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A FIXED POINT THEOREM FOR CYCLIC MAPPINGS IN COMPLETE METRIC SPACES

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Abstract. In this note, we introduce a new type of cyclic mappings – "eventual cyclic gross contraction" in complete metric spaces and present a fixed point theorem for the eventual cyclic gross contraction, which generalizes some existing results for cyclic mappings.

1. Introduction

The study of the existence of fixed points for various cyclic mappings is an important topic in the fixed point theory. There have been many related results in literature. In 2003, by virtue of the well-known fixed point theorem (cf, e.g., [7])

Theorem 1.1. *Let (X, d) be a complete metric space, $f : X \to X$ be continuous such that*

$$d(fx, f^2x) \le kd(x, fx), \quad \text{for all } x \in X, \text{ and } 0 < k < 1,$$

then f has a fixed point in X

[7] gave some new results for cyclic mappings in complete metric spaces. For example, the following fixed point theorem was proved.

Theorem 1.2. *Let A and B be two nonempty closed subsets of a complete metric space X, $f : A \cup B \to A \cup B$ be a cyclic mapping such that*

$$d(fx, fy) \le kd(x, y), \quad \text{for all } x \in A,\ y \in B \text{ and } 0 < k < 1.$$

Then f has a unique fixed point in $A \cap B$.

Recently, Karpagam and Agrawal [6] introduced the notion of cyclic orbital contraction, and obtained the following unique fixed point for such mappings.

Theorem 1.3. *Let A and B be two nonempty closed subsets of a complete metric space X, $f : A \cup B \to A \cup B$ be a cyclic orbital contraction. Then f has a unique fixed point in $A \cap B$.*

Meanwhile, we would like to mention the cyclic Meir-Keeler contraction, a well-known cyclic mapping. Actually,

Definition 1.4 ([8])**.** A mapping $\psi : R^+ \to R^+$ is said to be a Meir-Keeler-type mapping if for each $\eta > 0$, there exists $\delta > 0$ such that for $t \in R^+$ with $\eta \le t < \eta + \delta$, we have $\psi(t) < \eta$.

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Definition 1.5 ([3]). Let (X, d) be a metric space, and let A and B be nonempty subsets of X. Then $f : A \cup B \to A \cup B$ is called a cyclic Meir-Keeler contraction if the following are satisfied:

(1) f is a cyclic mapping;
(2) for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d(x, y) < d(A, B) + \varepsilon + \delta \qquad \text{implies } d(fx, fy) < d(A, B) + \varepsilon$$
for all $x \in A$ and $y \in B$.

There have appeared many fixed point theorems for the cyclic Meir-Keeler contraction so far (cf., e.g., [6]).

After reading the papers cited above as well as [1, 2, 4, 5], in this note, we introduce a new type of cyclic mappings – "eventual cyclic gross contraction" in complete metric spaces and present a fixed point theorem for the eventual cyclic gross contraction, which generalizes some existing results for cyclic mappings.

2. Result and proof

Throughout this paper, R^+ is the set of all nonnegative real numbers and N is the set of all natural numbers. Let A and B be nonempty subsets of a metric space$(X,\ d)$.

Recall that

Definition 2.1. A mapping T: $A \cup B \to A \cup B$, T is called a cyclic mapping if $T(A) \subset B,\ T(B) \subset A$.

Definition 2.2. A function $f :\ R^+ \to R^+$ is called an altering distance function if it satisfies the following conditions:

(1) f is monotone increasing and continuous;
(2) $f(t) = 0$ if and only if $t = 0$.

We introduce the following a new type of cycle mappings – "eventual cyclic gross contraction".

Definition 2.3. Let (X, d) be a metric space, and let A and B be nonempty subsets of X. Then $T : A \cup B \to A \cup B$ is called an eventual cyclic gross contraction if the following are satisfied:

(1) f is a cyclic mapping;
(2) for some $x \in A$,
$$\begin{aligned} f(d(T^{2n}x,\ Ty)) \ &\le\ f\Big(\frac{1}{2}\Big[d(T^{2n-1}x,\ Ty) + d(y,\ T^{2n}x)\Big]\Big) \\ &\quad -g(d(T^{2n-1}x,\ Ty),\ d(y,\ T^{2n}x)), \\ &\qquad n \ge K,\ y \in A, \end{aligned} \tag{2.1}$$
where $f : R^+ \to R^+$ is an altering distance function, $g :\ R^+ \times R^+ \to R^+$ is a lower semi-continuous mapping such that $g(a,\ b) = 0$ if and only if $a = b = 0$, and $K \in N$ is sufficiently large.

Now, we are ready to state our fixed point theorem for cycle mappings in complete metric spaces.

Theorem 2.4. *Let A and B be two nonempty closed subsets of a complete metric space $(X,\ d)$, and let $T:\ A\cup B\to\ A\cup B$ be an eventual cyclic gross contraction. Then $A\bigcap B$ is nonempty and T has a unique fixed point in $A\bigcap B$.*

Proof. Set $y=T^{2n}x$ for $n\geq K$. Then, by (2.1), we have

$$\begin{aligned} f(d(T^{2n}x,\ Ty)) &= f(d(T^{2n}x,\ T^{2n+1}x)) \\ &\leq f\Big(\frac{1}{2}\Big[d(T^{2n-1}x,\ T^{2n+1}x)+d(T^{2n}x,\ T^{2n}x)\Big]\Big) \\ &\quad -g\Big(d(T^{2n-1}x,\ T^{2n+1}x),d(T^{2n}x,\ T^{2n}x)\Big) \\ &= f\Big(\frac{1}{2}d(T^{2n-1}x,\ T^{2n+1}x)\Big)-g(d(T^{2n-1}x,\ T^{2n}x),0) \\ &\leq f\Big(\frac{1}{2}d(T^{2n-1}x,\ T^{2n+1}x)\Big). \end{aligned} \tag{2.2}$$

In view of that f is a monotone increasing, we obtain, for all $n\geq K$,

$$\begin{aligned} d(T^{2n}x,\ T^{2n+1}x) &\leq \frac{1}{2}d(T^{2n-1}x,\ T^{2n+1}x) \\ &\leq \frac{1}{2}d(T^{2n-1}x,\ T^{2n}x)+\frac{1}{2}d(T^{2n}x,\ T^{2n+1}x). \end{aligned} \tag{2.3}$$

From (2.3) it follows that

$$d(T^{2n}x,\ T^{2n+1}x)\leq d(T^{2n-1}x,\ T^{2n}x),\quad n\geq K.$$

Similarly, we can deduce that for all $n\in N$ with $n\geq K$,

$$d(T^{2n-1}x,\ T^{2n}x)\leq d(T^{2n-2}x,\ T^{2n-1}x).$$

Clearly, for all $n\geq K$, the sequence $\{d(T^nx,\ T^{n-1}x)\}$ is decreasing and bounded below. Letting

$$\lim_{n\to\infty}d(T^nx,\ T^{n-1}x)=\varphi\geq 0,$$

We will claim that $\varphi=0$, Otherwise if $\varphi>0$. Taking $n\to\infty$ in (2.3), we get

$$\varphi\leq\frac{1}{2}\lim_{n\to\infty}d(T^{2n-1}x,\ T^{2n+1}x)\leq\varphi,$$

thus

$$\lim_{n\to\infty}d(T^{2n-1}x,\ T^{2n+1}x)=2\varphi.$$

Using the continuity of f and the lower semi-continuous of g and taking $n\to\infty$ in (2.2), we have

$$f(\varphi)\ \leq\ f(\varphi)-g(2\varphi,0),$$

Therefore, $g(2\varphi,0)=0$, which implies that $\varphi=0$.

Next, we will show that $\{x_n\}$ is a Cauchy sequence. If this is not true, i.e., if $\{x_n\}$ is not a Cauchy sequence, then there is a number $\varepsilon_0>0$ such that for all $k>0$, we can find two subsequences $\{n_k\}$ and $\{m_k\}$ satisfying that:

(i) $m_k>n_k>k$,
(ii) $d(T^{n_k}x,T^{m_k}x)\geq\varepsilon_0$,

Moreover, taking the critical positive integer power of satisfying (ii) above into account, we can also let these two subsequences $\{n_k\}$ and $\{m_k\}$ satisfy

(iii) $d(T^{n_k-1}x, T^{m_k}x) < \varepsilon_0$.

By the assumptions, we get

$$\varepsilon_0 \leq d(T^{n_k}x, T^{m_k}x) \leq d(T^{n_k-1}x, T^{m_k}x) + d(T^{n_k x-1}, T^{n_k}x)$$

By taking $k \to \infty$, we have

$$\varepsilon_0 \leq d(T^{n_k x}, T^{m_k}x) \leq \varepsilon_0,$$

which yields that

$$\lim_{k\to\infty} d(T^{n_k}x, T^{m_k}x) = \varepsilon_0.$$

On the other hand, we can show that

$$\begin{aligned} \varepsilon_0 &\leq d(T^{n_k}x, T^{m_k}x) \\ &\leq d(T^{n_k}x, T^{m_k-1}x) + d(T^{m_k-1}x, T^{m_k}x) \\ &\leq d(T^{n_k}x, T^{m_k}x) + 2d(T^{m_k-1}x, T^{m_k}x), \end{aligned}$$

letting $k \to \infty$, one has

$$\varepsilon_0 \leq d(T^{n_k}x, T^{m_k-1}x) \leq \varepsilon_0.$$

Considering

$$\begin{aligned} f(\varepsilon_0) &\leq f(d(T^{n_k}x, T^{m_k}x)) \\ &\leq f\Big(\frac{1}{2}[(T^{n_k}x, T^{m_k-1}x) + d(T^{n_k-1}x, T^{m_k}x)]\Big) \\ &\quad -g\Big((T^{n_k}x, T^{m_k-1}x),\ d(T^{n_k-1}x, T^{m_k}x)\Big), \end{aligned}$$

provided $k \to \infty$, we get

$$f(\varepsilon_0) \leq f(\varepsilon_0) - g(\varepsilon_0,\ \varepsilon_0),$$

which means that

$$g(\varepsilon_0,\ \varepsilon_0) = 0.$$

By the property of g, we see that

$$\varepsilon_0 = 0,$$

which is contradiction with $\varepsilon_0 > 0$. Therefore $T^n x$ is a cauchy sequence.

Since $(X,\ d)$ is a complete metric space, there exists $x_0 \in A \bigcup B$ such that

$$\lim_{n\to\infty} T^n x = x_0.$$

It follows from our assumption that $T^{2n+1}x \in A$. Hence, $x_0 \in A$. Similarly, we know that $x_0 \in B$ by $T^{2n}x \in B$. Consequently,

$$x_0 \in A \bigcap B \neq \emptyset.$$

Next, we prove that x_0 is a fixed point of T. Taking advantage of the fact that

$$\begin{aligned} f(d(Tx_0, T^{2n+1}x)) &\leq f\Big(\frac{1}{2}[d(T^{2n}x, Tx_0) + d(T^{2n-1}x, x_0)]\Big) \\ &\quad -g(d(T^{2n}x, Tx_0),\ d(T^{2n-1}x, x_0)) \end{aligned}$$

Taking $n \to \infty$, we have

$$\begin{aligned} f(d(Tx_0, x_0)) &\leq f\Big(\frac{1}{2}[d(x_0, Tx_0) + d(x_0, x_0)]\Big) \\ &\quad -g(d(x_0, Tx_0),\ d(x_0, x_0)) \\ &\leq f\Big(\frac{1}{2}d(x_0, Tx_0)\Big). \end{aligned}$$

By the monotone increasing property of f, we get

$$d(Tx_0, x_0) \leq \frac{1}{2}(d(x_0, Tx_0)),$$

which yields that

$$d(Tx_0, x_0) = 0.$$

Thus, $Tx_0 = x_0$.

Finally, let us to prove the uniqueness of fixed point of T.

If there is $x_0' \in A \bigcap B$ such that $Tx_0' = x_0'$, then

$$\begin{aligned} f(d(x_0, x_0')) &= f(d(Tx_0, Tx_0')) \\ &\leq f\Big(\frac{1}{2}[d(x_0, Tx_0') + d(Tx_0, x_0')]\Big) \\ &\quad -g(d(x_0, Tx_0'),\ d(Tx_0, x_0')) \\ &\leq f(d(x_0, x_0')) - g(d(x_0, x_0'),\ d(x_0, x_0')). \end{aligned}$$

This implies that

$$d(x_0, x_0') = 0,$$

i.e.,

$$x_0 = x_0'$$

by the properties of f and g. □

References

[1] R. P. Agarwal, M. A. Alghamdi and N. Shahzad, *Fixed point theory for cyclic generalized contractions in partial metric spaces*, Fixed Point Theory Appl. **40** (2012), 11 pp.

[2] S. Chandok, J. Liang and D. O'Regan, *Common fixed points and invariant approximations for noncommuting contraction mappings in strongly convex metric spaces*, J. Nonlinear Convex Anal., to appear.

[3] A. A. Eldred, W. A. Kirk and P. Veeramani, *Proximal normal structure and relatively nonexpansive mappings*, Studia Mathematica **171** (2005), 283–293.

[4] N. J. Huang and H. X. Li, *Fixed point theorems of compatible mappings in convex metric spaces*, Soochow J. Math. **22** (1996), 439–447.

[5] N. Hussain, A. R. Khan and R. P. Agarwal, *Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces*, J. Nonlinear Convex Anal. **11** (2010), 475–489.

[6] S. Karpagam and S. Agrawal, *Best proximity point theorems for cyclic orbital Meir-Keeler contraction maps*, Nonlinear Anal. **74** (2011), 1040–1046.

[7] W. A. Kirk, P. S. Srinivasan and P. Veeramani, *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory **4** (2003), 79–89.

[8] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326–329.

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