

Then by (4.3),

$$-\sum_{i=1}^m f_i(x) + \langle x, x^*\rangle + \sum_{i=1}^m \varepsilon_i \geq \limsup_{\alpha\in I} \sum_{i=2}^m \sup\langle x_{i,\alpha}^*, V\rangle - mK. \tag{4.9}$$

Hence (4.5) holds.

Then by (4.5) and the Banach-Alaoglu Theorem (see [28, Theorem 3.15] or [31, Theorem 1.1.10]), there exists a weak* convergent *subnet* $(x_{i,\gamma}^*)_{\gamma\in\Gamma}$ of $(x_{i,\alpha}^*)_{\alpha\in I}$ such that

$$x_{i,\gamma}^* \rightharpoonup_{\mathrm{w}^*} x_{i,\infty}^* \in X^*, \quad i \in \{2, \ldots, m\}. \tag{4.10}$$

Since $\partial_{\varepsilon_i} f_i(x)$ is weak* closed by [31, Theorem 2.4.2], then

$$x_{i,\infty}^* \in \partial_{\varepsilon_i} f_i(x), \quad \forall i \in \{2, \ldots, m\}. \tag{4.11}$$

Then by (4.3),

$$x^* - \sum_{i=2}^m x_{i,\infty}^* \in \partial_{\varepsilon_1} f_1(x). \tag{4.12}$$

Combining the above two equations, we have

$$x^* \in \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x).$$

Hence $\sum_{i=1}^m \partial_{\varepsilon_i} f_i(x)$ is weak* closed.

Similarly, the set $\sum_{i=1}^m \partial_{\varepsilon_i} \overline{f_i}(z)$ is weak* closed for every $z \in \big(\bigcap_{i=1}^m \operatorname{dom} \overline{f_i}\big)$. □

Lemma 4.2. *Suppose that X is a Banach space. Let $m \in \mathbb{N}$, and $\varepsilon_i \geq 0$ and $f_i : X \to \,]-\infty, +\infty]$ be proper lower semicontinuous and convex, where $i \in \{1, 2, \ldots, m\}$. Assume that*

$$\operatorname{dom} f_1 \cap \Big(\bigcap_{i=2}^m \operatorname{int} \operatorname{dom} f_i\Big) \neq \varnothing.$$

Then for every $x \in \bigcap_{i=1}^m \operatorname{dom} f_i$*, the set* $\sum_{i=1}^m \partial_{\varepsilon_i} f_i(x)$ *is weak** *closed.*

Proof. By [21, Proposition 3.3], we conclude that f_i is continuous for $i \in \{2, \ldots, m\}$. Apply now Lemma 4.1 directly. $\square$

The following results recapture various known exactness results as consequences of our main results.

Corollary 4.3. (See [8, Theorem 3.5.8].) *Let* $m \in \mathbb{N}$*, and* $\varepsilon_i \geq 0$ *and* $f_i : X \to]-\infty, +\infty]$ *be proper convex, where* $i \in \{1, 2, \ldots, m\}$*. Assume that there exists* $x_0 \in \big(\bigcap_{i=1}^m \operatorname{dom} f_i\big)$ *such that* f_i *is continuous at* x_0 *for every* $i \in \{2, 3, \ldots, m\}$*. Then* $(\sum_{i=1}^m f_i)^* = f_1^* \Box \cdots \Box f_m^*$ *in* X^* *and the infimal convolution is exact everywhere. Furthermore,* $\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m$*.*

Proof. By [16, Lemma 15],

$$\overline{f_1 + f_2 \ldots + f_m} = \overline{f_1} + \overline{f_2 \ldots + f_m} = \ldots = \overline{f_1} + \overline{f_2} + \cdots + \overline{f_m}. \tag{4.13}$$

By the assumption, we have $x_0 \in \operatorname{dom} \overline{f_1} \cap \big(\bigcap_{i=2}^m \operatorname{int} \operatorname{dom} \overline{f_i}\big)$ and $\overline{f_i}$ is proper for every $i \in \{2, 3, \ldots, m\}$ by [31, Theorem 2.3.4(ii)].

We consider two cases.

Case 1: $\overline{f_1}$ is proper.

By (4.13), Lemma 4.1 and Theorem 3.2 (applied to $\overline{f_i}$), we have

$$\Big(\sum_{i=1}^m f_i\Big)^* = \Big(\overline{\sum_{i=1}^m f_i}\Big)^* = \Big(\sum_{i=1}^m \overline{f_i}\Big)^* = \overline{f_1}^* \Box \cdots \Box \overline{f_m}^* = f_1^* \Box \cdots \Box f_m^*. \tag{4.14}$$

Let $x^* \in X^*$. Next we will show that $(f_1^* \Box \cdots \Box f_m^*)(x^*)$ is achieved. This is clear when $x^* \notin \operatorname{dom}(\sum_{i=1}^m f_i)^*$ by (4.14). Now suppose that $x^* \in \operatorname{dom}(\sum_{i=1}^m f_i)^*$ and then $(\sum_{i=1}^m f_i)^*(x^*) \in \mathbb{R}$. By (4.14), there exists $(x_{i,n}^*)_{n\in\mathbb{N}}$ such that $\sum_{i=1}^m x_{i,n}^* = x^*$ and

$$f_1^*(x_{1,n}^*) + f_2^*(x_{2,n}^*) + \cdots + f_m^*(x_{m,n}^*) \leq \Big(\sum_{i=1}^m f_i\Big)^*(x^*) + \frac{1}{2n}. \tag{4.15}$$

Since $x^* \in \operatorname{dom}(\sum_{i=1}^m f_i)^*$, there exists $x \in X$ such that $x \in \partial_{\frac{1}{2n}}(\sum_{i=1}^m f_i)^*(x^*)$. Then by (4.13),

$$\begin{aligned}
\Big(\sum_{i=1}^m f_i\Big)^*(x^*) + \Big(\sum_{i=1}^m \overline{f_i}\Big)(x) &= \Big(\sum_{i=1}^m f_i\Big)^*(x^*) + \Big(\overline{\sum_{i=1}^m f_i}\Big)(x) \\
&= \Big(\sum_{i=1}^m f_i\Big)^*(x^*) + \Big(\sum_{i=1}^m f_i\Big)^{**}(x) \\
&\leq \langle x, x^* \rangle + \frac{1}{2n}.
\end{aligned}$$

Then by (4.15),

$$f_1^*(x_{1,n}^*) + f_2^*(x_{2,n}^*) + \cdots + f_m^*(x_{m,n}^*) + \Big(\sum_{i=1}^m \overline{f_i}\Big)(x) \leq \langle x, x^* \rangle + \frac{1}{n}.$$

Hence

$$x^*_{i,n} \in \partial_{\frac{1}{n}} \overline{f_i}(x), \quad \forall i \in \{1,2,\ldots,m\}, \forall n \in \mathbb{N}. \tag{4.16}$$

By the assumptions, there exist a neighbourhood V of 0 and $K > \max\{0, f_1(0)\}$ such that $V = -V$ (see [28, Theorem 1.14(a)]) and

$$V \subseteq \operatorname{dom} f_i \quad \text{and} \sup \overline{f_i}(V) \le \sup f_i(V) \le K, \quad \forall i \in \{2,3,\ldots,m\}.$$

As in the proof of Lemma 4.1, $\big(\sum_{i=2}^m \sup |\langle x^*_{i,n}, V\rangle|\big)_{n\in\mathbb{N}}$ is bounded and then there exists a weak* convergent *subnet* $(x^*_{i,\gamma})_{\gamma\in\Gamma}$ of $(x^*_{i,n})_{n\in\mathbb{N}}$ such that

$$x^*_{i,\gamma} \rightharpoonup_{\mathrm{w}^*} x^*_{i,\infty} \in X^*, \quad i \in \{2,\ldots,m\}$$

$$x^*_{1,\gamma} \rightharpoonup_{\mathrm{w}^*} x^* - \sum_{i=2}^m x^*_{i,\infty} \in X^*. \tag{4.17}$$

Combining (4.17) and taking the limit along the subnets in (4.15), we have

$$f_1^*\Big(x^* - \sum_{i=2}^m x^*_{i,\infty}\Big) + f_2^*(x^*_{2,\infty}) + \cdots + f_m^*(x^*_{m,\infty}) \le \Big(\sum_{i=1}^m f_i\Big)^*(x^*). \tag{4.18}$$

By (4.14) again and (4.18),

$$f_1^*\Big(x^* - \sum_{i=2}^m x^*_{i,\infty}\Big) + f_2^*(x^*_{2,\infty}) + \cdots + f_m^*(x^*_{m,\infty}) = (f_1^* \Box \cdots \Box f_m^*)(x^*).$$

Hence $f_1^* \Box \cdots \Box f_m^*$ is achieved at x^*.

By Lemma 3.7, we have $\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m$

Case 2: $\overline{f_1}$ is not proper.

Since $x_0 \in \operatorname{dom} \overline{f_1}$, we have there exists $y_0 \in X$ such that $\overline{f_1}(y_0) = -\infty$ and thus $\overline{f_1}(x) = -\infty$ for every $x \in \operatorname{dom} \overline{f_1}$ by [13, Proposition 2.4]. Thus by (4.13),

$$\overline{(f_1 + f_2 \ldots + f_m)}(x_0) = \overline{f_1}(x_0) + \overline{f_2}(x_0) + \cdots + \overline{f_m}(x_0) = -\infty \tag{4.19}$$

since $\overline{f_i}$ is proper for every $\in \{2,3,\ldots,m\}$ and $x_0 \in \operatorname{dom} \overline{f_1} \cap \big(\bigcap_{i=2}^m \operatorname{int} \operatorname{dom} \overline{f_i}\big)$.

We also have $f_1^* = +\infty$ and then

$$f_1^* \Box \cdots \Box f_m^* = +\infty. \tag{4.20}$$

Then by (4.19), we have

$$(\sum_{i=1}^m f_i)^* = \Big(\overline{\sum_{i=1}^m f_i}\Big)^* = +\infty = f_1^* \Box \cdots \Box f_m^*.$$

Hence $f_1^* \Box \cdots \Box f_m^*$ is exact everywhere.

Apply Lemma 3.7 directly to obtain that $\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m$.

Combining the above two cases, the result holds. $\square$

Corollary 4.4. *Suppose that X is a Banach space. Let $m \in \mathbb{N}$, and $f_i : X \to]-\infty, +\infty]$ be proper lower semicontinuous and convex with* $\operatorname{dom} f_1 \cap \big(\bigcap_{i=2}^m \operatorname{int} \operatorname{dom} f_i\big)$ *$\neq \varnothing$, where $i \in \{1,2,\ldots,m\}$. Then $(\sum_{i=1}^m f_i)^* = f_1^* \Box \cdots \Box f_m^*$ in X^* and the infimal convolution is exact everywhere. Furthermore, $\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m$.*

Proof. By [21, Proposition 3.3], f_i is continuous on $\operatorname{int}\operatorname{dom} f_i$ for $i \in \{2,\ldots,m\}$. Then apply Corollary 4.3 directly. □

Corollary 4.5 (Rockafellar)**.** (See [5, Theorem 4.1.19] [22, Theorem 3], or [31, Theorem 2.8.7(iii)].) *Let $f, g : X \to \left]-\infty,+\infty\right]$ be proper convex. Assume that there exists $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$ such that f is continuous at x_0. Then $(f+g)^* = f^* \Box g^*$ in X^* and the infimal convolution is exact everywhere. Furthermore, $\partial(f+g) = \partial f + \partial g$.*

Proof. Apply Corollary 4.3 directly. □

A *polyhedral set* is a subset of a Banach space defined as a finite intersection of halfspaces. A function $f : X \to \left]-\infty,+\infty\right]$ is said to be *polyhedrally convex* if $\operatorname{epi} f$ is a polyhedral set.

Corollary 4.6. *Let $m, k, d \in \mathbb{N}$ and suppose that $X = \mathbb{R}^d$, let $f_i : X \to \left]-\infty,+\infty\right]$ be a polyhedrally convex function for $i \in \{1,2,\ldots,k\}$. Let $f_j : X \to \left]-\infty,+\infty\right]$ be proper convex for every $j \in \{k+1,k+2,\ldots,m\}$. Assume that there exists $x_0 \in \bigcap_{i=1}^m \operatorname{dom} f_i$ such that f_i is continuous at x_0 for every $i \in \{k+1,k+2,\ldots,m\}$.*

Then $(\sum_{i=1}^m f_i)^* = f_1^* \Box \cdots \Box f_m^*$ in X^* and the infimal convolution is exact everywhere. Furthermore, $\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m$.

Proof. Set $g_1 := \sum_{i=1}^k f_i$ and $g_2 := \sum_{i=k+1}^m f_i$. By [23, Corollary 19.1.2], f_i is lower semicontinuous for every $i \in \{1,2,\ldots,k\}$, so is g_1. By Corollary 4.5, $(g_1+g_2)^* = g_1^* \Box g_2^*$ with the exact infimal convolution and $\partial(g_1+g_2) = \partial g_1 + \partial g_2$.

Let $i \in \{1,2,\ldots,k\}$. By [23, Theorem 19.2], f_i^* is a polyhedrally convex function. Hence $f_1^* \Box \cdots \Box f_m^*$ is polyhedrally convex by [23, Corollary 19.3.4] and hence $\sum_{i=1}^m \operatorname{epi} f_i^*$ is closed by [31, Theorem 2.1.3(ix)] and [23, Theorem 19.1]. Then applying Corollary 3.8, we have $g_1^* = f_1^* \Box \cdots \Box f_k^*$ with the infimal convolution is exact everywhere. Using now Lemma 3.7 we obtain $\partial g_1 = \partial(f_1 + f_2 + \cdots + f_k) = \partial f_1 + \cdots + \partial f_k$.

By Corollary 4.3, we have $g_2^* = f_{k+1}^* \Box \cdots \Box f_m^*$ with exact infimal convolution, and $\partial g_2 = \partial(f_{k+1} + f_{k+2} + \cdots + f_m) = \partial f_{k+1} + \cdots + \partial f_m$.

Combining the above results, we have $(\sum_{i=1}^m f_i)^* = (g_1+g_2)^* = f_1^* \Box \cdots \Box f_m^*$ with exact infimal convolution, and $\partial(f_1 + f_2 + \cdots + f_m) = \partial f_1 + \cdots + \partial f_m$. □

5. Conclusion

We have introduced a new dual condition for zero duality gap in convex programming. We have proved that our condition is less restrictive than all other conditions in the literature, and we have related it with (a) Bertsekas constraint qualification, (b) the closed epigraph condition, and (c) the interiority conditions. We have used our closedness condition to simplify the well-known expression for the subdifferential of the sum of convex functions. Our study has motivated the following open questions.

(1) Does the Closed Epigraph Condition imply Bertsekas Constraint Qualification?
(2) Are the conditions of Theorem 3.2 strictly more restrictive than Bertsekas Constraint Qualification?

(3) How do these results extend when, instead of the sum of convex functions, the objective of the primal problem has the form $f + g \circ A$, where f, g convex and A a linear operator?

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References

[1] H. Attouch and H. Brezis, *Duality for the sum of convex functions in general Banach spaces*, in Aspects of Mathematics and its Applications, J. A. Barroso (ed.), Elsevier Science Publishers, Amsterdam, 1986, pp. 125–133.

[2] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, Springer, New York, 2011.

[3] D. P. Bertsekas, *Extended monotropic programming and duality*, Lab. for Information and Decision Systems Report LIDS-2692, MIT; `http://web.mit.edu/dimitrib/www/Extended_Mono.pdf`, February 2010.

[4] J.M. Borwein, *Adjoint process duality*, Mathematics of Operations Research **8** (1983), 403–434.

[5] J. M. Borwein and J. D. Vanderwerff, *Convex Functions*, Cambridge University Press, Cambridge, 2010.

[6] R.I. Boţ and E. R. Csetnek, *On a zero duality gap result in extended monotropic programming*, Journal of Optimization Theory and Applications **147** (2010), 473–482.

[7] R. I. Boţ and S.-M. Grad, *Lower semicontinuous type regularity conditions for subdifferential calculus*, Optimization Methods and Software **25** (2010), 37–48.

[8] R.I. Boţ, S.-M. Grad, and G. Wanka, *Duality in Vector Optimization*, Springer, 2009.

[9] R.I. Boţ and G. Wanka, "A weaker regularity condition for subdifferential calculus and Fenchel duality in infinite dimensional spaces", *Nonlinear Anal.*, vol. 64 (2006), pp. 2787–2804, 2006.

[10] R. S. Burachik and A. N. Iusem, *Set-Valued Mappings and Enlargements of Monotone Operators*, vol 8, Springer, New York, 2008.

[11] R. S. Burachik and V. Jeyakumar, *A new geometric condition for Fenchels duality in infinite dimensional spaces*, Math. Programming **104** (2005), 229–233.

[12] R. S. Burachik, V. Jeyakumar and Z.-Y. Wu, *Necessary and sufficient conditions for stable conjugate duality*, Nonlinear Analysis: Theory Methods Appl. **64** (2006), 1998–2006.

[13] I. Ekeland and R. Témam, *Convex analysis and variational problems*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 1999.

[14] R. S. Burachik and S. N. Majeed, *Strong duality for generalized monotropic programming in infinite Dimensions*, Journal of Mathematical Analysis and Applications **400** (2013), 541–557.

[15] S. P. Fitzpatrick and S. Simons, *The Conjugates, Compositions and Marginals of Convex Functions* Journal of Convex Analysis **8** (2001), 423–446.

[16] A. Hantoutey, M.A. López and C. Zălinescu, *Subdiferential calculus rules in convex analysis: A unifying approach via pointwise supremum functions*, SIAM Journal on Optimization **19** (2008), 863–882.

[17] J.-B. Hiriart-Urruty, M. Moussaoui, A. Seeger and M. Volle, *Subdifferential calculus without qualification conditions, using approximate subdifferentials: a survey*, Nonlinear Anal. **24** (1995), 1727–1754.

[18] J.-B. Hiriart-Urruty and R. R. Phelps, *Subdifferential Calculus Using ε-Subdifferentials*, Journal of Functional Analysis **118** (1993), 154–166.
[19] P. J. Laurent, *Approximation et Optimisation*, Hermann, Paris, 1972.
[20] G. Li and K.F. Ng, *On extension of Fenchel duality and its application*, SIAM Journal on Optimization **19** (2008), 1489–1509.
[21] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, 2nd Edition, Springer-Verlag, Berlin, 1993.
[22] R. T. Rockafellar, *Extension of Fenchel's duality theorem for convex functions*, Duke Mathematical Journal **33** (1966), 81–89.
[23] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, 1970.
[24] R. T. Rockafellar, *Monotropic programming: descent algorithms and duality*, in Nonlinear Programming 4, O.L. Mangasarian (ed.), Academic Press, 1981, pp. 327–366.
[25] R. T. Rockafellar, *Network Flows and Monotropic Optimization*, Wiley, New York, 1984.
[26] R. T. Rockafellar and R.J-B Wets, *Variational Analysis*, 3rd Printing, Springer-Verlag, New York, 2009.
[27] B. Rodrigues and S. Simons, *Conjugate functions and subdifferentials in non-normed situations for operators with complete graphs*, Nonlinear Anal: Theory Methods Appl. **12** (1998), 1069–1078.
[28] W. Rudin, *Functional Analysis*, Second Edition, McGraw-Hill, New York, 1991.
[29] P. Tseng, *Some convex programs without a duality gap*, Mathematical Progamming Ser. B (2009), 553–578.
[30] C. Zălinescu, *A comparison of constraint qualifications in infinite-dimensional convex programming revisited*, Australian Mathematical Society. Journal. Series B. Applied Mathematics **40** (1999), 353–378.
[31] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing, River Edge, NJ, 2002.

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