Journal of Nonlinear and Convex Analysis
Volume 15, Number 1, 2014, 129–135

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

A PROJECTION METHOD FOR APPROXIMATING FIXED POINTS OF QUASI NONEXPANSIVE MAPPINGS WITHOUT THE USUAL DEMICLOSEDNESS CONDITION

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Abstract. We introduce and analyze an abstract algorithm that aims to find the projection onto a closed convex subset of a Hilbert space. When specialized to the fixed point set of a quasi nonexpansive mapping, the required sufficient condition (termed "fixed-point closed") is less restrictive than the usual conditions based on the demiclosedness principle. A concrete example of a subgradient projector is presented which illustrates the applicability of this generalization.

1. Introduction

Throughout this note, we assume that

(1.1) X is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$.

Suppose that

(1.2) C is a closed convex subset of X, and $x_0 \in X$.

We are interested in finding the projection (nearest point mapping) $P_C x_0$, i.e., the unique solution to the optimization problem

$$d(x_0, C) := \min_{c\in C} \|x_0 - c\|, \tag{1.3}$$

especially when C is the fixed point set of some operator $T\colon X \to X$. It will be convenient to set, for arbitrary given vectors x and y in X,

$$H(x,y) := \big\{z \in X \;\big|\; \|y-z\| \le \|x-z\|\big\} = \big\{z \in X \;\big|\; 2\langle z, x-y\rangle \le \|x\|^2 - \|y\|^2\big\}. \tag{1.4}$$

Note that $H(x,y)$ is equal to either X (if $x = y$) or a halfspace; in any case, the projection onto $H(x,y)$ is easy to compute and has a well known closed form. In order to solve (1.3), we shall study the following simple abstract iteration:

Algorithm 1.1. Recall the assumption (1.2), and set $C_0 = X$. Given $n \in \mathbb{N}$ and $x_n \in X$, pick $y_n \in X$, and set

$$C_{n+1} := C_n \cap H(x_n, y_n) \text{ and } x_{n+1} = P_{C_{n+1}} x_0. \tag{1.5}$$

2010 *Mathematics Subject Classification.* Primary 47H09; Secondary 52B55, 65K10, 90C25.

Key words and phrases. Convex function, demiclosedness principle, fixed point, nonexpansive, polyhedron, projection, quasi nonexpansive, subgradient projection.

This research was carried out during a visit of JC in Kelowna in Fall 2012. HHB was partially supported by the Natural Sciences and Engineering Research Council of Canada and by the Canada Research Chair Program. JC was partially supported by the Academic Award for Excellent Ph.D. Candidates Funded by Wuhan University, the Fundamental Research Fund for the Central Universities, and the Ph.D. short-time mobility program by Wuhan University. XW was partially supported by the Natural Sciences and Engineering Research Council of Canada.

Observe that if the sequence is well defined, then

$$C_0 \supseteq C_1 \supseteq \cdots C_n \supseteq C_{n+1} \supseteq \cdots \tag{1.6}$$

and so

$$\|x_0 - x_n\| = d(x_0, C_n) \le d(x_0, C_{n+1}) = \|x_0 - x_{n+1}\| \tag{1.7}$$

for every $n \in \mathbb{N}$. It then follows that

$$\beta := \lim_{n\in\mathbb{N}} \|x_0 - x_n\| = \sup_{n\in\mathbb{N}} \|x_0 - x_n\| \in [0, +\infty] \tag{1.8}$$

is well defined. Furthermore, if $m < n$, then $x_n \in C_m$ which implies

$$\langle x_n - x_m, x_0 - x_m\rangle \le 0 \tag{1.9}$$

as well as

$$\|y_m - x_n\| \le \|x_m - x_n\| \tag{1.10}$$

because $x_n \in C_n \subseteq C_{m+1} \subseteq H(x_m, y_m)$.

Lemma 1.2. *Suppose that the sequence $(x_n)_{n\in\mathbb{N}}$ is generated by Algorithm 1.1. Suppose also that for every subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of (x_n), we have*

$$\left.\begin{array}{r} x_{k_n} \to \bar{x} \\ x_{k_n} - y_{k_n} \to 0 \end{array}\right\} \;\Rightarrow\; \bar{x} \in C. \tag{1.11}$$

Then every bounded subsequence of $(x_n)_{n\in\mathbb{N}}$ must converge to a point in C.

Proof. Let $(x_{k_n})_{n\in\mathbb{N}}$ be a bounded subsequence of $(x_n)_{n\in\mathbb{N}}$. It follows from (1.7) that $\beta < +\infty$. Let $n > m$. Using (1.9), we obtain

$$\|x_{k_n} - x_{k_m}\|^2 = \|x_{k_n} - x_0\|^2 - \|x_{k_m} - x_0\|^2 + 2\langle x_{k_n} - x_{k_m}, x_0 - x_{k_m}\rangle \tag{1.12a}$$

$$\le \|x_{k_n} - x_0\|^2 - \|x_{k_m} - x_0\|^2 \tag{1.12b}$$

$$\to \beta^2 - \beta^2 = 0 \text{ as } n \ge m \to +\infty. \tag{1.12c}$$

Hence $(x_{k_n})_{n\in\mathbb{N}}$ is a Cauchy sequence. Thus, there exists $\bar{x} \in X$ such that $x_{k_n} \to \bar{x}$. Now, from (1.10), we obtain $\|y_{k_n} - x_{k_{n+1}}\| \le \|x_{k_n} - x_{k_{n+1}}\| \to \|\bar{x} - \bar{x}\| = 0$ and thus $y_{k_n} - x_{k_{n+1}} \to 0$. It follows that $x_{k_n} - y_{k_n} = (x_{k_n} - x_{k_{n+1}}) + (x_{k_{n+1}} - y_{k_n}) \to 0$. Now apply (1.11). □

The previous result allows us to derive the following dichotomy result.

Theorem 1.3 (dichotomy)**.** *Suppose that $(x_n)_{n\in\mathbb{N}}$ is generated by Algorithm 1.1, that $(\forall n \in \mathbb{N})$ $C \subseteq C_n$, and that for every subsequence $(x_{k_n})_{n\in\mathbb{N}}$ of (x_n), we have*

$$\left.\begin{array}{r} x_{k_n} \to \bar{x} \\ x_{k_n} - y_{k_n} \to 0 \end{array}\right\} \;\Rightarrow\; \bar{x} \in C. \tag{1.13}$$

Then exactly one of the following holds:

(i) *$C \ne \varnothing$ and $x_n \to P_C x_0$.*
(ii) *$C = \varnothing$ and $\|x_n\| \to +\infty$.*

Proof. Note that

$$(1.14) \qquad (\forall n \in \mathbb{N}) \quad \|x_0 - x_n\| = d(x_0, C_n) \le d(x_0, C).$$

(i): Assume that $C \neq \varnothing$. Then $(x_n)_{n\in\mathbb{N}}$ is bounded by (1.14). By Lemma 1.2, $\bar{x} := \lim_{n\in\mathbb{N}} x_n \in C$. In turn, (1.14) yields $\|x_0 - \bar{x}\| \le d(x_0, C)$. Therefore, $\bar{x} = P_C x_0$, as claimed.

(ii): Suppose that $\|x_n\| \not\to +\infty$. Then $(x_n)_{n\in\mathbb{N}}$ contains a bounded subsequence which, by Lemma 1.2, must converge to a point in C. Hence if $C = \varnothing$, then $\|x_n\| \to +\infty$. ☐

Remark 1.4. Several comments regarding Theorem 1.3 are in order.

(i) Algorithm 1.1 is related to a method studied by Takahashi et al in [13, Theorem 4.1]. (See also [11, 12, Theorem 2] for Bregman-distance based variants.) While that method is more flexible in some ways, our method has the advantage of requiring neither nonexpansiveness of the given operator nor the nonemptiness of the target set.

(ii) Our proofs are different because we establish strong convergence directly via a Cauchy sequence argument. The proofs mentioned in the previous item are based on a Kadec-Klee property or on Opial's property. (We expect that our proof will generalize to Bregman distances, possibly incorporating errors and families of operators.)

(iii) As we shall see in Section 3 below, our framework encompasses subgradient projectors which are important in optimization.

(iv) The computation of the sequence $(x_n)_{n\in\mathbb{N}}$ requires to compute projections of the *same* initial point x_0 onto polyhedra (intersections of finitely many halfspaces). While this is not necessarily an easy task, this is considered to be a standard quadratic programming problem in convex optimization. Moreover, since C_{n+1} is constructed from C_n by intersecting with the halfspace $H(x_n, y_n)$, it seems plausible to apply *active set methods* (with a warm start) to solve these projections. While a detailed excursion on this matter is beyond the scope of this paper, we do refer the reader to [1, 9, 10] for references on computing projections onto polyhedra.

2. An application to finding nearest fixed points

Recall that $T\colon X \to X$ is called *nonexpansive* if

$$(2.1) \qquad (\forall x \in X)(\forall y \in X) \quad \|Tx - Ty\| \le \|x - y\|;$$

moreover, T is *quasi nonexpansive* if

$$(2.2) \qquad (\forall x \in X)(\forall y \in \operatorname{Fix} T) \quad \|Tx - y\| \le \|x - y\|,$$

where $\operatorname{Fix} T := \big\{x \in X \bigm| x = Tx\big\}$. See [7, 8, 5] for further information on the fixed point theory of nonexpansive mappings.

The next result is readily checked.

Lemma 2.1. *Let $T\colon X \to X$ be quasi nonexpansive. Consider the following properties:*

(i) *T is nonexpansive.*

(ii) T *is continuous.*
(iii) T *is* fixed-point closed, *i.e., if* $x_n \to \bar{x}$ *and* $x_n - Tx_n \to 0$, *then* $\bar{x} \in \operatorname{Fix} T$.

Then (i)⇒(ii)⇒(iii).

Remark 2.2. It is well known that if $T\colon X \to X$ is nonexpansive, then

$$\left.\begin{array}{r} x_n \rightharpoonup \bar{x} \\ x_n - Tx_n \to 0 \end{array}\right\} \;\Rightarrow\; \bar{x} \in \operatorname{Fix} T; \tag{2.3}$$

this is the famous demiclosedness principle — to be precise, this states that $\operatorname{Id} - T$ is demiclosed at 0. For recent results on this principle, see [2] and the references therein. It is clear that demiclosedness of $\operatorname{Id} - T$ at 0 implies that T is fixed-point closed; the converse, however, is false (see Example 3.2 below).

Our main result now yields easily the following result, which by Lemma 2.1 is applicable in particular when T is nonexpansive. (See also [13, Theorem 4.1] for extensions in the nonexpansive case.)

Theorem 2.3 (trichotomy)**.** *Let* $T\colon X \to X$ *be quasi nonexpansive and fixed-point closed, let* $x_0 \in X$*, and set* $C_0 := X$*. Given* $n \in \mathbb{N}$ *and* x_n*, set*

$$C_{n+1} := C_n \cap H(x_n, Tx_n) \ \text{ and } \ x_{n+1} = P_{C_{n+1}} x_0. \tag{2.4}$$

Then exactly one of the following holds:

(i) $\operatorname{Fix} T \neq \varnothing$ *and* $x_n \to P_{\operatorname{Fix} T} x_0$.
(ii) $\operatorname{Fix} T = \varnothing$ *and* $\|x_n\| \to +\infty$.
(iii) $\operatorname{Fix} T = \varnothing$ *and the sequence is not well defined (i.e.,* C_{n+1} *is empty for some* n*).*

Proof. Set $C = \operatorname{Fix} T$, and $(y_n)_{n\in\mathbb{N}} = (Tx_n)_{n\in\mathbb{N}}$ provided that $(x_n)_{n\in\mathbb{N}}$ is well defined. In this case, it is clear that (1.11) holds because T is fixed-point closed.

(i): Assume that $C \neq \varnothing$. If $C_n \neq \varnothing$ and $C \subseteq C_n$, then $(\forall c \in C)$ $\|Tx_n - c\| \le \|x_n - c\|$ and so $c \in H(x_n, Tx_n)$. It follows that $C \subseteq C_{n+1}$ and the sequence $(x_n)_{n\in\mathbb{N}}$ is well defined. The conclusion thus follows from Theorem 1.3.

(ii)&(iii): Assume that $C = \varnothing$. If $(x_n)_{n\in\mathbb{N}}$ is not well defined, then (iii) happens. Finally, if $(x_n)_{n\in\mathbb{N}}$ is well defined, then (ii) occurs again by Theorem 1.3. ☐

Let us now illustrate the three alternatives in Theorem 2.3.

Example 2.4. Suppose that $X = \mathbb{R}$ and set $T := \alpha \operatorname{Id}$, where $\alpha \in [0,1[$. Then T is nonexpansive with $\operatorname{Fix} T = \{0\}$. Let $x_0 \ge 0$. Then $Tx_0 = \alpha x_0$ and $C_1 =]-\infty, (\alpha+1)/2x_0]$. Thus, $x_1 = (\alpha+1)/2x_0$. It follows inductively that $(x_n)_{n\in\mathbb{N}}$ is well defined and

$$(\forall n \in \mathbb{N}) \quad x_n = \big((\alpha+1)/2\big)^n x_0 \to 0 = P_{\operatorname{Fix} T} x_0, \tag{2.5}$$

as is also guaranteed by Theorem 2.3(i).

Example 2.5. Suppose that $X = \mathbb{R}$ and set $T\colon X \to X\colon x \mapsto x + \alpha$, where $\alpha > 0$. Clearly, T is nonexpansive and $\operatorname{Fix} T = \varnothing$. One checks that $x_n = x_0 + n\alpha/2$; hence, $|x_n| \to +\infty$.

Example 2.6. Suppose $X = \mathbb{R}$, let $\sigma\colon X \to \{-1, +1\}$, and set $T_\sigma\colon X \mapsto X : x \mapsto x + \sigma(x)$. For trivial reasons, T_σ is quasi nonexpansive (since $\operatorname{Fix} T_\sigma = \varnothing$) and T_σ is fixed-point closed (since $\operatorname{ran}(\operatorname{Id} - T_\sigma) \subseteq \{+1, -1\}$). We now assume that $\sigma(0) = 1$ and $\sigma(1/2) = -1$. Let $x_0 = 0$. Then $C_1 = [1/2, +\infty[$, $x_1 = 1/2$ and $C_2 = C_1 \cap \,]-\infty, 0] = \varnothing$, which means the algorithm terminates.

3. Subgradient projector

The astute reader will ask whether the fairly general assumptions on T in Theorem 2.3, i.e., that "T be quasi nonexpansive and fixed-point closed", are really needed in applications. In this section, we provide an example that not only requires this generality but that also does not satisfy the usual demiclosedness type assumptions seen in this area.

To this end, let

$$f\colon X \to \mathbb{R} \tag{3.1}$$

be convex, continuous, and Gâteaux differentiable such that $f \geq 0$ and

$$C := \big\{x \in X \;\big|\; f(x) \leq 0\big\} = \{0\}. \tag{3.2}$$

Write $g := \nabla f$ for convenience. The *subgradient projector* in this case is defined by

$$T\colon X \to X : x \mapsto \begin{cases} x, & \text{if } x = 0;\\ x - \frac{f(x)}{\|g(x)\|^2} g(x), & \text{if } x \neq 0. \end{cases} \tag{3.3}$$

Then it follows (from e.g., [4, Proposition 2.3]) that T is *quasi firmly nonexpansive*, i.e.,

$$(\forall x \in X)(\forall y \in \operatorname{Fix} T) \quad \|Tx - y\|^2 + \|x - Tx\|^2 \leq \|x - y\|^2. \tag{3.4}$$

Lemma 3.1. *The following hold:*

(i) *T is quasi nonexpansive.*
(ii) *T is fixed-point closed.*
(iii) *T is continuous at 0.*
(iv) *If f is Fréchet differentiable, then T is continuous.*

Proof. (i): This follows immediately from (3.4).

(ii): Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \to \bar{x}$ and $x_n - Tx_n \to 0$. We assume that $\bar{x} \neq 0$ (for if $\bar{x} = 0$, then the conclusion is trivially true) and that $(x_n)_{n\in\mathbb{N}}$ lies in $X \smallsetminus \{0\}$. To reach the required contradiction, observe first that the continuity of f yields $f(x_n) \to f(\bar{x}) > 0$. Now $x_n - Tx_n \to 0 \Leftrightarrow \|x_n - Tx_n\| \to 0 \Leftrightarrow f(x_n)/g(x_n) \to 0$; thus,

$$\lim_{n\in\mathbb{N}} \|g(x_n)\| = +\infty. \tag{3.5}$$

On the other hand, g is strong-to-weak continuous (see, e.g., [5, Proposition 17.31]); therefore, the sequence $(g(x_n))_{n\in\mathbb{N}}$ converges weakly to $g(\bar{x})$. In particular, $(g(x_n))_{n\in\mathbb{N}}$ is bounded — but this contradicts (3.5).

(iii): Convexity yields $(\forall x \in X \smallsetminus \{0\})$ $\langle 0 - x, \nabla f(x)\rangle \leq f(0) - f(x)$, which implies $f(x) \leq \langle x, g(x)\rangle \leq \|x\|\|g(x)\|$; thus, $f(x)/\|g(x)\| \leq \|x\|$. Hence $\lim_{x\to 0} Tx = 0 = T0$, as claimed.

(iv): If f is Fréchet differentiable, then g is strong-to-strong continuous (see, e.g., [5, Proposition 17.32]), which in turn yields the continuity of T on

$$\big\{x \in X \;\big|\; g(x) \neq 0\big\} = X \smallsetminus \{0\}.$$

□

Note that Lemma 3.1 guarantees the applicability of Theorem 2.3 to the subgradient projector T.

Example 3.2. Suppose that $X = \ell^2 = \big\{\mathbf{x} = (x_n)_{n\geq 1} \;\big|\; \sum_{n\geq 1} |x_n|^2 < +\infty\big\}$ and set

$$f\colon X \to \mathbb{R}\colon \mathbf{x} = (x_n)_{n\geq 1} \mapsto \sum_{n\geq 1} n x_n^{2n}. \tag{3.6}$$

Then f is well defined, convex, and continuous (see [3, Example 7.11]). Moreover, f is Gâteaux differentiable with $g(\mathbf{x}) = \nabla f(\mathbf{x}) = (2n^2 x_n^{2n-1})_{n\geq 1}$. Denote the sequence of standard unit vectors by $(\mathbf{e}_n)_{n\geq 1}$, and set

$$(\forall n \geq 1) \quad \mathbf{x}_n := \mathbf{e}_1 + \mathbf{e}_n \rightharpoonup \mathbf{e}_1 \tag{3.7}$$

For $n \geq 2$, we have $f(\mathbf{x}_n) = 1 + n$, $g(\mathbf{x}_n) = 2\mathbf{e}_1 + 2n^2\mathbf{e}_n$; hence $\|g(\mathbf{x}_n)\| = \sqrt{4 + 4n^4}$ and thus $f(\mathbf{x}_n)/\|g(\mathbf{x}_n)\| \to 0$. It follows that $\mathbf{x}_n - T(\mathbf{x}_n) \to 0$. Since

$$\left.\begin{array}{c} \mathbf{x}_n \rightharpoonup \mathbf{e}_1 \\ \mathbf{x}_n - T(\mathbf{x}_n) \to 0 \end{array}\right\} \;\not\Rightarrow\; \mathbf{e}_1 = 0, \tag{3.8}$$

we see that $\operatorname{Id} - T$ is *not* demiclosed at 0 and that T is not weak-to-weak continuous however, T is fixed-point closed by Lemma 3.1(ii).

Remark 3.3. Some comments regarding Example 3.2 are in order.

- (i) This example illustrates that some of the sufficient conditions demiclosedness type conditions provided in the literature (see, e.g., [6, Proposition 2.2]) to guarantee convergence are actually not applicable to the subgradient projector T of the function f defined in Example 3.2. However, Theorem 2.3 is applicable with T because of Lemma 3.1.
- (ii) Some additional work (which we omit here) shows that f is actually Fréchet differentiable on X. Thus, by Lemma 3.1(iv), T is actually strong-to-strong continuous.
- (iii) It also follows from the classical demiclosedness principle that T is not nonexpansive.

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Manuscript received March 20, 2013
revised April 24, 2013

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