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THE MINIMAL DISPLACEMENT AND EXTREMAL SPACES

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Abstract. We show that both separable preduals of L_1 and non-type I C^*-algebras are strictly extremal with respect to the minimal displacement of k-Lipschitz mappings acting on the unit ball of a Banach space. In particular, every separable $C(K)$ space is strictly extremal.

1. Introduction

Throughout the paper $(X, \|\cdot\|)$ denotes a real or complex infinite-dimensional Banach space. The notion of the minimal displacement was introduced by K. Goebel in [2]. Let C be a bounded closed and convex subset of X and $T : C \to C$ a mapping. The minimal displacement of T is the number

$$d_T = \inf \{\|x - Tx\| : x \in C\}.$$

Goebel showed that if T is k-Lipschitz then

$$d_T \leq \left(1 - \frac{1}{k}\right) r(C) \qquad \text{for } k \geq 1,$$

where $r(C) = \inf_{x \in C} \sup_{y \in C} \|x - y\|$ denotes the Chebyshev radius of C. There are some spaces and sets with $d_T = \left(1 - \frac{1}{k}\right) r(C)$. The minimal displacement characteristic of X is a function

$$\psi_X(k) = \sup \{d_T : T : B_X \to B_X,\ T \in L(k)\},\ k \geq 1,$$

where B_X denotes the closed unit ball of X and $L(k)$ is the class of k-Lipschitz mappings. It is known that

$$\psi_X(k) \leq 1 - \frac{1}{k}$$

for any space X and the spaces with $\psi_X(k) = 1 - \frac{1}{k}$ are said to be extremal. Among extremal spaces are some sequence and function spaces such as $c_0, c, C[0,1], BC(\mathbb{R}), BC_0(\mathbb{R})$ (see [5,9]).

Recently Bolibok [1] proved that in ℓ_∞,

$$\psi_{\ell_\infty}(k) \geq \begin{cases} \left(3 - 2\sqrt{2}\right)(k-1), & \text{if } 1 \leq k \leq 2 + \sqrt{2}, \\ 1 - \frac{2}{k}, & \text{if } k > 2 + \sqrt{2}. \end{cases}$$

It is still an open problem whether the space ℓ_∞ is extremal with respect to the minimal displacement. Recall that ℓ_∞ is isometric to $C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$. In this note we show that every separable $C(K)$ space, where K is compact Hausdorff, is strictly extremal (see Definition 2.1). This is a consequence of a more general Theorem 2.5, which states that all separable preduals

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of L_1 are (strictly) extremal. An analogous result holds in the non-commutative case of separable non-type I C^*-algebras.

2. Results

We begin by recalling the arguments which show that (a real or complex) c_0 is extremal (see [3, 5]). Fix $k \geq 1$ and define a mapping $T : B_{c_0} \to B_{c_0}$ by

$$Tx = T(x_1, x_2, x_3, ...) = (1, k\,|x_1| \wedge 1, k\,|x_2| \wedge 1, ...).$$

It is clear that $T \in L(k)$ and for any $x = (x_1, x_2, x_3, ...) \in B_{c_0}$, $\|Tx - x\| > 1 - \frac{1}{k}$, since the reverse inequality implies $|x_1| \geq \frac{1}{k}$, $k\,|x_1| \wedge 1 = 1$ and, consequently, $|x_i| \geq \frac{1}{k}$ for $i = 1, 2, 3, ...$, which contradicts $x \in c_0$. Notice that the minimal displacement $d_T = 1 - \frac{1}{k}$ is not achieved by T at any point of B_{c_0}. This suggests the following definition.

Definition 2.1. A Banach space X is said to be strictly extremal if for every $k > 1$, there exists a mapping $T : B_X \to B_X$, $T \in L(k)$, such that $\|Tx - x\| > 1 - \frac{1}{k}$ for every $x \in B_X$.

It follows from the above that c_0 is strictly extremal. On the other hand we have the following result.

Proposition 2.2. *Suppose that B_X has the fixed point property for nonexpansive mappings (i.e., every nonexpansive mapping $S : B_X \to B_X$ has a fixed point). Then for every k-Lipschitz mapping $T : B_X \to B_X$, $k \geq 1$, there exists $x \in B_X$ such that $\|Tx - x\| \leq 1 - \frac{1}{k}$. In particular, X is not strictly extremal.*

Proof. Let $T : B_X \to B_X$ be k-Lipschitz. Then $\frac{1}{k}T$ is nonexpansive and consequently there exists $\|x\| \leq \frac{1}{k}$ such that $Tx = kx$. Hence $\|Tx - x\| = (k - 1)\,\|x\| \leq 1 - \frac{1}{k}$. □

Proposition 2.2 applies to all uniformly nonsquare Banach spaces, uniformly noncreasy spaces as well as to ℓ_∞.

In what follows we need the following observation.

Lemma 2.3. *Suppose that Y is a subspace of a Banach space X and there exists an m-Lipschitz retraction $R : B_X \to B_Y$. Then*

$$\psi_X(k) \geq \frac{1}{m}\psi_Y\left(\frac{k}{m}\right)$$

for every $k \geq 1$.

Proof. Fix $\varepsilon > 0$ and select a k-Lipschitz mappping $T : B_Y \to B_Y$ such that $\|Ty - y\| > \psi_Y(k) - \varepsilon$ for every $y \in B_Y$. Define $\widetilde{T} : B_X \to B_X$ by putting $\widetilde{T}x = (T \circ R)x$, $x \in B_X$. Then

$$\psi_Y(k) - \varepsilon < \|TRx - Rx\| = \|RTRx - Rx\| \leq m\left\|\widetilde{T}x - x\right\|$$

for every $x \in B_X$. Notice that $\widetilde{T}$ is km-Lipschitz and hence

$$\psi_X(km) \geq \frac{1}{m}(\psi_Y(k) - \varepsilon).$$

This completes the proof since ε is arbitrary. □

Recall that Y is said to be a k-complemented subspace of a Banach space X if there exists a (linear) projection $P : X \to Y$ with $\|P\| \le k$. The well known Sobczyk theorem asserts that c_0 is 2-complemented in any separable Banach space X containing it.

Proposition 2.4. *Let X be a separable space which contains c_0. Then*

$$\psi_X(k) \ge \frac{1}{2} - \frac{1}{k},\ k \ge 1.$$

Proof. Let $P : X \to c_0$ be a projection with $\|P\| \le 2$ and define a retraction $R : X \to c_0$ by

$$(Rx)(i) = \begin{cases} (Px)(i), & \text{if } |(Px)(i)| \le 1, \\ \frac{(Px)(i)}{|(Px)(i)|}, & \text{if } |(Px)(i)| > 1. \end{cases}$$

Then R is 2-Lipschitz and $R(B_X) \subset R(B_{c_0})$. It is enough to apply Lemma 2.3 since c_0 is extremal. □

The most interesting case is if X contains a 1-complemented copy of c_0.

Theorem 2.5. *Let X be a separable infinite-dimensional Banach space whose dual is an $L_1(\mu)$ space over some measure space (Ω, Σ, μ). Then $\psi_X(k) = 1 - \frac{1}{k}$, i.e., X is an extremal space.*

Proof. It follows from the Zippin theorem (see [11, Theorem 1]) that if X is a separable infinite-dimensional real Banach space whose dual is an $L_1(\mu)$, then X contains a 1-complemented copy of c_0. If X is complex, consider its real part to obtain a 1-Lipschitz retraction $R : X \to c_0$. It is now enough to apply Lemma 2.3. □

By examining the proof of Lemma 2.3 we conclude that every separable predual of $L_1(\mu)$ is in fact strictly extremal. It is well known that $C(K)$, the Banach space of scalar-valued continuous functions on the Hausdorff compact space K, is a predual of some $L_1(\mu)$ (see, e.g., [6]). Hence we obtain the following corollary.

Corollary 2.6. *Every separable infinite-dimensional $C(K)$ space, for some compact Hausdorff space K, is strictly extremal.*

There exists an extensive literature regarding complemented subspaces of Banach spaces. Our next simple observation is concerned with the connection between the existence of a nonexpansive retraction and the fixed point property.

Proposition 2.7. *Suppose that Y is a subspace of a Banach space X and there exists a nonexpasive retraction $R : X \to Y$ such that $R(B_X) \subset B_Y$. If B_X has the fixed point property for nonexpasive mappings, then B_Y has the fixed point property, too.*

Proof. Suppose, contrary to our claim, that there exists a nonexpansive mapping $T : B_Y \to B_Y$ without a fixed point. Then the mapping $T \circ R : B_X \to B_X$ is nonexpansive and fixed point free which contradicts our assumption. □

It is well-known that B_{ℓ_∞} has the fixed point property, whereas B_{c_0} does not have the fixed point property for nonexpansive mappings (see, e.g., [4]). Hence we obtain another proof of the well-known result that there is no nonexpansive retraction from B_{ℓ_∞} into B_{c_0}. In the same way, we conclude that there is no nonexpansive retraction from B_{ℓ_∞} into $B_{\hat{c}_0}$, where $\hat{c}_0$ denotes the space of scalar-valued sequences converging to 0 with respect to a Banach limit. Notice that $\hat{c}_0$ is a strictly extremal subspace of ℓ_∞ of codimension one.

The space $C(K)$ of complex-valued continuous functions on K forms a commutative C^*-algebra under addition, pointwise multiplication and conjugation. The remainder of this paper deals with a special class of C^*-algebras. Recall (see, e.g., [10, Definition 1.5]) that a C^*-algebra $\mathcal{A}$ is called type I if every irreducible $*$-representation $\varphi : \mathcal{A} \to B(H)$ on a Hilbert space H satisfies $K(H) \subset \varphi(\mathcal{A})$ (φ is called irreducible if $\varphi(\mathcal{A})$ has no invariant (closed linear) subspaces other than $\{0\}$ and H). A fundamental example of non-type I C^*-algebra is the CAR (canonical anti-commutation relations) algebra, defined as follows. Identify $B(\ell_2)$ with infinite matrices and define CAR_d to be all $T \in B(\ell_2)$ so that there exist $n \geq 0$ and $A \in M_{2^n}$ (the space of all $2^n \times 2^n$ matrices over $\mathbb{C}$) such that

$$T = \begin{bmatrix} A & & & \\ & A & & \\ & & A & \\ & & & \ddots \end{bmatrix}.$$

The CAR algebra is the norm-closure of CAR_d.

Theorem 2.8 (W. Lusky [8])**.** *Every separable L_1-predual space X (over $\mathbb{C}$) is isometrically isomorphic to a 1-complemented subspace of the CAR algebra.*

Combining a remark after Theorem 2.5 with Theorem 2.8 we deduce that the CAR algebra is strictly extremal. It turns out that the same is true for all separable non-type I C^*-algebras.

Theorem 2.9. *Every separable non-type I C^*-algebra is strictly extremal.*

Proof. It follows from [10, Corollary 1.7] that if $\mathcal{A}$ is a separable non-type I C^*-algebra then the CAR algebra is isometric to a 1-complemented subspace of $\mathcal{A}$. It suffices to follow the arguments of Lemma 2.3 since the CAR algebra is strictly extremal. □

References

[1] K. Bolibok, *The minimal displacement problem in the space ℓ_∞*, in Topics in Nonlinear Analysis: dedicated to Kazimierz Goebel and Lech Górniewicz on the occasion of their 70th birthday, W. Kryszewski and R. Skiba (eds.), Cent. Eur. J. Math. **12** (2012), 2211–2214.

[2] K. Goebel, *On the minimal displacement of points under lipschitzian mappings*, Pacific J. Math. **48** (1973), 151–163.

[3] K. Goebel, *Concise Course on Fixed Point Theorems*, Yokohama Publishers, Yokohama, 2002.

[4] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[5] K. Goebel, G. Marino, L. Muglia and R. Volpe, *The retraction constant and the minimal displacement characteristic of some Banach spaces*, Nonlinear Anal. **67** (2007), 735–744.

[6] H. E. Lacey, *The Isometric Theory of Classical Banach Spaces*, Springer-Verlag, New York-Heidelberg, 1974.
[7] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II. Function spaces*, Springer-Verlag, Berlin-New York, 1979.
[8] W. Lusky, *Every separable L_1-predual is complemented in a C^*-algebra*, Studia Math. **160** (2004), 103–116.
[9] Ł. Piasecki, *Retracting ball onto a sphere in some Banach spaces*, Nonlinear Anal. **74** (2011), 396-399.
[10] H. Rosenthal, *Banach and operator space structure of C^*-algebras*, in Trends in Banach spaces and operator theory, A. Kamińska (ed.), Contemp. Math. 321, Amer. Math. Soc., Providence, RI, 2003, pp. 275–294.
[11] M. Zippin, *On some subspaces of Banach spaces whose duals are L_1 spaces*, Proc. Amer. Math. Soc. **23** (1969), 378–385.

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