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SUFFICIENT OPTIMALITY CONDITIONS FOR GLOBAL PARETO SOLUTIONS TO MULTIOBJECTIVE PROBLEMS WITH EQUILIBRIUM CONSTRAINTS

TRUONG Q. BAO AND BORIS S. MORDUKHOVICH

Dedicated to Simeon Reich in honor of his 65th birthday

Abstract. We present new sufficient conditions for Pareto optimal solutions to constrained multiobjective problems with and without equilibrium constraints. They are derived by using advanced tools and techniques of variational analysis and generalized differentiation.

1. Introduction

This paper concerns the study of sufficient conditions for Pareto minimizers to multiobjective optimization problems, which typically have at least two conflicting objectives in the sense that the gain of one objective is the expense of the other. The concept of Pareto efficiency/optimality has been widely recognized to be useful in the theory and applications of multiobjective/vector optimization problems. A number of variants and extensions of this fundamental concept have been introduced in the literature; see, e.g., [10, 15, 18, 19, 24] and the references therein.

Let Z be a vector space ordered by a closed convex cone Θ. Denoting the ordering relation on Z under consideration by "$\leq_\Theta$", we have its description

$$z_1 \leq_\Theta z_2 \ \text{ if and only if } \ z_1 \in z_2 - \Theta. \tag{1.1}$$

Given a set Ξ in Z. We say that a point $\bar z \in \Xi$ is a (global) *Pareto maximal point* of Ξ if and only if there is no points $z \in \Xi \setminus \{\bar z\}$ such that $\bar z \leq_\Theta z$ or, equivalently,

$$\Xi \cap (\bar z + \Theta) = \{\bar z\}.$$

Similarly, $\bar z \in \Xi$ is a *Pareto minimal point* of Ξ if there is no points $z \in \Xi \setminus \{\bar z\}$ with $z \leq_\Theta \bar z$, i.e., we have

$$\Xi \cap (\bar z - \Theta) = \{\bar z\}.$$

Denote the set of all the maximal points and the collection of all the minimal points to Ξ with respect to the ordering cone Θ by $\mathrm{Max}\,(\Xi;\Theta)$ and $\mathrm{Min}\,(\Xi;\Theta)$, respectively. Obviously, $\bar z \in \mathrm{Max}\,(\Xi;\Theta)$ if and only if $-\bar z \in \mathrm{Min}\,(-\Xi;\Theta)$. Therefore, it is sufficient

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to study optimality conditions for maximality only since the corresponding results for the other can be easily derived from them.

We pay our main attention to the following two general classes of constrained multiobjective optimization problems. The first one concerns *multiobjective optimization problems* with *geometric constraints* (MOPGC in short) given by

$$\begin{cases} \Theta\text{-maximize} & F(x) \\ \text{subject to} & x \in \Omega, \end{cases} \tag{1.2}$$

where the *cost* $F : X \rightrightarrows Z$ is a set-valued mapping between Banach spaces with the range space Z *ordered* by a convex and closed cone $\Theta \subset Z$, and where $\Omega \subset X$ is a nonempty *constraint set*. The second class consists of *multiobjective optimization problems* with *equilibrium constraints* (MOPEC in short) written as

$$\begin{cases} \Theta\text{-maximize} & F(x) \\ \text{subject to} & \mathbf{0} \in G(x,y) + Q(x,y), \\ & x \in \Omega. \end{cases} \tag{1.3}$$

We see that, in addition to geometric constraints, MOPEC (1.3) contain "equilibrium constraints" modeled by the *extended generalized equations* (EGE)

$$\mathbf{0} \in G(x,y) + Q(x,y), \tag{1.4}$$

where the set-valued mappings $G\colon X \times Y \rightrightarrows W$ and $Q\colon X \times Y \rightrightarrows W$ are between Banach spaces. For convenience we use the terms *base* and *field* referring to the mappings G and Q of EGE (1.4), respectively, which are commonly used for the single-valued function $g : X \times Y \to W$ and a set-valued mapping Q. In the latter case, they play essentially distinct roles; cf. [23] for more discussions. Model (1.4) is a base set-valued extension of the generalized equation formalism by Robinson [27] well recognized in optimization theory and applications. This extension has been suggested in [2] in order to describe solution mappings to certain classes of nonsmooth optimization problems including bilevel programming with nonsmooth lower-level problems, set-valued parametric variational inequalities, etc. Then it has been further discussed in [3, 7] that the EGE framework (1.4) is a convenient formalism to model important classes of constraints in optimization, namely:

- Solution maps $y \in S(x)$ for various *equilibrium constraints* [20, 23, 24] can be converted to the EGE form with $G(x,y) := -y$ and $Q(x,y) := S(x)$.
- Parametric *generalized equations* of the type $\mathbf{0} \in g(x,y) + Q(x,y)$ with single-valued base mappings $g\colon X \times Y \to W$ depending on the decision variable $y \in Y$ and the parameter $x \in X$ are specifications of (1.4) with $G = g : X \times Y \to W$.
- Generalized equations of the *variational types* described via a normal cone operator $\mathbf{0} \in g(x,y) + N(y;\Omega)$ or via a subdifferential operator $\mathbf{0} \in g(x,y) + \partial\varphi(y)$ are (1.4) with $G = g : X \times Y \to Y^*$ and $Q(x,y) := N(\cdot;\Omega)\colon Y \rightrightarrows Y^*$ or $Q(x,y) := \partial\varphi(\cdot)\colon Y \rightrightarrows Y^*$. Robinson's original framework [27] corresponds to the normal cone form $Q(y) = N(y;\Omega)$ with a convex set Ω. This covers classical variational inequalities and complementarity problems, KKT systems in nonlinear programming, etc.

- *Operator constraints* of the type $x \in G^{-1}(\Lambda) \Longleftrightarrow G(x) \cap \Lambda \neq \emptyset$ with a set-valued mapping $G\colon X \rightrightarrows W$ and a subset $\Lambda \subset W$ can be formulated in form (1.4) with both set-valued mappings $G(x,y) := G(x)$ and $Q(x,y) \equiv -\Lambda$.
- *Fixed-point constraints* $x \in Q(x)$ with a set-valued mapping $Q\colon X \rightrightarrows X$ can be written in form (1.4) with $Q(x)$ and $G(x) := -x$.
- *Inequality constraints* of $\varphi_i(x) \le 0,\ i = 1, \ldots, m$, are given by (1.4) with

$$G(x) := \prod_{i=1}^{m} \big[\varphi_i(x), \infty\big) \quad \text{and} \quad Q(x) := I\!R^m_+.$$

- *Equality constraints* of $\varphi_i(x) = 0,\ i = 1, \ldots, p$, can be combined to one extended equation (1.4) with the problem data

$$G(x) := \big(\varphi_{m+1}(x), \ldots, \varphi_{m+r}(x)\big) \quad \text{and} \quad Q(x) := \{\mathbf{0}\} \in I\!R^p.$$

- *Geometric constraints* of the type $x \in \Omega$, where Ω is a subset of a Banach space X, can be considered in form (1.4) with $G(x) := \{-x\}$ and $Q(x) \equiv \Omega$.

The last observation indicates the possibility to exclude explicit geometric constraints from the formulation of MOPEC (1.3). However, the results obtained in this paper show that keeping the MOPEC form as in (1.3) allows us to treat geometric constraints more efficiently; in particular, in infinite-dimensional settings, where a certain Lipschitzian behavior of cost mappings allows us to avoid restrictive compactness assumptions on geometric constraints.

Since MOPGC (1.2) is obviously a special case of MOPEC (1.3), it seems natural to focus on the study of the latter problem. However, we choose the opposite way concentrating first on the study of the simpler problem (1.2) and then deriving the corresponding results for (1.3) from those for (1.2). This approach has been employed in [3] to obtain necessary optimality conditions for general MOPEC (1.3) by implementing extensive *calculus* for the generalized differential constructions involved; see Section 2. In this paper we proceed in a similar way to derive sufficient optimality conditions.

To motivate our interest to the study of MOPEC, recall that in the case of real-valued cost functions $F = \varphi\colon X \to I\!R$ problems (1.3) reduce to the well-recognized in optimization theory class of *mathematical programs with equilibrium constraints* (MPEC), which is challenging theoretically while having numerous applications to operations research, engineering, mechanics, economics, multilevel games, and other areas; see [14, 20, 24, 26] for more details and discussions. One of the most important sources of MPEC in optimization theory relates to problems of parametric optimization

$$\text{(1.5)} \qquad \underset{y}{\text{minimize}} \quad \varphi(x,y) \quad \text{subject to} \quad y \in \Xi(x) \subset Y,$$

where y is the decision variable and x is the parameter. If φ is differentiable in y, the corresponding KKT system is given by the equilibrium constraint

$$\text{(1.6)} \qquad \mathbf{0} \in \nabla_y \varphi(x,y) + N\big(y; \Xi(x)\big)$$

expressed via the appropriate normal cone to the moving constraint set $\Xi(x)$; see Section 2. If the set Ξ is parameter-independent and convex, the KKT system (1.6) amounts to the classical variational inequalities as in the original Robinson's model, while the case of moving convex sets in (1.6) relates to the study of quasi-variational inequalities in the way initiated by Outrata; see [26] and also [25] for more details. Considering in this way the parametric problem (1.5) with a *nonsmooth* cost φ brings us to the generalized KKT system

$$\mathbf{0} \in \partial_y \varphi(x,y) + N\big(y; \Xi(x)\big), \tag{1.7}$$

which is of the EGE type (1.4); see [2, 3]. Note that more general fields in (1.4) in comparison with (1.6) and (1.7) are given in the composite subdifferential forms (cf. [23, Chapter 4]):

- $Q := \partial(\psi \circ g)$ with $g : X \times Y \to W$ and $\psi : W \to I\!R \cup \{\infty\}$.
- $Q := (\partial\psi \circ g)$ with $g : X \times Y \to W$ and $\psi : W \to I\!R \cup \{\infty\}$.

In this paper we develop a *dual-space approach* to study multiobjective optimization problems as MOPGC and MOPEC by employing the basic constructions of generalized differentiation in variational analysis discussed in Section 2. Elements of this approach concerning *necessary* optimality conditions for various classes of multiobjective problems can be found in [1–8, 11, 12, 16, 24, 30] and the references therein. In contrast to the aforementioned works we concentrate now on deriving *sufficient* optimality conditions for the appropriate classes of MOPGC and MOPEC considered below. Some results in this direction can be found in [8,9] for *weak Pareto* solutions to MOPGC models. Now we address the *Pareto optimality*, concerning global Pareto maximizers without interiority assumptions on the ordering cone in both MOPGC and MOPEC models; this is more challenging theoretically and much more important for applications.

The rest of the paper is organized as follows. Section 2 contains some basic definitions and preliminary results on generalized differentiation, which are widely used in the sequel. In Section 3 we establish new *sufficient optimality conditions* for *Pareto maximizers* of the multiobjective problem (1.2). It is important to emphasize that the study of Pareto optimality is much more involved, especially in infinite-dimensional spaces, in comparison with the weak Pareto counterpart considered in [9]. Indeed, in the Pareto case we do not have the nonempty interiority property of ordering cones, which allows us to employ separation theorems for nonconvex sets in primal-space approaches to vector optimization or automatically ensures some kinds of compactness in the dual space used in the dual-space developments. The approach of this paper is based on variational principles and related calculus rules of generalized differentiation. Generalized differential calculus is instrumental in deriving sufficient conditions for Pareto maximizers in the main class of MOPEC (1.3) considered in Section 4. The final Section 5 contains concluding remarks and some open questions of the future research.

2. Tools of variational analysis and generalized differentiation

Throughout the paper we use the conventional notation of variational analysis and generalized differentiation; see, e.g., [23, 29]. Given a Banach space X, denote its norm by $\|\cdot\|$, its dual space equipped with the weak* topology w^* by X^*, and the canonical pairing between X and X^* by $\langle \cdot, \cdot \rangle$. The symbols $\mathbb{B}$ and $\mathbb{B}^*$ stand for the closed unit balls of the space in question and its topological dual. Given a set Ω in X, the expressions $\mathrm{cl}\,\Omega$, $\mathrm{bd}\,\Omega$, and $\mathrm{int}\,\Omega$ signify the closure, boundary, and interior of Ω, respectively. We use the notation $x \xrightarrow{\Omega} \bar{x}$ to indicate that $x \to \bar{x}$ with $x \in \Omega$. For a set-valued mapping $F : X \rightrightarrows Z$ between two Banach spaces, its domain and graph are given by

$$\operatorname{dom} F := \big\{x \in X \;\big|\; F(x) \neq \emptyset\big\} \text{ and } \operatorname{gph} F := \big\{(x,z) \in X \times Z \;\big|\; z \in F(x)\big\},$$

respectively. The inverse mapping of F is defined by $F^{-1}(z) := \{x \in X |\; z \in F(x)\}$.

Recall that a Banach space X is *Asplund* if any of its separable subspace has a separable dual. This is a broad subclass of Banach spaces including, in particular, every reflexive space; see, e.g., [23] and the references therein. The classical spaces c_0, ℓ^p, and $L^p[0,1]$ with $1 < p < \infty$ are Asplund spaces while $C[0,1]$, ℓ_1, and ℓ_∞ are not. Since the main results of this paper hold in Asplund spaces, we assume that all the spaces under consideration are Asplund.

Following [23], let us begin with the definition of the basic construction of generalized normals to nonempty sets. Given a set $\emptyset \neq \Omega \subset X$, the *prenormal cone* (known also as the regular or Fréchet normal cone) to Ω at $\bar{x} \in \Omega$ is defined by

$$\widehat{N}(\bar{x};\Omega) := \Big\{x^* \in X^* \;\Big|\; \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \le 0\Big\}$$

and the *normal cone* to Ω at $\bar{x}$, known also as the basic/limiting/Mordukhovich normal cone, is defined by

$$\begin{aligned} N(\bar{x};\Omega) := \Big\{x^* \in X^* \;\Big|\; &\exists\, x_k \xrightarrow{\Omega} \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ as } k \to \infty \\ &\text{with } x_k^* \in \widehat{N}(x_k;\Omega) \text{ for all } k \in \mathbb{N} := \{1,2,\ldots\}\Big\}. \end{aligned} \tag{2.1}$$

When $\Omega = I\!R^n$ and the set Ω is locally closed around $\bar{x}$, by [23, Theorem 1.6], the cone (2.1) can be equivalently described as

$$\begin{aligned} N(\bar{x};\Omega) = \Big\{v \in I\!R^n \Big| \;&\exists \text{ sequences } x_k \to \bar{x},\ \alpha_k \ge 0,\ w_k \in \Pi(x_k;\Omega) \\ &\text{such that } \alpha_k(x_k - w_k) \to v \text{ as } k \to \infty\Big\}, \end{aligned}$$

via the Euclidean projector $\Pi(x;\Omega)$ of Ω; this was in fact the original definition in [21]. Note that the normal cone (2.1) and the associated coderivative and subdifferential constructions for mappings presented below are often *nonconvex* while enjoying *full calculus* based on *variational/extremal principles* of variational analysis; see [23, 29] and the references therein.

Given a set-valued mapping $F\colon X \rightrightarrows Z$ between arbitrary (Asplund) spaces, consider two limiting coderivative constructions reduced to the original one [22] in finite dimensions; see [23] for more details. The *normal coderivative* $D_N^*F(\bar{x},\bar{z}) : Z^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{z})$ is defined by

$$\begin{aligned} D_N^*F(\bar{x},\bar{z})(z^*) &:= \big\{x^* \in X^* \big|\; (x^*, -z^*) \in N\big((\bar{x},\bar{z});\operatorname{gph} F\big)\big\} \\ &= \Big\{x^* \in X^* \;\Big|\; \exists\, (x_k, z_k) \xrightarrow{\operatorname{gph} F} (\bar{x},\bar{z}),\ (x_k^*, z_k^*) \xrightarrow{w^*} (x^*, z^*) \\ &\qquad \text{with } (x_k^*, -z_k^*) \in \widehat{N}\big((x_k,z_k);\operatorname{gph} F\big) \text{ for all } k \in \mathbb{N}\Big\}. \end{aligned} \tag{2.2}$$

The *mixed coderivative* $D_M^*F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{z})$ is defined by replacing the weak* convergence for sequences of dual elements in Z^* in (2.2) with the norm convergence of them, namely:

$$
\begin{aligned}
D^*_M F(\bar{x},\bar{z})(z^*) := \Big\{ x^* \in X^* \Big|\ & \exists\, (x_k,z_k) \stackrel{\mathrm{gph}\, F}{\longrightarrow} (\bar{x},\bar{z}),\ x^*_k \stackrel{w^*}{\longrightarrow} x^*, \\
z^*_k \stackrel{\|\cdot\|}{\longrightarrow} z^* \ \text{ with }\ (x^*_k,-z^*_k) &\in \widehat{N}\big((x_k,z_k);\mathrm{gph}\, F\big) \ \text{ for all }\ k\in\mathbb{N} \Big\}.
\end{aligned} \tag{2.3}
$$

We omit $\bar{z}=f(\bar{x})$ in the coderivative notation if $F=f\colon X\to Z$ is single-valued. It immediately follows from definitions (2.2) and (2.3) that

$$
D^*_M F(\bar{x},\bar{z})(z^*) \subset D^*_N F(\bar{x},\bar{z})(z^*) \ \text{ for all }\ z^*\in Z^*, \tag{2.4}
$$

where the equality surely holds when $\dim Z<\infty$ while not in general. We say that F is *strongly coderivatively normal* at $(\bar{x},\bar{z})$ if inclusion (2.4) holds as equality. Some classes of strongly coderivatively normal mappings with images in infinite-dimensional spaces are listed in [23, Proposition 4.9].

Let now the image space Z of F be *ordered* by some order relation $\le_\Theta$ from (1.1) via a closed and convex ordering cone $\Theta\subset Z$. This setting allows us to formulate the following *subdifferential* notions for set-valued mappings defined and studied in [3–5] as vector counterparts of the corresponding subdifferentials by Mordukhovich [21,23] for scalar (extended-real-valued) functions. Given $F\colon X\rightrightarrows Z$ with Z ordered by Θ, consider first the *epigraph* of F with respect to Θ given by

$$
\mathrm{epi}\, F := \big\{(x,z)\in X\times Z \big|\ z\in F(x)+\Theta\big\}
$$

and observe that $\mathrm{epi}\, F=\mathrm{gph}\, F$ if $\Theta=\{\mathbf{0}\}$ while $\mathrm{gph}\, F\subset\mathrm{epi}\, F$ otherwise. Let $\mathcal{E}_F\colon X\rightrightarrows Z$ be the *epigraphical multifunction* associated with F by $\mathcal{E}_F(x):=F(x)+\Theta$. We obviously have $\mathrm{gph}\,\mathcal{E}_F=\mathrm{epi}\, F$ and then omit Θ in the epigraph notation $\mathrm{epi}_\Theta F$ and the epigraphical multifunction $\mathcal{E}_{F,\Theta}$ for simplicity. Applying the coderivative operators to the epigraphical multifunction $\mathcal{E}_F$, define the following:

- The *basic subdifferential* of F at $(\bar{x},\bar{z})\in\mathrm{epi}\, F$ *in direction* z^* ($\|z^*\|=1$) is

$$
\begin{aligned}
\partial F(\bar{x},\bar{z})(z^*) &:= D^*_N\mathcal{E}_F(\bar{x},\bar{z})(z^*) \\
&= \big\{x^*\in X^* \big|\ (x^*,-z^*)\in N\big((\bar{x},\bar{z});\mathrm{epi}\, F\big)\big\}.
\end{aligned} \tag{2.5}
$$

- The *basic subdifferential* of F at $(\bar{x},\bar{z})$ is

$$
\begin{aligned}
\partial F(\bar{x},\bar{z}) &:= \bigcup_{\|z^*\|=1} \partial F(\bar{x},\bar{z})(z^*) \\
&= \bigcup\Big\{\partial F(\bar{x},\bar{z})(z^*) \mid -z^*\in N(\mathbf{0};\Theta),\ \|z^*\|=1\Big\}.
\end{aligned} \tag{2.6}
$$

- The *singular subdifferential* of F at $(\bar{x},\bar{z})$ is

$$
\partial^\infty F(\bar{x},\bar{z}) := D^*_M\mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0}). \tag{2.7}
$$

As usual, we drop $\bar{z}=f(\bar{x})$ in the subdifferential notation if $F=f\colon X\to Z$ is single-valued and do not mention the ordering cone Θ therein for simplicity. It follows from [23, Theorem 4.10] that $\partial^\infty F(\bar{x},\bar{z})=\{\mathbf{0}\}$ if F is *epigraphically Lipschitz-like* (*ELL*) around this point, which means that its epigraphical multifunction $\mathcal{E}_F$ is Lipschitz-like around this point. The Lipschitz-like (known also Aubin or pseudo-Lipschitz) property has been recognized as a fundamental property of set-valued mappings equivalent to metric regularity and linear openness of the inverses; see [23, 29] and the references therein for more details.

It is not hard to check the validity of the implication

$$\partial F(\bar{x},\bar{z})(z^*) \neq \emptyset \Longrightarrow -z^* \in N(\mathbf{0};\Theta), \tag{2.8}$$

which shows that the requirement $-z^* \in N(\mathbf{0};\Theta)$ in construction (2.6) is abundant. Observe also that [7, Proposition 3.2] ensures the relationships

$$\partial f(\bar{x},\bar{z})(z^*) \subset \partial f\big(\bar{x},f(\bar{x})\big)(z^*) \subset D_N^* f(\bar{x})(z^*)$$

for all $(\bar{x},\bar{z}) \in \operatorname{epi} f$ and $z^* \in Z^*$ in the case of single-valued mappings $f\colon X \to Z$.

In some results below we impose as an alternative assumption the following property: $F\colon X \rightrightarrows Z$ exhibits *singular subdifferential regularity* at $(\bar{x},\bar{z}) \in \operatorname{epi} F$ if the mixed coderivative in (2.7) can be replaced by the normal one, i.e.,

$$\partial^\infty F(\bar{x},\bar{z}) = D_N^* \mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0}). \tag{2.9}$$

This is automatic provided that the ordering cone Θ is SNC at the origin (see below); in particular, either $\dim Z < \infty$ or $\operatorname{int}\Theta \neq \emptyset$. To justify it, observe that

$$\partial^\infty F(\bar{x},\bar{z}) = D_M^* \mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0}) \subset D_N^* \mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0}).$$

Hence it remains to show that $D_N^* \mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0}) \subset D_M^* \mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0})$. Pick any $x^* \in D_N^* \mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0})$, which is equivalent to $(x^*,\mathbf{0}) \in N\big((\bar{x},\bar{z});\operatorname{epi} F\big)$. By the definition of limiting normals there are sequences $\{(x_k,z_k,x_k^*,z_k^*)\}$ such that

$$(x_k,z_k) \xrightarrow{\text{epi F}} (\bar{x},\bar{z}),\ (x_k^*,z_k^*) \in \widehat{N}\big((x_k,z_k);\operatorname{epi} F\big) \ \text{ with } \ (x_k^*,z_k^*) \xrightarrow{w^*} (x^*,\mathbf{0}), \tag{2.10}$$

which ensure the relationships

$$z_k^* \in \widehat{N}(\mathbf{0};\Theta) \ \text{ with } \ z_k^* \xrightarrow{w^*} \mathbf{0}$$

due to (2.8). Taking into account the SNC property of Θ at $\mathbf{0}$, we obtain $\|z_k^*\| \to 0$ as $k \to \infty$. The strong convergence of the sequence $\{z_k^*\}$ together with (2.10) clearly justifies that $x^* \in D_M^* \mathcal{E}_F(\bar{x},\bar{z})(\mathbf{0})$.

Next we recall several "sequential normal compactness" properties of sets and mappings, which are automatic in finite dimensions while being a crucial ingredient of variational analysis in infinite dimensions; see the books [23, 24] for a comprehensive theory and numerous applications of various properties of this type. Let Ω be a subset of the product space $X \times Z$. Then:

- Ω is *sequentially normally compact* (SNC) at $\bar{v} := (\bar{x},\bar{z}) \in \Omega$ if for any sequences

$$v_k \xrightarrow{\Omega} \bar{v} \ \text{ and } \ (x_k^*,z_k^*) \in \widehat{N}(v_k;\Omega), \quad \forall\, k \in \mathbb{N}, \tag{2.11}$$

we have the implication $(x_k^*,z_k^*) \xrightarrow{w^*} \mathbf{0} \Longrightarrow (x_k^*,z_k^*) \xrightarrow{\|\cdot\|} \mathbf{0}$ as $k \to \infty$.

- Ω is *partially SNC* (PSNC) with respect to X at $\bar{v} \in \Omega$ if for any sequences (v_k,x_k^*,z_k^*) satisfying (2.11) we have the implication

$$\big[x_k^* \xrightarrow{w^*} \mathbf{0} \ \text{ and } \ z_k^* \xrightarrow{\|\cdot\|} \mathbf{0}\big] \Longrightarrow x_k^* \xrightarrow{\|\cdot\|} \mathbf{0} \ \text{ as } \ k \to \infty.$$

- Ω is *strongly PSNC* with respect to X at $\bar{v} \in \Omega$ if for any sequences (v_k,x_k^*,z_k^*) satisfying (2.11) we have the implication

$$\big[(x_k^*,z_k^*) \xrightarrow{w^*} \mathbf{0}\big] \Longrightarrow x_k^* \xrightarrow{\|\cdot\|} \mathbf{0} \ \text{ as } \ k \to \infty.$$

If $F\colon X \rightrightarrows Z$ is a general *mapping*, its *SNC* and *PSNC* properties at $(\bar{x},\bar{z}) \in \operatorname{gph} F$ are induced by those for its graph. When in addition Z is ordered by the cone Θ, we can involve the epigraph of F and get the following versions:

- F is *sequentially normally epigraphically compact* (SNEC) at $(\bar{x},\bar{z}) \in \operatorname{epi} F$ if the epigraphical multifunction $\mathcal{E}_F$ of F with respect to Θ is SNC at $(\bar{x},\bar{z})$.
- F is *partially SNEC* (PSNEC) at $(\bar{x},\bar{z}) \in \operatorname{epi} F$ if $\mathcal{E}_F$ is PSNC at $(\bar{x},\bar{z})$.

It follows from [23, Theorem 4.10] that F is PSNC at $(\bar{x},\bar{z}) \in \operatorname{gph} F$ (resp. PSNEC at $(\bar{x},\bar{z}) \in \operatorname{epi} F$) provided that F is Lipschitz-like (resp. epigraphically Lipschitz-like) around this point. Note finally that F is SNEC at $(\bar{x},\bar{z}) \in \operatorname{epi} F$ if and only if Θ is SNC at the origin and F is PSNEC at $(\bar{x},\bar{z})$; see Remark (iii) below.

To conclude this section, we list *calculus rules* from [23, 24], which are needed in the proofs of our main results in Sections 3 and 4:

- [23, Theorem 1.44] mixed coderivatives of Lipschitzian mappings;
- [23, Theorem 3.4] basic normals to set intersections in product spaces;
- [23, Corollary 3.5] intersection rule under the SNC condition;
- [23, Corollary 3.80] PSNC sets in products of two spaces;
- [23, Corollary 3.81] SNC property of set intersections;
- [23, Theorem 4.10] pointbased characterizations of Lipschitz-like property;
- [24, Proposition 5.3] lower subdifferential conditions for local minima under geometric constraints.

3. Sufficient conditions in set-valued optimization with geometric constraints

In this section we derive new sufficient optimality conditions for Pareto maximizers of multiobjective optimization problems with geometric constraints described in (1.2), where the definition of Pareto maximization is understood in the sense described in what follows.

Let $F\colon X \rightrightarrows Z$ be a set-valued cost in (1.2), where Z is ordered by the ordering relation $\le_\Theta$ from (1.1) generated by an ordering cone $\Theta \subset Z$. In contrast to single-valued costs (there is at most one output for each input), for each $\bar{x} \in \operatorname{dom} F$ the image $F(\bar{x})$ of F at $\bar{x}$ is not singleton in general. Therefore we need to specify a $\bar{z} \in F(\bar{x})$ and consider a pair of input and output $(\bar{x},\bar{z})$ as a feasible solution of F. Then a pair $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is a *(global Pareto) maximizer* of F over Ω—or simply a maximizer of the constrained problem (1.2)—if $\bar{z} \in \operatorname{Max}\big(F(\Omega);\Theta\big)$, i.e., $\bar{z}$ is a maximal point of the image set of F over Ω, or equivalently

$$(\bar{z}+\Theta) \cap F(\Omega) = \{\bar{z}\} \;\text{ with }\; F(\Omega) := \cup\,\{F(x) \mid x \in \Omega\}. \tag{3.1}$$

When $\Omega = X$, the pair $(\bar{x},\bar{z})$ is said to be a maximizer of F. When $F = f : X \to Z$ is single-valued, we omit $\bar{z} = f(\bar{x})$ in the notion of maximizers, i.e., we simply say that $\bar{x}$ is a maximizer of f instead of $(\bar{x}, f(\bar{x}))$ is a maximizer of f. We add the adjective "local" to maximizers if there is a neighborhood U of $\bar{x}$ such that $\bar{z} \in \operatorname{Max}\big(F(\Omega \cap U);\Theta\big)$. In this way the pair $(\bar{x},\bar{z})$ is a local minimizer of the constrained problem (1.2) if $\bar{z} \in \operatorname{Max}\big(F(\Omega\cap U);-\Theta\big)$ or $-\bar{z} \in \operatorname{Max}\,(-F(\Omega\cap U);\Theta)$.

Some sufficient conditions for *weak* Pareto maximizers of MOPGC (1.2) have been obtained in our recent papers [8, 9] under the assumption on the nonempty interior of the ordering cone Θ. Now we drop this assumption and address proper Pareto maximizers by further developing our dual-space variational approach involving more advanced results of generalized differential and SNC calculi.

First recall the following technical assumption on geometric constraints introduced and discussed in [9].

Definition 3.1. (normal independence). Let X be a Fréchet smooth space (i.e., there exists an equivalent norm $\|\cdot\|$ on X that is Fréchet differentiable at nonzero points). A subset $\Omega \subset X$ has the *normal independence* property if for any $x \in \Omega$ we have the condition

$$-\nabla\|\cdot - x\|(u) \notin N(u;\Omega) \ \text{ whenever } \ u \in \Omega \setminus \{x\}. \tag{3.2}$$

It is easy to check that every closed and convex subset of a Fréchet smooth space has the normal independence property; see [9, Proposition 3.1]. On the other hand, there are many nonconvex sets enjoying this property; e.g., $\{(x,y) \in I\!R^2 \mid y \ge -|x|\}$ and $\{(x,y) \in I\!R^2 \mid y \ge x^3\}$.

The *standing assumptions* of this paper are as follows.

(H1) **(ordering cone assumptions)** $\Theta \subset Z$ is a closed and convex cone with $\Theta \setminus (-\Theta) \ne \emptyset$, i.e., Θ is not a subspace of Z.

(H2) **(cost assumptions)** $F : X \rightrightarrows Z$ is an epiclosed mapping from a Fréchet smooth space X to an Asplund space Z, i.e., $\operatorname{epi} F$ is a closed set in $X \times Z$.

(H3) **(geometric constraint assumptions)** Ω is a closed subset of X satisfying the normal independence property in the sense of Definition 3.1.

Now we are ready to formulate and prove the main result of this section providing sufficient conditions for global Pareto maximizers in MOPGC (1.2). The following theorem is given under the most general assumptions holding in more common and standard settings; see the discussions and consequences below. Recall again that the imposed SNC requirements are automatic in finite dimensions.

Theorem 3.2. (sufficient conditions for global maximizers of MOPGC). *Consider the constrained multiobjective problem* (1.2) *under the standing assumptions* (*H*1)–(*H*3) *and fix* $(\bar{x},\bar{z}) \in \operatorname{gph} F$ *such that*

$$\mathbf{0} \notin \partial F(\bar{u},\bar{z}) + N(\bar{u};\Omega) \ \textit{ whenever } \ \bar{u} \in \Omega \ \textit{ with } \ \bar{z} \in \mathcal{E}_F(\bar{u}); \tag{3.3}$$

this automatically holds when the pair $(\bar{u},\bar{z})$ *is not a local minimizer of* (1.2). *Assume that the* $\bar{z}$*-level set*

$$\operatorname{lev}_{\Omega}(F;\bar{z}) := \operatorname{lev}(F;\bar{z}) \cap \Omega = \big\{x \in \Omega \mid F(x) \cap (\bar{z} - \Theta) \ne \emptyset\big\} \tag{3.4}$$

is compact in X *and impose the implication*

$$\Big[\bar{v} \in F(\bar{t}),\ \bar{v} \ne \bar{z},\ \bar{z} \le_{\Theta} \bar{v}\Big] \Longrightarrow \Big[\bar{t} \notin \operatorname{lev}_{\Omega}(F;\bar{z})\Big], \tag{3.5}$$

which holds automatically for single-valued mappings. Suppose finally that for any $\bar{u} \in \Omega$ *with* $\bar{z} \in \mathcal{E}_F(\bar{u})$ *the problem data* F *and* Ω *satisfy the qualification condition*

$$\partial^{\infty} F(\bar{u},\bar{z}) \cap \big(-N(\bar{u};\Omega)\big) = \{\mathbf{0}\}, \tag{3.6}$$

which is automatic when F *is ELL around* $(\bar{u},\bar{z})$, *and that either one of the following requirements* (a)–(c) *of the SNC type holds in the infinite dimensions:*

(a) *Θ is SNC at the origin and Ω is SNC at $\bar{u}$.*
(b) *F is PSNEC at $(\bar{u}, \bar{z})$ and Θ is SNC at the origin.*
(c) *Ω is SNC at $\bar{u}$, $(\mathcal{E}_F)^{-1}$ is PSNC at $(\bar{z}, \bar{u})$, and F exhibits the singular subdifferential regularity property* (2.9) *at $(\bar{u}, \bar{z})$.*

Then the validity of the conditions

$$\partial^\infty F(\bar{u}, \bar{z}) + N(\bar{u}; \Omega) \subset N(\bar{u}; \Omega) \quad \text{and} \tag{3.7}$$

$$\partial F(\bar{u}, \bar{z}) + N(\bar{u}; \Omega) \subset N(\bar{u}; \Omega) \tag{3.8}$$

for all $\bar{u} \in \Omega$ with $\bar{z} \in \mathcal{E}_F(\bar{u})$ is sufficient *for the global Pareto maximality of $(\bar{x}, \bar{z})$.*

Proof. Arguing by contradiction, suppose that $(\bar{x}, \bar{z})$ is not a global Pareto maximizer for problem (1.2). Then there are $\bar{t} \in \Omega$ and $\bar{v} \in F(\bar{t})$ such that $\bar{z} \leq_\Theta \bar{v}$ while $\bar{z} \neq \bar{v}$. This yields $\bar{t} \notin \mathrm{lev}_\Omega(\mathrm{F}; \bar{\mathrm{z}})$ by implication (3.5). Observe that

$$\begin{aligned} u \in \mathrm{lev}(F; \bar{z}) \cap \Omega \quad &\Longleftrightarrow \quad u \in \mathrm{lev}(F; \bar{z}) \;\text{ and }\; u \in \Omega \\ &\Longleftrightarrow \quad \big[\exists\; v \in F(u) \;\big|\; v \leq_\Theta \bar{z}\big] \;\text{ and }\; u \in \Omega \\ &\Longleftrightarrow \quad \big[\exists\; \theta \in \Theta \;\big|\; \bar{z} = v + \theta \in F(u) + \Theta\big] \;\text{ and }\; u \in \Omega \\ &\Longleftrightarrow \quad (u, \bar{z}) \in \mathrm{epi}\, F \;\text{ and }\; (u, \bar{z}) \in \Omega \times \{\bar{z}\} \\ &\Longleftrightarrow \quad (u, \bar{z}) \in \mathrm{epi}\, F \cap (\Omega \times \{\bar{z}\}). \end{aligned}$$

Next we note that the problem of finding the shortest distance from $\bar{t}$ and the compact $\bar{z}$-level set $\mathrm{lev}_\Omega(F; \bar{z})$ of F over Ω attains its minimum at some $\bar{u} \in \mathrm{lev}_\Omega(F; \bar{z})$ by the classical Weierstrass theorem. Thus $(\bar{u}, \bar{z})$ is an optimal solution to the problem of *scalar constrained optimization*:

$$\begin{cases} \text{minimize} & \varphi(u, v) := \|u - \bar{t}\| \\ \text{subject to} & (u, v) \in \Xi := \Xi_1 \cap \Xi_2, \end{cases} \tag{3.9}$$

where the sets Ξ_1 and Ξ_2 are defined by

$$\Xi_1 := \mathrm{epi}\, F \quad \text{and} \quad \Xi_2 := \Omega \times \{\bar{z}\}. \tag{3.10}$$

These sets are obviously closed in $X \times Z$ due to $(H1)$ and $(H2)$. Since φ in (3.9) is Lipschitz continuous, all the assumptions ensuring the lower subdifferential necessary condition from [24, Proposition 5.3] are satisfied. Applying this necessary condition to the minimizer $(\bar{u}, \bar{z})$ of problem (3.9), we have

$$\mathbf{0} \in \partial \|\bar{u} - \bar{t}\| \times \{\mathbf{0}\} + N\big((\bar{u}, \bar{z}); \Xi_1 \cap \Xi_2\big). \tag{3.11}$$

It follows that $\bar{u} - \bar{t} \neq \mathbf{0}$, since $\bar{t} \notin \mathrm{lev}_\Omega(F; \bar{z})$ and $\bar{u} \in \mathrm{lev}_\Omega(F; \bar{z})$. The convexity and Fréchet differentiability of renorm $\|\cdot\|$ on $X \setminus \{\mathbf{0}\}$ ensure that

$$\partial \|\bar{u} - \bar{t}\| = \{\nabla \|\bar{u} - \bar{t}\|\}.$$

Denoting $x^* := -\nabla \|\bar{u} - \bar{t}\|$ allows us to derive from (3.11) the inclusion

$$(x^*, \mathbf{0}) \in N\big((\bar{u}, \bar{z}); \Xi_1 \cap \Xi_2\big) \;\text{ with }\; \Xi_1 := \mathrm{epi}\, F \;\text{ and }\; \Xi_2 := \Omega \times \{\bar{z}\}. \tag{3.12}$$

We proceed further by employing in (3.12) the fundamental intersection rule from [23, Theorem 3.4] under the following two assumptions on the sets Ξ_1 and Ξ_2 in (3.12), where we identify $\{X, Z\}$ with the index set in the product $X \times Z$:

(A) There are two subsets J_1 and J_2 of $\{X,Z\}$ with $J_1 \cup J_2 = \{X,Z\}$ such that one of the sets is PSNC at $(\bar{u},\bar{z})$ with respect to J_1 while the other is strongly PSNC at $(\bar{u},\bar{z})$ with respect to J_2.

(B) The limiting qualification condition for Ξ_1 and Ξ_2 at $(\bar{u},\bar{z})$ holds: for any sequences

$$(x_{ik},z_{ik}) \overset{\Xi_i}{\to} (\bar{u},\bar{z}),\ (x^*_{ik},z^*_{ik}) \overset{w^*}{\to} (x^*_i,z^*_i),\ (x^*_{ik},z^*_{ik}) \in \widehat{N}\big((x_{ik},z_{ik});\Xi_i\big), \tag{3.13}$$

for $i = 1,2$, we have the implication

$$\|(x^*_{1k},z^*_{1k}) + (x^*_{2k},z^*_{2k})\| \to 0 \implies \big[x^*_1 = x^*_2 = \mathbf{0} \text{ and } z^*_1 = z^*_2 = \mathbf{0}\big]. \tag{3.14}$$

Let us first check the validity of condition (A). This step easily follows from the imposed conditions (a)–(c). Precisely, for each of these cases we have:

(a) Ξ_1 is strongly PSNC at $(\bar{u},\bar{z})$ with respect to $J_1 = \{Z\}$ and Ξ_2 is PSNC at $(\bar{u},\bar{z})$ with respect to $J_2 = \{X\}$.

(b) In this case we have $\Xi_1 = \operatorname{epi} F$ is SNC at $(\bar{u},\bar{z})$, and hence we can put $J_1 = \{X,Z\}$ and $J_2 = \emptyset$. To check this property, take sequences $(x_k,z_k) \overset{\text{epi F}}{\longrightarrow} (\bar{u},\bar{z})$ and $(x^*_k,z^*_k) \in X^* \times Z^*$ satisfying

$$(x^*_k,z^*_k) \overset{w^*}{\longrightarrow} (\mathbf{0},\mathbf{0}) \text{ with } (x^*_k,z^*_k) \in \widehat{N}\big((x_k,z_k);\operatorname{epi} F\big). \tag{3.15}$$

It follows from implication (2.8) that $z^*_k \overset{w^*}{\longrightarrow} \mathbf{0}$ with $z^*_k \in \widehat{N}(z_k;\Theta)$. By the assumed SNC property of Θ at the origin we get $\|z^*_k\| \to 0$, which yields together with (3.15) that $\|x^*_k\| \to 0$ due to the PSNEC assumption on F. Thus $\operatorname{epi} F$ is SNC at $(\bar{u},\bar{z})$.

(c) In this case we check directly that Ξ_1 is PSNC at $(\bar{u},\bar{z})$ with respect to $J_1 = \{Z\}$ and Ξ_2 is strongly PSNC at $(\bar{u},\bar{z})$ with respect to $J_2 = \{X\}$.

Next we verify the validity of condition (B). Take arbitrary sequences in (3.13) and get by the structures of Ξ_1 and Ξ_2 in (3.10) that

$$\begin{cases} (x^*_{1k},z^*_{1k}) \in \widehat{N}\big((x_{1k},z_{1k});\operatorname{epi} F\big) & \implies z^*_{1k} \in \widehat{N}(\mathbf{0};\Theta), \\ (x^*_{2k},z^*_{2k}) \in \widehat{N}\big((x_{2k},z_{2k});\Omega \times \{\bar{z}\}\big) & \implies x^*_{2k} \in \widehat{N}(x_{2k};\Omega) \text{ and } z_{2k} \equiv \bar{z}. \end{cases} \tag{3.16}$$

Passing to the limit in (3.16) as $k \to \infty$ allows us to obtain from the hypothesis of implication (3.15) that

$$\mathbf{0} \in D^*_N \mathcal{E}_F(\bar{u},\bar{z})(z^*) + N(\bar{u};\Omega) \text{ and } -z^* \in N(\mathbf{0};\Theta). \tag{3.17}$$

Observe that if $z^* \neq \mathbf{0}$ in (3.17), then it can be scaled to a unit vector in order to equivalently express the inclusions in (3.17) via the subdifferential of F at $(\bar{u},\bar{z})$:

$$\mathbf{0} \in D^*_N \mathcal{E}_F(\bar{u},\bar{z})\Big(\frac{z^*}{\|z^*\|}\Big) + N(\bar{u};\Omega) \subset \partial F(\bar{u},\bar{z}) + N(\bar{u};\Omega),$$

which surely contradicts the assumption in (3.3). Thus $z^* = \mathbf{0}$.

Using further either the SNC property of Θ in conditions (a) and (b), or the singular subdifferential regularity of F in condition (c) allows us to deduce from (3.17) the existence of $u^* \in X^*$ such that

$$u^* \in \partial^\infty F(\bar{u},\bar{z}) \cap \big(-N(\bar{u};\Omega)\big),$$

which implies the conclusion of implication (3.15) $u^* = \mathbf{0}$ by the assumed qualification condition (3.6). Thus condition (B) is also satisfied.

The validity of hypotheses (A) and (B) for the intersection rule in (3.12) ensures that the pair $(x^*, \mathbf{0})$ satisfies the more elaborated inclusion

$$(x^*, \mathbf{0}) \in N\big((\bar{u}, \bar{z}); \operatorname{epi} F\big) + N\big((\bar{u}, \bar{z}); \Omega \times \{\bar{z}\}\big) = N\big((\bar{u}, \bar{z}); \operatorname{epi} F\big) + N(\bar{u}; \Omega) \times Z^*,$$

which justifies the existence of dual elements $(u_1^*, -z^*) \in N\big((\bar{u}, \bar{z}); \operatorname{epi} F\big)$ and $u_2^* \in N(\bar{u}; \Omega)$ such that

$$x^* = u_1^* + u_2^* \text{ and } u_1^* \in D_N^* \mathcal{E}_F(\bar{u}, \bar{z})(z^*). \tag{3.18}$$

Recall that the inclusion in (3.18) implies that $-z^* \in N(\mathbf{0}; \Theta)$. To finish the proof of the theorem, we consider the following two cases:

• **Case 1:** $z^* = \mathbf{0}$. In this case we can show that $D_N^* \mathcal{E}_F(\bar{u}, \bar{z})(\mathbf{0}) = \partial^\infty F(\bar{u}, \bar{z})$ by using the same arguments as in verifying (B) above. Thus (3.18) yields

$$-\nabla \|\bar{u} - \bar{t}\| = x^* = u_1^* + u_2^* \in \partial^\infty F(\bar{u}, \bar{z}) + N(\bar{u}; \Omega) \subset N(\bar{u}; \Omega),$$

where the last inclusion holds due to (3.7). This contradicts the normal independence property of Ω.

• **Case 2:** $z^* \neq \mathbf{0}$. In this case we get from (3.18) that

$$\frac{x^*}{\|z^*\|} \in \partial F(\bar{u}, \bar{u}) + N(\bar{u}; \Omega) \subset N(\bar{u}, \Omega),$$

where the last inclusion holds due to (3.8). Therefore

$$x^* = -\nabla \|\bar{u} - \bar{t}\| \in N(\bar{u}; \Omega)$$

since $N(\bar{u}; \Omega)$ is a cone. Again we arrive at the contradiction with the normal independence property of Ω.

Summing up, we conclude that the pair $(\bar{x}, \bar{z})$ is a global Pareto maximizer of MOPGC (1.2) and thus complete the proof of the theorem. □

Now let us compare the results obtained in Theorem 3.2 with the known ones in the literature and also discuss the major assumptions made.

Remark 3.3 (on the assumptions of Theorem 3.2)**.**

(i) The fulfilment of implication (3.5) is automatic if F enjoys the property

$$F(x) = \operatorname{Max}\big(F(x); \Theta\big);$$

in particular, when $F = f : X \to Z$ is single-valued.

(ii) Both sufficient conditions (3.7) and (3.8) can be combined into one condition in terms of coderivative of the epigraphical multifunction of the cost mapping

$$D_N^* \mathcal{E}_F(\bar{u}, \bar{z})(z^*) + N(\bar{u}; \Omega) \subset N(\bar{u}; \Omega) \text{ for all } z^* \in Z^*.$$

(iii) The SNC condition (b) is equivalent to that F is SNEC at the point under consideration. In the proof of Theorem 3.2 we justified one direction: the former implies the latter. Thus it remains to justify the reverse implication. Assume that F is SNEC at $(\bar{u}, \bar{z})$. Then it is PSNEC at $(\bar{u}, \bar{z})$. Next we show that Θ is SNC at the origin. Indeed, for any sequence $\{(z_k, z_k^*)\} \subset Z \times Z^*$ satisfying

$$z_k \in \Theta \quad \text{and} \quad z_k^* \in \widehat{N}(z_k; \Theta) \subset \widehat{N}(\mathbf{0}; \Theta)$$

we have $\{(\bar{u}, \bar{z}, \mathbf{0}, z_k^*)\} \subset X \times Z \times X^* \times Z^*$ with $(\mathbf{0}, z_k^*) \in \widehat{N}\big((\bar{u}, \bar{z}); \operatorname{epi} F\big)$. It follows further from the SNEC property of F at $(\bar{u}, \bar{z})$ that

$$\text{if } \; z_k^* \xrightarrow{w^*} \mathbf{0}, \quad \text{then } \; \|z_k^*\| \to 0$$

verifying the SNC property of Θ. It is important to recall that in establishing necessary conditions for constrained multiobjective optimization problems via coderivatives of cost mappings (see, e.g., [6, Theorem 4.6] and cf. [8, Theorem 6.1]) we need the validity of the four SNC conditions: (a) Θ is SNC and Ω is SNC; (b) F is PSNC and Θ is SNC; (c) Ω is SNC at $\bar{u}$, F^{-1} is PSNC; and (d) F is SNC. Observe that conditions (b) and (d) for F are independent while their counterparts for the epigraphical multifunction $\mathcal{E}_F$ are equivalent.

(iv) Sufficient conditions for Pareto maximizers can be established in general Banach settings by imposing some generalized convexity requirements on the $\bar{z}$-level set of F over Ω; see [9, Remark 3.5] and [9, Corollary 3.6] for details.

(v) The proof of Theorem 3.2 is based on calculus rules for generalized differentiation and SNC properties. It occurs to be much simpler than those of [8, Theorem 7.1] and [9, Theorem 3.3] in which we used a more fundamental tool of variational analysis, the extremal principle, that is the driving force in deriving a number of calculus results including sum rules and chain rules for coderivatives of set-valued mappings.

Remark 3.4 (comparisons with previous results on sufficient conditions)**.**

(i) When $\operatorname{int} \Theta \neq \emptyset$ for the ordering cone $\Theta \subset Z$, we can consider a weak counterpart of the relation $\leq_\Theta$ in (1.1) with replacing Θ by its interior and denote it by $<_\Theta$. A feasible solution $(\bar{x}, \bar{z})$ is a *weak maximizer* of MOPGC (1.2) if

$$F(\Omega) \cap (\bar{z} + \operatorname{int} \Theta) = \emptyset.$$

It is obvious that every Pareto maximizer is weak Pareto provided that $\operatorname{int} \Theta \neq \emptyset$ and that the sufficient result in Theorem 3.2 is still valid for weak Pareto maximizers to the problems under consideration provided that the weaker version

$$\Big[\bar{v} \in F(\bar{t}),\ \bar{z} <_\Theta \bar{v}\Big] \Longrightarrow \Big[\bar{t} \notin \operatorname{lev}_\Omega(F; \bar{z})\Big],$$

of (3.5) holds; the latter is equivalent to the implication in [9, Theorem 3.3]:

$$\Big[F(\bar{t}) \cap (\bar{z} + \operatorname{int} \Theta) = \emptyset\Big] \Longrightarrow \Big[\bar{t} \notin \operatorname{lev}_\Omega(F; \bar{z})\Big].$$

Observe that when $\operatorname{int} \Theta \neq \emptyset$, the ordering cone Θ is SNC at the origin. Thus the SNC condition (a) implies the SNC condition (c) and the SNC conditions (a)–(c) reduce to that either Ω is SNC at $\bar{u}$ or F is PSNEC at $(\bar{u}, \bar{z})$. Note also that the four conditions (3.3), (3.6), (3.7), and (3.8) are of the same forms of (3.6), (3.8), (3.9), and (3.7) in [9, Theorem 3.3]; cf. [8, Theorem 7.1]. Observe finally that the conditions in this paper are imposed at each point $\bar{u} \in \Omega$ with $\bar{z} \in \mathcal{E}_F(\bar{u}) = F(\bar{u}) + \Theta$, i.e., $(\bar{u}, \bar{z}) \in \operatorname{epi} F$, while those in [9, Theorem 3.3] are needed at each pair of input and output $(\bar{u}, \bar{v}) \in \operatorname{gph} F$ with $\bar{u} \in \Omega$ and $\bar{v} \in \bar{z} - \operatorname{bd} \Theta$. It is important to emphasize that by the input-based conditions of Theorem 3.2 are more efficient than the input-output-based conditions in [8, 9] since we have the inclusions

$$\partial F(\bar{x}, \bar{z}) \subset \partial F(\bar{x}, \bar{v}) \quad \text{and} \quad \partial^\infty F(\bar{x}, \bar{z}) \subset \partial^\infty F(\bar{x}, \bar{v}) \tag{3.19}$$

for every $\bar{v} \in F(\bar{x}) + \Theta$ provided that F is in the class of set-valued mappings enjoying the following compactness property at $(\bar{x}, \bar{z})$: for any sequence $\{(x_k, z_k)\} \subset \operatorname{epi} F$ satisfying $(x_k, z_k) \to (\bar{x}, \bar{z})$ as $k \to \infty$ there exists a sequence $\{v_k\}$ with $v_k \in F(x_k)$ and $z_k \in v_k + \Theta$ such that it contains a subsequence converging to some $\bar{v} \in F(\bar{x})$. To justify the validity of (3.19), we use the definitions of the limiting subdifferential and normal cone directly. It is easy to check that this class of mappings includes single-valued continuous ones and extended-real-valued lower semicontinuous functions and that the inclusions in (3.19) are strict in general, e.g., when $\bar{z}$ happens to be in the interior of the image of F at $\bar{x}$. Roughly speaking, the fulfilment of either the four conditions (3.3), (3.6), (3.7), and (3.8) in Theorem 3.2 or conditions (3.6)—(3.9) in [9, Theorem 3.3] ensures the optimality of the point under consideration. Each set of the conditions is divided into two groups due to the employed tools of generalized differentiation.

(ii) Note that the sufficient conditions of Theorem 3.2 do not cover in general standard sufficient conditions in nonlinear programming and convex optimization. Our approach is different from the conventional one as, e.g., in [19, 28], where sufficient conditions are derived by strengthening known necessary conditions assuming certain local convexity and the like. In contrast, we employ tools of variational analysis to obtain new sufficient conditions for problems of vector and set-valued optimization in fully nonconvex settings. Note also that, being efficient for broad classes of problems in constrained optimization, the sufficient conditions obtained in this paper may not be generally applied to unconstrained problems. Some results for unconstrained problems via our approach can be found in [9, Section 4].

To conclude this section, we present two useful consequences of Theorem 3.2 for problems of vector optimization and scalar optimization, respectively.

Corollary 3.5 (sufficient conditions in nonconvex vector optimization). *Consider the multiobjective problem* (1.2) *with a single-valued cost mapping* $f : X \to Z$, *which is Lipschitz continuous on the Fréchet smooth space* X. *Fix* $\bar{x} \in \Omega$ *with* $\bar{z} := f(\bar{x})$ *and, in addition to the standing assumptions* $(H1)$ *and* $(H3)$, *suppose that either* Θ *is SNC at the origin or* Ω *is SNC at* $\bar{u}$, $(\mathcal{E}_f)^{-1}$ *is PSNC at* $(\bar{z}, \bar{u})$, *and* f *exhibits the singular subdifferential regularity at* $\bar{u}$ *for every* $\bar{u} \in \Omega$ *with* $\bar{z} \in f(\bar{u}) + \operatorname{bd} \Theta$. *Suppose also that*

$$\mathbf{0} \notin \partial f(\bar{u}, \bar{z}) + N(\bar{u}; \Omega) \quad \text{and} \quad \partial f(\bar{u}, \bar{z}) + N(\bar{u}; \Omega) \subset N(\bar{u}; \Omega) \tag{3.20}$$

for all such elements $\bar{u}$. *Then* $\bar{x}$ *is a global Pareto maximizer for the vector optimization problem* (1.2).

Proof. This result is a particular case of Theorem 3.2 with the following specifications of its assumptions and conclusions:

- Implication (3.5) is automatic since f is single-valued.

- Conditions (3.6) and (3.7) hold since f is Lipschitz continuous around $\bar{u}$. Precisely, the Lipschitz continuity of f implies the Lipschitz-like property of its epigraphical multifunction $\mathcal{E}_f$ by [7, Corollary 3.5]. The latter property ensures the injectivity property of the mixed coderivative of $\mathcal{E}_F$ at $\mathbf{0}$ by [23, Theorem 1.44]:
$$D^*_M\mathcal{E}_f(\bar{u},\bar{z})(\mathbf{0}) = \partial^\infty f(\bar{u}) = \{\mathbf{0}\},$$
which in turn verifies the validity of the sufficient optimality conditions of Theorem 3.2:
$$\partial^\infty f(\bar{u}) \cap \big(-N(\bar{u};\Omega)\big) = \{\mathbf{0}\} \cap \big(-N(\bar{u};\Omega)\big) = \{\mathbf{0}\} \text{ and}$$
$$\partial^\infty f(\bar{u}) + N(\bar{u};\Omega) = \{\mathbf{0}\} + N(\bar{u};\Omega) = N(\bar{u};\Omega).$$
- We can use $\bar{z} \in f(\bar{u}) + \operatorname{bd}\Theta$ instead of $\bar{z} \in f(\bar{u}) + \Theta$ due to $\partial^\infty f(\bar{u}) = \emptyset$ for $\bar{z} \in f(\bar{u}) + \operatorname{int}\Theta$.

□

Corollary 3.6 (sufficient conditions in nonsmooth scalar optimization)**.** *Let $\varphi : X \to I\!R$ be a locally Lipschitzian, $\Omega \subset X$ be a closed and convex set in the reflexive Banach space X, and $\bar{x} \in \Omega$. Assume that for every $\bar{u} \in \Omega$ with $\varphi(\bar{u}) = \varphi(\bar{x})$ we have*

$$\mathbf{0} \notin \partial\varphi(\bar{u}) + N(\bar{u};\Omega) \quad \textit{and} \tag{3.21}$$

$$\partial\varphi(\bar{u}) \subset N(\bar{u};\Omega) \Leftrightarrow \partial\varphi(\bar{u}) + N(\bar{u};\Omega) \subset N(\bar{u};\Omega). \tag{3.22}$$

Then $\bar{x}$ is a global maximum of φ over Ω.

Proof. It follows directly from Corollary 3.5 since every reflexive Banach space is Fréchet smooth and the convexity property of Ω implies the normal independence condition for this set by [9, Proposition 3.1]. □

If we assume further that $X = I\!R^n$, then Corollary 3.6 reduces to the result by Dutta [13, Theorem 3.2]. If the data is convex, it recaptures that by Hiriart-Urruty and Ledyaev [17, Theorem 1.1].

4. Sufficient conditions for multiobjective problems with equilibrium constraints

This section is devoted to deriving sufficient conditions for global Pareto maximizers in MOPEC (1.3). For simplicity we consider the case of single-valued cost mappings and replace hypothesis $(H2)$ by following stronger version:

(H2′) **(modified cost assumptions)** $F = f : X \to Z$ is a Lipschitz continuous mapping from a Fréchet smooth space X to an ordered Asplund space Z.

Note that under $(H2')$ assumptions (3.5), (3.6), and (3.7) hold. The next assumption is imposed on the equilibrium constraint data in (1.3):

(H4) **(equilibrium constraint assumptions)** $G\colon X \times Y \rightrightarrows W$ and $Q\colon X \times Y \rightrightarrows W$ are closed-graph mappings between Asplund spaces.

Denoting S the set of all the feasible solutions to MOPEC (1.3) by

$$S := \big\{x \in \Omega \,\big|\, \exists\, y \in Y \text{ with } \mathbf{0} \in G(x,y) + Q(x,y)\big\}, \tag{4.1}$$

we say that $\bar{x}$ is a (global) *maximizer* of MOPEC (1.3) if it is a maximizer in the following MOPGC (1.2):

$$\begin{cases} \Theta\text{-maximizer} & f(x) \\ \text{subject to} & x \in S \subset X. \end{cases} \tag{4.2}$$

To establish sufficient conditions for multiobjective optimization problems with equilibrium constraints we follow the scheme developed by Bao and Mordukhovich [3] for deriving necessary optimality conditions in MOPEC. In contrast to the conventional approach of reducing MOPEC to the equivalent problem (4.2), we introduce another equivalent problem of type (1.2) as well by adding *extra variables* into the cost to describe a new geometric constraint in form of set intersections in an appropriate product space; see [3, Theorem 3.4 and Theorem 4.3] and cf. the constraint of the auxiliary problem (4.7) in the proof of Theorem 4.1 below. Due to this reduction it is possible to derive optimality conditions for MOPEC (1.3) from those in (1.2) by using refined calculus rules for generalized differential objects including normal cones to sets and coderivatives of set-valued mappings. Here is the main result of this section.

Theorem 4.1 (sufficient conditions for global Pareto maximizers in MOPEC)**.** *Consider MOPEC* (1.3) *under hypotheses* $(H1)$, $(H2')$, $(H3)$, *and* $(H4)$ *and take* $\bar{x} \in S$ *from the set of feasible solutions* (4.1) *with* $\bar{z} := f(\bar{x})$. *Assume that for every* $\bar{u} \in \Omega$ *with* $f(\bar{u}) = f(\bar{x})$ *and every* $\overline{w} \in G(\bar{u},\bar{y})$ *with* $-\overline{w} \in Q(\bar{u},\bar{y})$ *corresponding to some* $\bar{y} \in Y$ *we have the condition*

$$\mathbf{0} \notin \partial f(\bar{u}) + D^*_N G(\bar{u},\bar{y},\overline{w})(y^*,w^*) + D^*_N Q(\bar{u},\bar{y},-\overline{w})(y^*,w^*) + N(\bar{u};\Omega), \tag{4.3}$$

which holds, in particular, when $\bar{x}$ *is not a local minimizer of* (1.3). *Suppose also that the* $\bar{z}$*-level set*

$$\operatorname{lev}_S(f;\bar{z}) = \operatorname{lev}(f;\bar{z}) \cap S$$

is compact in X *and that for every* $(\bar{u},\bar{y},\bar{z},\overline{w})$ *with* $\bar{u} \in S$, $\bar{z} = f(\bar{u})$ *and* $\overline{w} \in G(\bar{x},\bar{y}) \cap \big(-Q(\bar{x},\bar{y})\big)$ *the following two groups of assumptions are satisfied:*

SNC*: G and Q are SNC at $(\bar{u},\bar{y},\overline{w})$ and $(\bar{u},\bar{y},-\overline{w})$, respectively. Either Θ is SNC at $\mathbf{0}$ or Ω is SNC at $\bar{u}$, $(\mathcal{E}_f)^{-1}$ is PSNC at $(\bar{z},\bar{u})$, and f exhibits the singular subdifferential regularity at $\bar{u}$.*

Constraint Qualification*:*

$$\begin{bmatrix} x^*_G \in D^*_N G(\bar{x},\bar{y},\overline{w})(y^*,w^*) \\ x^*_Q \in D^*_N Q(\bar{x},\bar{y},-\overline{w})(-y^*,w^*) \\ x^*_\Omega \in N(\bar{x};\Omega),\ x^*_G + x^*_Q + x^*_\Omega = \mathbf{0} \end{bmatrix} \Longrightarrow \begin{bmatrix} y^* = \mathbf{0},\ w^* = \mathbf{0} \\ x^*_G = x^*_Q = x^*_\Omega = \mathbf{0} \end{bmatrix}. \tag{4.4}$$

Then the validity of the conditions

$$D^*_N G(\bar{u},\bar{y},\overline{w})(y^*,w^*) + D^*_N Q(\bar{u},\bar{y},\overline{w})(y^*,w^*) + N(\bar{u};\Omega) \subset N(\bar{u};\Omega), \tag{4.5}$$

$$\partial f(\bar{u}) + D^*_N G(\bar{u},\bar{y},\overline{w})(y^*,w^*) + D^*_N Q(\bar{u},\bar{y},\overline{w})(y^*,w^*) + N(\bar{u};\Omega) \subset N(\bar{u};\Omega) \tag{4.6}$$

is sufficient *for the global Pareto maximality of* $\bar{x}$ *in MOPEC* (1.3).

Proof. Arguing by contradiction, suppose that $\bar{x}$ is not a global Pareto maximizer for MOPEC (1.3). Then, by the equivalence mentioned above, $\bar{x}$ is not an optimal solution to problem (4.2), i.e., there is a $\bar{t} \in S$ such that $f(\bar{x}) \leq_\Theta f(\bar{t})$ and $f(\bar{x}) \neq f(\bar{t})$. This yields that $\bar{t} \notin \mathrm{lev}_S(f;\bar{z}) = \mathrm{lev}\,(f;\bar{z}) \cap S$. Considering now the problem of finding the shortest distance from the point $\bar{t}$ to the compact set $\mathrm{lev}_S(f;\bar{z})$, we find its optimal solution $\bar{u} \in \mathrm{lev}_S(f;\bar{z})$ by the classical Weierstrass theorem. Observe that

$$\begin{aligned} & x \in \mathrm{lev}(f;\bar{z}) \cap S \\ \iff & \left[\begin{array}{l} \exists\, x \in \Omega,\ \theta \in \Theta,\ y \in Y,\ w \in W \ \text{ with} \\ f(x) + \theta = \bar{z},\ \mathbf{0} = w - w \in G(x,y) + Q(x,y) \end{array} \right] \\ \iff & (x,\bar{z}) \in \mathrm{epi}\, f,\ (x,y,w) \in \mathrm{gph}\, G \cap \big(\mathrm{gph}\,(-Q)\big), \ \text{ and } \ x \in \Omega \end{aligned}$$

and thus $\bar{u}$ is a minimizer of *scalar* optimization problem with geometric constraints of the intersection type:

$$(4.7) \qquad \begin{cases} \text{minimize} & \varphi(x) := \|x - \bar{t}\| \\ \text{subject to} & (x,y,z,w) \in \Xi := \Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4, \end{cases}$$

where Ξ_1, Ξ_2, Ξ_3, and Ξ_4 are subsets in the (Asplund) product space $P := X \times Y \times Z \times W$ defined by

$$(4.8) \qquad \begin{cases} \Xi_1 := \big\{(x,y,z,w) \in P \;\big|\; (x,z) \in \mathrm{epi}\, f\big\}, \\ \Xi_2 := \big\{(x,y,z,w) \in P \;\big|\; x \in \Omega \ \text{ and } \ z = \bar{z}\big\}, \\ \Xi_3 := \big\{(x,y,z,w) \in P \;\big|\; (x,y,w) \in \mathrm{gph}\, G\big\}, \\ \Xi_4 := \big\{(x,y,z,w) \in P \;\big|\; (x,y,w) \in \mathrm{gph}\,(-Q)\big\}. \end{cases}$$

As in the proof of Theorem 3.2, we conclude by using the necessary optimality condition in problem (4.7) that

$$(4.9) \qquad \mathbf{0} \in \partial\varphi(\bar{u}) \times \{(\mathbf{0},\mathbf{0},\mathbf{0})\} + N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi\big).$$

Next we use several intersection rules to represent the normal cone $N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi\big)$ as a sum of the normal cones to the components in the set intersection. It is shown in Proposition 4.2 below that the validity of the assumed constraint qualification (4.4) implies the fulfilment of the qualification conditions needed for the calculus rules employed. Observe first that both the sets Ξ_3 and Ξ_4 are SNC at $(\bar{u},\bar{y},\bar{z},\overline{w})$ since the mappings G and Q are SNC at $(\bar{u},\bar{z},\overline{w})$ and $(\bar{u},\bar{z},-\overline{w})$, respectively, and that the qualification condition (4.4) implies the normal qualification condition [23, Corollary 3.81] for the system $\{\Xi_3,\Xi_4\}$, which thus ensures the SNC property for the intersection $\Xi_3 \cap \Xi_4$ at $(\bar{u},\bar{y},\bar{z},\overline{w})$.

By Proposition 4.2 the normal qualification condition for $(\Xi_1 \cap \Xi_2)$ and $(\Xi_3 \cap \Xi_4)$ is satisfied by the constraint qualification (4.4). Thus applying the intersection rule from [23, Corollary 3.5] to $(\Xi_1 \cap \Xi_2)$ and $(\Xi_3 \cap \Xi_4)$ gives us

$$(4.10) \quad N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi\big) \subset N\big((\bar{u},\bar{y},\bar{z},\overline{w});(\Xi_1 \cap \Xi_2)\big) + N\big((\bar{u},\bar{y},\bar{z},\overline{w});(\Xi_3 \cap \Xi_4)\big).$$

Furthermore, applying the aforementioned intersection rule to the sets Ξ_3 and Ξ_4 under the normal qualification condition (4.4) gives us

$$N\big((\bar{u},\bar{y},\bar{z},\overline{w});(\Xi_3\cap\Xi_4)\big)\subset N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_3\big)+N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_4\big). \tag{4.11}$$

It remains unfolding the normal cone to $\Xi_1\cap\Xi_2$ in (4.10). To proceed, we employ the intersection rule from [23, Theorem 3.4] to the PSNC sets Ξ_1 and Ξ_2 by taking into account that the PSNC assumptions of that rule hold under the assumptions made in this theorem and observing that the fulfilment of the mixed qualification condition for Ξ_1 and Ξ_2 is guaranteed by the imposed qualification condition (4.4) due to Proposition 4.2. In this way we arrive at the inclusion

$$N\big((\bar{u},\bar{y},\bar{z},\overline{w});(\Xi_1\cap\Xi_2)\big)\subset N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_1\big)+N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_2\big). \tag{4.12}$$

It follows from the structures of the sets Ξ_i for $i=1,2,3,4$ in (4.8) and the inclusion in (4.9) that

$$\begin{aligned}
&(x_f^*,\mathbf{0},-z^*,\mathbf{0})\in N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_1\big)\Longrightarrow(x_f^*,-z^*)\in N\big((\bar{u},\bar{z});\operatorname{epi}f\big),\\
&(x_G^*,-y^*,\mathbf{0},-w^*)\in N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_2\big)\Longrightarrow(x_G^*,-y^*,-w^*)\in N\big((\bar{u},\bar{y},\overline{w});\operatorname{gph}G\big),\\
&(x_Q^*,y^*,\mathbf{0},w^*)\in N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_3\big)\Longrightarrow(x_Q^*,y^*,-w^*)\in N\big((\bar{u},\bar{y},-\overline{w});\operatorname{gph}Q\big),\\
&(x_\Omega^*,\mathbf{0},z^*,\mathbf{0})\in N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_4\big)\Longrightarrow x_\Omega^*\in N(\bar{u};\Omega)\ \text{ and }\ z^*\in Z^*.
\end{aligned}$$

Using this and substituting the relationships in (4.11), (4.12), and (4.10) into the necessary optimality condition (4.9) give us the inclusion

$$\begin{aligned}
x^*\in D_N^*\mathcal{E}_f(\bar{u},\bar{z})(z^*)\ &+\ D_N^*G(\bar{u},\bar{y},\overline{w})(y^*,w^*)\\
&+\ D_N^*Q(\bar{u},\bar{y},\overline{w})(-y^*,w^*)+N(\bar{u};\Omega),
\end{aligned}\tag{4.13}$$

where $x^*:=-\nabla\|\cdot-\bar{y}\|(\bar{u})$. Finally, we combine (4.13) with (4.5) when $z^*=\mathbf{0}$ and with (4.6) when $z^*\neq\mathbf{0}$ and arrive in this way at $x^*\in N(\bar{u};\Omega)$, which obviously contradicts the normal independence condition for Ω from Definition 3.1. This contradiction ensures that $\bar{x}$ is a global Pareto maximizer for MOPEC (1.3) and thus completes the proof of the theorem. □

Now we formulate and prove Proposition 4.2 used above. Moreover, we obtain an extended version of it that is valued for non-Lipschitzian and set-valued cost mappings. This would allow the reader to derive extended versions of Theorem 4.1 for more general MOPEC (1.3). It is easy to see that the general qualification condition (4.14) reduces to that of (4.4) in the setting of Theorem 4.1. In what follows we consider the sets Ξ_i defined in (4.8) with the replacement of the single-valued cost f therein by its set-valued counterpart F.

Proposition 4.2 (fulfillment of qualification conditions)**.** *Let $F\colon X\rightrightarrows Z$ be a set-valued mapping with a closed graph, and let the sets Ξ_1, Ξ_2, Ξ_3, and Ξ_4 be defined* *in* (4.8). *Assume that the qualification condition*

$$\text{(4.14)}\qquad \begin{bmatrix} x_F^* \in \partial^\infty F(\bar{x},\bar{z}),\ x_\Omega^* \in N(\bar{x};\Omega), \\ x_G^* \in D_N^* G(\bar{x},\bar{y},\overline{w})(y^*,w^*), \\ x_Q^* \in D_N^* Q(\bar{x},\bar{y},-\overline{w})(-y^*,w^*), \\ x_F^* + x_G^* + x_Q^* + x_\Omega^* = \mathbf{0} \end{bmatrix} \Longrightarrow \begin{bmatrix} y^* = \mathbf{0}, w^* = \mathbf{0}, \\ x_F^* = \mathbf{0}, \\ x_G^* = x_Q^* = x_\Omega^* = \mathbf{0} \end{bmatrix}$$

is satisfied. Assume also that (4.3) *holds and that conditions* (a)–(c) *of Theorem* 3.2 *are fulfilled. Then we have:*

- **mixed qualification condition for** Ξ_1 **and** Ξ_2*:*

$$\text{(4.15)}\qquad \partial^\infty F(\bar{x},\bar{z}) \cap \big(-N(\bar{x};\Omega)\big) = \{\mathbf{0}\}.$$

- **normal qualification condition for** Ξ_3 **and** Ξ_4*:*

$$\text{(4.16)}\qquad N\big((\bar{x},\bar{y},\overline{w});\operatorname{gph} G\big) \cap \Big(-N\big((\bar{x},\bar{y},\overline{w});\operatorname{gph}(-Q)\big)\Big) = \{\mathbf{0}\}.$$

- **normal qualification condition for** $\Xi_1 \cap \Xi_2$ **and** $\Xi_3 \cap \Xi_4$*:*

$$\text{(4.17)}\qquad \begin{bmatrix} x^* \in \partial^\infty F(\bar{x},\bar{z}) + N(\bar{x};\Omega) \\ -x^* \in D_N^* G(\bar{x},\bar{y},\overline{w})(y^*,w^*) \\ \qquad + D_N^* Q(\bar{x},\bar{y},-\overline{w})(-y^*,w^*) \end{bmatrix} \Longrightarrow \begin{bmatrix} x^* = \mathbf{0},\ y^* = \mathbf{0} \\ w^* = \mathbf{0} \end{bmatrix}.$$

Proof. Putting $y^* = \mathbf{0}$, $w^* = \mathbf{0}$, and $x_G^* = x_Q^* = \mathbf{0}$ in (4.14) gives us (4.15). In a similar way condition (4.14) implies (4.16) by taking $x_F^* = x_\Omega^* = \mathbf{0}$. It remains to verify the validity of the normal qualification condition (4.17) for $\Xi_1 \cap \Xi_2$ and $\Xi_3 \cap \Xi_4$, which can be equivalently written as

$$\text{(4.18)}\qquad N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_1 \cap \Xi_2\big) \cap \Big(-N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_3 \cap \Xi_4\big)\Big) = \{\mathbf{0}\}.$$

Remembering the discussions above, we have under the assumed SNC that

$$N\big((\bar{u},\bar{y},\bar{z},\overline{w});(\Xi_1 \cap \Xi_2)\big) \subset N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_1\big) + N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_2\big) \text{ and}$$
$$N\big((\bar{u},\bar{y},\bar{z},\overline{w});(\Xi_3 \cap \Xi_4)\big) \subset N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_3\big) + N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_4\big).$$

It is clear that the fulfilment of (4.18) is ensured by the validity of the implication

$$\text{(4.19)}\qquad \begin{bmatrix} (x^*,\mathbf{0},\mathbf{0},\mathbf{0}) \in N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_1\big) + N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_2\big) \\ (-x^*,\mathbf{0},\mathbf{0},\mathbf{0}) \in N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_3\big) + N\big((\bar{u},\bar{y},\bar{z},\overline{w});\Xi_4\big) \end{bmatrix} \Longrightarrow x^* = \mathbf{0}.$$

The hypothesis of implication (4.19) is equivalent, by taking into the descriptions of the normal cones to Ξ_i for $i = 1,2,3,4$ in (4.8), to the fulfillment of the inclusion

$$\mathbf{0} \in D_N^* \mathcal{E}_F(\bar{u},\bar{z})(z^*) + D_N^* G(\bar{u},\bar{y},\overline{w})(y^*,w^*) + D_N^* Q(\bar{u},\bar{y},-\overline{w})(-y^*,w^*) + N(\bar{u};\Omega).$$

Note that the assumed condition (4.3) yields $z^* = \mathbf{0}$. Then either the SNC property of Θ or the singular subdifferential regularity (2.9) of F ensures that $D_N^* \mathcal{E}_F(\bar{u},\bar{z})(\mathbf{0}) = \partial^\infty F(\bar{u},\bar{z})$. This together with the hypothesis of (4.19) gives us

$$\begin{cases} x_F^* \in \partial^\infty F(\bar{u},\bar{z}),\ x_G^* \in D_N^* G(\bar{x},\bar{y},\overline{w})(y^*,w^*), \\ x_Q^* \in D_N^* Q(\bar{x},\bar{y},-\overline{w})(-y^*,w^*),\ x_\Omega^* \in N(\bar{x};\Omega), \\ x_F^* + x_G^* + x_Q^* + x_\Omega^* = \mathbf{0} \ \text{ with } \ x^* = x_F^* + x_\Omega^* = -(x_G^* + x_Q^*), \end{cases}$$

and thus $x^*_F = x^*_G = x^*_Q = x^*_\Omega = \mathbf{0}$ by the qualification condition (4.14). Therefore $x^* = \mathbf{0}$, which verifies the normal qualification condition for $\Xi_1 \cap \Xi_4$ and $\Xi_2 \cap \Xi_3$ and completes the proof of the proposition. □

The last result of the paper provides new sufficient conditions for the conventional model in *multiobjective mathematical programming* with finitely many objectives, inequality and equality constraints:

$$
(4.20) \qquad \begin{cases} I\!R^n_+\text{–maximize} & f(x) = \big(f_1(x), \ldots, f_n(x)\big) \\ \text{subject to} & g_i(x) \le 0 \;\text{ for }\; i = 1, \ldots, m, \\ & h_j(x) = 0 \;\text{ for }\; j = 1, \ldots, p, \\ & x \in \Omega, \end{cases}
$$

where the real-valued functions f_k as $k = 1, \ldots, n$, g_i as $i = 1, \ldots, m$, and h_j as $j = 1, \ldots, p$ are assumed to be Lipschitz continuous, and where the Pareto maximization is determined by the positive ordering cone of $I\!R^n$. That is, $\bar{x}$ is a global maximizer of problem (4.20) if there is no feasible solution $x \in S := \{x \in \Omega \mid g_i(x) \le 0 \;\text{ for }\; i = 1, \ldots, m \;\text{ and }\; h_j(x) = 0 \;\text{ for }\; j = 1, \ldots, p\}$ such that

$$
\begin{cases} f_k(x) \ge f_k(\bar{x}) \;\text{ for all }\; k \in \{1, \ldots, n\} \;\text{ and} \\ f_{k_0}(x) > f_{k_0}(\bar{x}) \;\text{ for some }\; k_0 \in \{1, \ldots, n\}. \end{cases}
$$

We can see that problem (4.20) is a particular case of (1.3) with

$$
G(x) := \prod_{i=1}^{m} \Big[g_i(x), \infty \Big) \times \prod_{j=1}^{p} \Big\{ h_j(x) \Big\} \;\text{ and }\; Q(x) \equiv I\!R^m_+ \times \{\mathbf{0}_{I\!R^p}\}.
$$

It is not hard to check that

$$
D^*G(\bar{u})(\lambda, \kappa) \subset \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{u}) + \sum_{j=1}^{p} \kappa_j \partial h_j(\bar{u})
$$

under the assumed Lipschitz continuity and that

$$
D^*Q(\bar{u})(\lambda, \kappa) \equiv \{\mathbf{0}\} \;\text{ for all }\; \lambda \in I\!R^m_+ \;\text{ and }\; \kappa \in I\!R^p.
$$

This allows us to deduce sufficient conditions for the multiobjective mathematical programs (4.20) from those for MOPEC obtained in Theorem 4.1.

Corollary 4.3 (sufficient conditions for global Pareto maximizers to multiobjective mathematical programs)**.** *Consider problem* (4.20) *on a Fréchet smooth space X with the feasible solution set S defined above. Assume that all the functions involved are locally Lipschitzian around the reference points and that the constraint set Ω is locally closed while enjoying the normal independence property from Definition* 3.1. *Fix $\bar{x} \in S$ with $\bar{z} := f(\bar{x})$ and assume that the $\bar{z}$-level set*

$$
\operatorname{lev}_S(f; \bar{z}) := \big\{ x \in S \;\big|\; f_i(x) \le \bar{z}_i, \;\; \textit{for} \;\; i = 1, \ldots, n \big\} \;\; \textit{with} \;\; \bar{z}_i := f_i(\bar{x})
$$

is compact in X. Suppose also that any $u \in S$ with $f(u) = \bar{z}$ is not a local minimizer of f over S and thus

$$(4.21)\quad \begin{cases} \mathbf{0} \notin \sum_{k=1}^{n} \mu_k \partial f_k(u) + \sum_{i=1}^{m} \lambda_i \partial g_i(u) + \sum_{j=1}^{p} \kappa_j \partial h_j(u) + N(u;\Omega), \text{ where} \\ \sum_{i=1}^{n} \mu_k = 1,\ \mu_k \geq 0 \text{ as } k = 1,\dots,n,\ \lambda_i \geq 0 \text{ as } i = 1,\dots,m, \text{ and} \\ \kappa_j \in I\!R \text{ as } j = 1,\dots,p. \end{cases}$$

Assume finally that the following constraint qualification

$$(4.22)\quad \begin{bmatrix} \mathbf{0} \in \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{u}) + \sum_{j=1}^{p} \kappa_j \partial h_j(\bar{u}) + N(\bar{u};\Omega) \\ \lambda_i \geq 0 \text{ as } i = 1,\dots,m \text{ and} \\ \kappa_j \in I\!R \text{ as } j = 1,\dots,p, \end{bmatrix} \Rightarrow \begin{bmatrix} \lambda_i = 0,\ i = 1,\dots,m \\ \kappa_j = 0,\ j = 1,\dots,p \end{bmatrix}$$

holds for any $\bar{u} \in S$ with $f(\bar{u}) = \bar{z}$. Then the validity of the conditions

$$(4.23)\quad \begin{cases} \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{u}) + \sum_{j=1}^{p} \kappa_j \partial h_j(\bar{u}) + N(\bar{u};\Omega) \subset N(\bar{u};\Omega) \text{ for all} \\ \lambda_i \geq 0 \text{ as } i = 1,\dots,n,\ \kappa_j \in I\!R \text{ as } j = 1,\dots,p. \end{cases}$$

$$(4.24)\quad \begin{cases} \sum_{k=1}^{n} \mu_k \partial f_k(\bar{u}) + \sum_{i=1}^{m} \lambda_i \partial g_i(\bar{u}) + \sum_{j=1}^{p} \kappa_j \partial h_j(\bar{u}) + N(\bar{u};\Omega) \subset N(\bar{u};\Omega) \\ \text{whenever } \sum_{k=1}^{n} \mu_k = 1,\ \mu_k \geq 0 \text{ as } k = 1,\dots,n, \\ \lambda_i \geq 0 \text{ as } i = 1,\dots,n,\ \kappa_j \in I\!R \text{ as } j = 1,\dots,p. \end{cases}$$

is sufficient *for the global Pareto maximality of $\bar{x}$ in problem* (4.20).

Proof. It is straightforward from Theorem 4.1. □

We conclude this section with a simple example illustrating Corollary 4.3.

Example 4.4. (illustration of sufficient optimality conditions). Consider the following multiobjective programming problem of type (4.20):

$$\begin{cases} I\!R^2_+\text{–maximize} & f(x_1,x_2) = (-x_1,-x_2)\big(=: (f_1(x), f_2(x))\big) \\ \text{subject to} & g(x_1,x_2) := |x_1| - x_2 \leq 0, \\ & (x_1,x_2) \in \Omega := I\!R_+ \times I\!R \subset I\!R^2 \end{cases}$$

and the feasible solution $\bar{x} = (0,0)$ to this problem. It is easy to check that there is no other feasible solutions $\bar{u}$ such that $f(\bar{u}) = f(\bar{x}) = (0,0)$. We directly calculate the subdifferentials of the objective and constraint functions and the normal cone to the geometric constraint set by

$$\partial f_1(0,0) = \{(-1,0)\},\ \ \partial f_2(0,0) = \{(-1,0)\},$$

$$\partial g(0,0) = [-1,1] \times \{-1\},\ \text{ and } N\big((0,0);\Omega\big) = \{0\} \times I\!R_-.$$

There are two multipliers $\mu_1 =: \mu$ and $\mu_2 = 1 - \mu$ with $\mu \in [0, 1]$ corresponding the two objective functions. For any $\mu \in [0, 1]$ and $\lambda \geq 0$, all the four conditions (4.21), (4.22), (4.23), and (4.24) are satisfied:

- $$\begin{aligned} \mathbf{0} &\notin \mu\partial f_1(0,0) + (1-\mu)\partial f_2(0,0) + \lambda\partial g(0,0) + N\big((0,0);\Omega\big) \\ &= (-\mu, 0) + (0, \mu - 1) + [-\lambda, \lambda] \times \{-\lambda\} + I\!R \times I\!R_-; \end{aligned}$$
- $$\mathbf{0} \in \lambda\partial g(0,0) + N\big((0,0);\Omega\big) = [-\lambda, \lambda] \times \{-\lambda\} + I\!R \times I\!R_- \Longrightarrow \lambda = 0;$$
- $$\begin{aligned} &\lambda\partial g(0,0) + N\big((0,0);\Omega\big) \subset [-\lambda, \lambda] \times \{-\lambda\} + I\!R \times I\!R- \\ &\qquad \subset I\!R \times I\!R_- = N\big((0,0);\Omega\big); \end{aligned}$$
- $$\begin{aligned} &\mu\partial f_1(0,0) + (1-\mu)\partial f_2(0,0) + \lambda\partial g(0,0) + N\big((0,0);\Omega\big) \\ &= (-\mu, 0) + (0, \mu - 1) + [-\lambda, \lambda] \times \{-\lambda\} + I\!R \times I\!R_- \\ &\subset I\!R \times I\!R_- = N\big((0,0);\Omega\big). \end{aligned}$$

Thus $\bar{x} = (0, 0)$ is a maximizer of the problem under consideration by Corollary 4.3.

5. Concluding remarks

New sufficient conditions for Pareto optimal solutions to constrained multiobjective problems with and without equilibrium constraints are established in Theorems 3.2, 4.1 and their consequences by using advanced techniques of variational analysis and generalized differentiation in the novel dual-space approach. It is important to emphasize that we do not seek sufficient optimality conditions by strengthening necessary optimality ones with additional convexity-type assumptions while considering fully nonconvex settings. It is a challenging issue of our future research to unify these two independent approaches and to derive in this way the most appropriate sufficient optimality conditions in both frameworks of scalar and multiobjective optimization.

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Truong Q. Bao
Department of Mathematics & Computer Science, Northern Michigan University, Marquette, Michigan 49855, USA
E-mail address: `btruong@nmu.edu`

Boris S. Mordukhovich
Department of Mathematics, Wayne State University, Detroit, Michigan 48202, USA
E-mail address: `boris@math.wayne.edu`