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A NOTE ON HALPERN'S ALGORITHM IN THE HILBERT BALL

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Abstract. We establish a strong convergence theorem for Halpern's algorithm in the Hilbert ball. This iterative process concerns the approximation of fixed points of those mappings which are nonexpansive with respect to the hyperbolic metric.

1. Introduction

It is well known that many theoretical and practical problems can be formulated as a fixed point problem,

$$x = Tx, \tag{1.1}$$

where T is an appropriate (nonlinear) operator. The solutions of this equation are called fixed points of T. If T is a strict contraction defined on a complete metric space (X, d), then the Banach contraction principle guarantees that there exists a unique solution to equation (1.1) and for any $x \in X$, the sequence of iterates $\{T^n x\}_{n=0}^{\infty}$ converges to it. If the operator T is merely nonexpansive (that is, 1-Lipschitz), then we must assume additional conditions in order to ensure the existence of fixed points of T, and even when such fixed points exist, the sequence of iterates might not, in general, converge to one of them. These facts motivate the study of the asymptotic behavior of nonexpansive mappings and of related iterative schemes, a topic which has become one of the more active areas in Nonlinear Functional Analysis. Much of the research has so far focused on the case where T is a self-mapping of a closed and convex subset K of a Banach space. Basically, two types of algorithms have been considered: the Halpern algorithm [3] and the Mann algorithm [6].

In this note we focus on Halpern's algorithm, which we now recall. Let points $u, x_0 \in K$ and a sequence $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$ be given. Halpern's algorithm generates a sequence $\{x_n\}_{n=0}^{\infty}$ via the recursion

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n, \quad n \geq 0. \tag{1.2}$$

In the case where the nonexpansive mapping T has fixed points, the sequence generated by Halpern's algorithm strongly converges to one of them in Hilbert space [3] and in certain Banach spaces [8] (under appropriate assumptions on the sequence $\{\alpha_n\}_{n=0}^{\infty}$).

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Since the operator T is nonlinear and the algorithm has a convex structure, it is natural to ask whether it is possible to develop corresponding results for nonexpansive mappings in nonlinear spaces, such as Hadamard manifolds (that is, complete and simply connected Riemannian manifolds of nonpositive curvature), by extending concepts and techniques which originate in Euclidean, Hilbert and Banach spaces. Indeed, in [5] an analogue of (1.2) is studied in Hadamard manifolds of finite dimension. We are interested in proving analogous results for a particular infinite dimensional manifold, namely the Hilbert ball $\mathbb{B}$ endowed with the hyperbolic metric ρ [2]. In this paper we study the analogue of (1.2) in this space and prove a theorem regarding the strong convergence of sequences generated by it to a fixed point of a ρ-nonexpansive mapping T, when such a point exists (see Theorem 3.1 below).

2. Preliminaries

Let (X,ρ) be a complete metric space. A mapping $c:\mathbb{R}\to X$ is called a metric embedding of $\mathbb{R}$ into X if $\rho(c(s),c(t))=|s-t|$ for all $s,t\in\mathbb{R}$. The image of $\mathbb{R}$ under a metric embedding is called a *metric line* and the image of an interval $[a,b]$ under such a mapping is called a *metric segment*. Assume that (X,ρ) contains a family M of metric lines such that for each pair of points $x,y\in X$, there is a unique metric line in M passing through them. This metric line determines a unique metric segment connecting x to y, which we denote by $[x,y]$. For each $0\le t\le 1$, there is a unique point $z\in[x,y]$ such that

$$\rho(x,z)=t\rho(x,y)\ \text{ and }\ \rho(z,y)=(1-t)\rho(x,y).$$

This point is denoted by $(1-t)x\oplus ty$. We say that (X,ρ,M) is a *hyperbolic space* [9] if

$$\rho\left(\frac{1}{2}x\oplus\frac{1}{2}y,\frac{1}{2}x\oplus\frac{1}{2}z\right)\le\frac{1}{2}\rho(y,z)$$

for all $x,y,z\in X$. An equivalent condition is that the inequality

$$\rho((1-t)x\oplus ty,(1-t)w\oplus tz)\le(1-t)\rho(x,w)+t\rho(y,z)$$

holds for all $0\le t\le 1$ and $x,y,z,w\in X$.

A subset $K\subset X$ is called ρ-convex if $x,y\in K\Rightarrow[x,y]\subset K$.

Clearly, all Banach spaces are hyperbolic spaces. In addition, Hadamard manifolds are also hyperbolic spaces. We are mainly interested in the infinite dimensional Hilbert ball endowed with the hyperbolic metric, some properties of which we now recall.

Let $\mathbb{B}$ be the open unit ball of a complex Hilbert space $(H,\langle\cdot,\cdot\rangle)$ with induced norm $\|\cdot\|$. For $a\in\mathbb{B}$, the Möbius transformations M_a on $\mathbb{B}$ have the form

$$M_a(z)=(P_a+\sqrt{1-\|a\|^2}Q_a)m_a(z),\ z\in\mathbb{B},$$

where P_a is the orthogonal projection onto the subspace spanned by a, $Q_a=I-P_a$ and

$$m_a(z)=(z+a)/(1+\langle z,a\rangle),\ z\in\mathbb{B}.$$

The metric ρ on $\mathbb{B}$ is given by $\rho(x,y)=\tanh^{-1}\|M_{-x}(y)\|$, where $\tanh^{-1}$ denotes the inverse hyperbolic tangent. The Hilbert ball $\mathbb{B}$ endowed with this metric ρ is a hyperbolic space.

With each $x \in \mathbb{B}$ we associate a tangent Hilbert space H_x the elements of which are denoted by $\{[x,y]\}_{y\in\mathbb{B}}$. In the language of differential geometry, the vector $[x,y] \in H_x$ is identified with the vector v in the tangent space at x for which $\exp_x v = y$, where $\exp_x$ is the exponential map at x. The linear structure and the inner product in H_x are determined by the (surjective) mapping $i : H_x \to H$ defined by

$$i([x,y]) = \begin{cases} (\rho(x,y)/\|M_{-x}(y)\|)M_{-x}(y) & y \neq x \\ 0 & y = x \end{cases}.$$

Note that the norm of the vector $[x,y]$ in H_x is $\rho(x,y)$ and that for every $x,y \in \mathbb{B}$, the Hilbert spaces H_x and H_y are isometric via the isometry $U_{xy} : H_x \to H_y$ given by $U_{xy}[x,z] = [y, M_y(M_{-x}(z))]$ for all $z \in \mathbb{B}$.

For more details concerning hyperbolic spaces and the Hilbert ball, see [2], [9] and [11].

Let K be a nonempty, ρ-closed and ρ-convex subset of $\mathbb{B}$. We denote by P_K the *nearest point projection* onto K defined by

$$P_K(p) = \{p_0 \in K \;:\; \rho(p,p_0) \leq \rho(p,q) \;\; \forall q \in K\}, \;\; \forall p \in \mathbb{B}.$$

Lemma 2.1. *For any point $p \in \mathbb{B}$, $P_K(p)$ is a singleton and the following inequality holds for all $q \in K$:*

$$\mathrm{Re}\langle [P_K(p),p],[P_K(p),q]\rangle \leq 0.$$

Proof. A proof of the existence and uniqueness of the point $P_K(p)$ can be found in [2, page 108]. The fact that such a point is unique also implies that for $r = \rho(p, P_K(p))$ the closed ball $B(p,r)$ of radius r about p intersects K only at $P_K(p)$.

If there were a point $q \in K$ with $\mathrm{Re}\langle [P_K(p),p],[P_K(p),q]\rangle > 0$, then we would have points $(1-t)P_K(p) \oplus tq$ on the metric segment $[P_K(p),q]$ with positive small parameter t which are both in K and $B(p,r)$. This contradicts our previous conclusion and completes the proof (*cf.* [13, page 96]).

Alternatively, recall [12, page 642] that a self-mapping T of $\mathbb{B}$ is firmly nonexpansive (of the first kind) if and only if

$$\mathrm{Re}\{\langle [Tx,Ty],[Tx,x]\rangle + \langle [Ty,Tx],[Ty,y]\rangle\} \leq 0$$

for all $x,y \in \mathbb{B}$. Since $P_K : \mathbb{B} \to \mathbb{B}$ is known to be firmly nonexpansive (of the first kind) [2, page 124], we may take $x = p \in \mathbb{B}$ and $q \in K$, and obtain that $P_K q = q$ and $\mathrm{Re}\langle [P_K(p),p],[P_K(p),q]\rangle \leq 0$, as claimed.

Let $\{x_n\}_{n=0}^{\infty}$ be a ρ-bounded sequence in $\mathbb{B}$, and let K be a ρ-closed and ρ-convex subset of $\mathbb{B}$. Consider the functional $f : \mathbb{B} \to [0,\infty)$ defined by

$$f(x) = \limsup_{n\to\infty} \rho(x_n, x).$$

A point z in K is said to be an *asymptotic center* of the sequence $\{x_n\}_{n=0}^{\infty}$ with respect to K if $f(z) = \min\{f(x) \;:\; x \in K\}$. The minimum of f over K is called the *asymptotic radius* of $\{x_n\}_{n=0}^{\infty}$ with respect to K.

Proposition 2.2 ([2, page 116])**.** *Every ρ-bounded sequence in $(\mathbb{B},\rho)$ has a unique asymptotic center with respect to any ρ-closed and ρ-convex subset of $\mathbb{B}$.*

The asymptotic center of $\{x_n\}_{n=0}^{\infty}$ with respect to K will be denoted by $A(K, \{x_n\})$ and its asymptotic radius by $r(K, \{x_n\})$. If $K = \mathbb{B}$ we shall write $A(\{x_n\})$ and $r(\{x_n\})$, respectively.

Lemma 2.3 ([2, page 116])**.** *If $\{x_n\} \subset K$, then $A(\{x_n\}) = A(K, \{x_n\})$.*

Proposition 2.4 ([2, page 117])**.** *If a ρ-bounded sequence $\{x_n\}_{n=0}^{\infty}$ converges weakly to x, then $x = A(\{x_n\})$.*

We say that a mapping $T : K \to K$ is *ρ-nonexpansive* if for any two points $x, y \in K$, the following inequality holds:

$$\rho(Tx, Ty) \leq \rho(x, y).$$

It is known that every holomorphic self-mapping of $\mathbb{B}$ is ρ-nonexpansive [2, page 118].

Let $T : K \to K$ be a ρ-nonexpansive mapping. We shall call a sequence $\{y_n\}_{n=0}^{\infty} \subset K$ an approximating sequence for T if $\lim_{n\to\infty} \rho(y_n, Ty_n) = 0$.

Theorem 2.5 ([2, page 120])**.** *Let $T : K \to K$ be a ρ-nonexpansive mapping. The following statements are equivalent:*

(a) *T has a fixed point;*
(b) *There exists a point x in K such that the sequence of iterates $\{T^n x\}_{n=0}^{\infty}$ is ρ-bounded;*
(c) *The sequence of iterates $\{T^n x\}_{n=0}^{\infty}$ is ρ-bounded for all x in K;*
(d) *There exists a ρ-bounded approximating sequence for T.*

The asymptotic centers of the sequences in parts (b) and (d) are fixed points of T.

We also need the following result concerning the structure of the fixed point set $\mathrm{Fix}(T)$ of T.

Theorem 2.6 ([2, page 120])**.** *The fixed point set of a ρ-nonexpansive mapping $T : K \to K$ is ρ-closed and ρ-convex.*

Since the CN inequality holds in the Hilbert ball $(\mathbb{B}, \rho)$ [9, page 541], we know that $(\mathbb{B}, \rho)$ is a CAT(0) space [1, page 163].

A *metric triangle* $\Delta(p_1, p_2, p_3)$ in the Hilbert ball $\mathbb{B}$ is the set consisting of three points p_1, p_2 and p_3 in $\mathbb{B}$, and the three metric segments joining p_i and p_{i+1}, where $i = 1, 2, 3 \pmod 3$.

The following lemma is a basic tool for comparing the geometry of the Hilbert ball to that of the Euclidean plane.

Lemma 2.7 ([1, page 24])**.** *Let $\Delta(p, q, r)$ be a metric triangle in $\mathbb{B}$. Then there exist points $p', q', r' \in \mathbb{R}^2$ such that*

$$\rho(p, q) = \|p' - q'\|, \quad \rho(q, r) = \|q' - r'\|, \quad \rho(r, p) = \|r' - p'\|.$$

The triangle $\Delta(p', q', r')$ is called the *comparison triangle* of the metric triangle $\Delta(p, q, r)$. It is unique up to isometry. A point $x' \in [q', r']$ is called the comparison point for $x \in [q, r]$ if $\rho(q, x) = \|q' - x'\|$. Comparison points on $[p', q']$ and $[r', p']$ are defined in the same way. If $p \neq q$ and $p \neq r$, then the angle at p is the Riemannian angle between the vectors $[p,q]$ and $[p,r]$ in H_p, namely the unique $\alpha \in [0,\pi]$ for which

$$\cos\alpha = \frac{\operatorname{Re}\langle [p,q],[p,r]\rangle}{\rho(p,q)\cdot\rho(p,r)}.$$

The interior angle α' of $\Delta(p',q',r')$ at p' is called the comparison angle of α.

For every pair of points $x, y \in \Delta(p,q,r)$ and their comparison points $x', y' \in \Delta(p',q',r')$, the CAT(0) inequality holds:

$$\rho(x,y) \le \|x'-y'\|.$$

Lemma 2.8 ([1, page 161])**.** *Let $\Delta(p,q,r)$ be a metric triangle in the Hilbert ball $\mathbb{B}$ and let $\Delta(p',q',r')$ be its comparison triangle.*

(i) *Let α,β,γ (respectively, α',β',γ') be the angles of $\Delta(p,q,r)$ (respectively, $\Delta(p',q',r')$) at the vertices p,q,r (respectively, p',q',r'). Then the following inequalities hold:*

$$\alpha' \ge \alpha, \quad \beta' \ge \beta, \quad \gamma' \ge \gamma.$$

(ii) *Let z be a point on the metric segment $[p,q]$ and z' its comparison point on the side $[p',q']$. Then the following inequality holds:*

$$\rho(z,r) \le \|z'-r'\|.$$

Lemma 2.9 ([14])**.** *Let $\{\beta_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two real sequences satisfying the following conditions:*

(i) $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$*;*
(ii) $\limsup_{n\to\infty} b_n \le 0$.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of non-negative real numbers such that

$$a_{n+1} \le (1-\beta_n)a_n + \beta_n b_n, \quad n \ge 0.$$

Then $\lim_{n\to\infty} a_n = 0$.

3. Main result

Let K be a ρ-closed and ρ-convex subset of $\mathbb{B}$ and let $T : K \to K$ be a ρ-nonexpansive mapping. Let points $u, x_0 \in K$ and a sequence $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ be given. Then Halpern's algorithm in $\mathbb{B}$ generates the sequence of iterations defined by the recursion

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n, \quad \forall n \ge 0. \tag{3.1}$$

Theorem 3.1. *Let K be a ρ-closed and ρ-convex subset of $\mathbb{B}$ and let $T : K \to K$ be a ρ-nonexpansive mapping with $F := Fix(T) \ne \emptyset$. Let points $u, x_0 \in K$ be given. Suppose that a sequence $\{\alpha_n\}_{n=0}^{\infty} \subset (0,1)$ satisfies*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$*;*
(ii) $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ *or* $\lim_{n\to\infty}(\alpha_n-\alpha_{n-1})/\alpha_n = 0$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by algorithm (3.1) strongly converges to $P_F(u)$.

A similar result under stronger assumptions on the sequence $\{\alpha_n\}_{n=0}^{\infty}$ was proved in [4] by using different techniques. Closely related theorems, established by employing other methods, can also be found in [10, 7].

Proof. The proof of Theorem 3.1 is divided into four steps.

Step 1. The sequences $\{x_n\}_{n=0}^{\infty}$ and $\{Tx_n\}_{n=0}^{\infty}$ are ρ-bounded.
Take $x \in F$ and set $M := \max\{\rho(u,x), \rho(x_0,x)\}$. We prove that $\rho(x_n,x) \leq M$ for all n by mathematical induction on n. It is clear that $\rho(x_0,x) \leq M$. Using now the hyperbolic property of the metric ρ, the ρ-nonexpansivity of T and the inductive hypothesis, we obtain

$$\begin{aligned}\rho(x_{n+1},x) &= \rho(\alpha_n u \oplus (1-\alpha_n)Tx_n, x)\\ &\leq \alpha_n \rho(u,x) + (1-\alpha_n)\rho(Tx_n,x)\\ &\leq \alpha_n \rho(u,x) + (1-\alpha_n)\rho(x_n,x)\\ &\leq M.\end{aligned}$$

Thus $\{x_n\}_{n=0}^{\infty}$ is indeed ρ-bounded and the ρ-boundedness of $\{Tx_n\}_{n=0}^{\infty}$ is a direct consequence.

Step 2. $\lim_{n\to\infty} \rho(x_{n+1},x_n) = 0$.
By Step 1, we can find a constant C such that

$$\rho(x_n,x_{n-1}) \leq C \quad\text{and}\quad \rho(u,Tx_n) \leq C \tag{3.2}$$

for all $n \geq 1$ and $n \geq 0$, respectively.

For each $n \geq 1$, consider the metric segment $[u, Tx_{n-1}]$ in $\mathbb{B}$ as the metric embedding of the real interval $[s,t]$ under $c : \mathbb{R} \to \mathbb{B}$. Namely, $c(s) = u$ and $c(t) = Tx_{n-1}$.

Using the hyperbolic property of the metric ρ, the nonexpansivity of T and (3.2), we obtain for each $n \geq 1$,

$$\begin{aligned}\rho(x_{n+1},x_n) &= \rho(\alpha_n u \oplus (1-\alpha_n)Tx_n, \alpha_{n-1}u \oplus (1-\alpha_{n-1})Tx_{n-1})\\ &\leq \rho(\alpha_n u \oplus (1-\alpha_n)Tx_n, \alpha_n u \oplus (1-\alpha_n)Tx_{n-1})\\ &\quad + \rho(\alpha_n u \oplus (1-\alpha_n)Tx_{n-1}, \alpha_{n-1}u \oplus (1-\alpha_{n-1})Tx_{n-1})\\ &\leq \alpha_n \rho(u,u) + (1-\alpha_n)\rho(Tx_n, Tx_{n-1})\\ &\quad + \rho(c(\alpha_n s + (1-\alpha_n)t), c(\alpha_{n-1}s + (1-\alpha_{n-1})t))\\ &\leq (1-\alpha_n)\rho(x_n,x_{n-1})\\ &\quad + |(\alpha_n s + (1-\alpha_n)t) - (\alpha_{n-1}s + (1-\alpha_{n-1})t)|\\ &= (1-\alpha_n)\rho(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}||t-s|\\ &= (1-\alpha_n)\rho(x_n,x_{n-1}) + |\alpha_n - \alpha_{n-1}|\rho(u,Tx_{n-1}),\end{aligned}$$

and finally,

$$\rho(x_{n+1},x_n) \leq (1-\alpha_n)\rho(x_n,x_{n-1}) + C|\alpha_n - \alpha_{n-1}|. \tag{3.3}$$

Thus, if $\lim_{n\to\infty}(\alpha_n - \alpha_{n-1})/\alpha_n = 0$, we can apply Lemma 2.9 to conclude that $\lim_{n\to\infty} \rho(x_{n+1},x_n) = 0$.

As to the case where $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, let $k \leq n$. From (3.3) we see that

$$\rho(x_{n+1},x_n) \leq C\prod_{i=k}^{n}(1-\alpha_i) + C\sum_{i=k}^{n}|\alpha_i - \alpha_{i-1}|.$$

Since $\prod_{i=k}^{\infty}(1-\alpha_i) = 0$ for all $k \geq 1$, letting $n \to \infty$, we get

$$\limsup_{n\to\infty} \rho(x_{n+1}, x_n) \leq C \sum_{i=k}^{\infty} |\alpha_i - \alpha_{i-1}|.$$

Now letting $k \to \infty$ and using the limit

$$\lim_{k\to\infty} \sum_{i=k}^{\infty} |\alpha_i - \alpha_{i-1}| = 0,$$

we see that in this case too we have $\lim_{n\to\infty} \rho(x_{n+1}, x_n) = 0$, as claimed.

Step 3. $\limsup_{n\to\infty} \mathrm{Re}\langle [P_F(u), u], [P_F(u), Tx_n]\rangle \leq 0$.
By Step 1, the sequence $\{\mathrm{Re}\langle [P_F(u), u], [P_F(u), Tx_n]\rangle\}$ is bounded; hence its upper limit is finite. Evidently, we can find a subsequence $\{z_k\}_{k=1}^{\infty}$ of $\{x_n\}_{n=0}^{\infty}$, $z_k := x_{n_k}$, $k \geq 1$, so that, defining

$$c_k := \mathrm{Re}\langle [P_F(u), u], [P_F(u), Tz_k]\rangle,$$

we have

$$\limsup_{n\to\infty} \mathrm{Re}\langle [P_F(u), u], [P_F(u), Tx_n]\rangle = \lim_{k\to\infty} c_k.$$

Since $\{x_n\}_{n=0}^{\infty}$ is ρ-bounded by Step 1, we may assume, without any loss of generality, that $z_k \rightharpoonup \bar{x}$ as $k \to \infty$ for some $\bar{x} \in \mathbb{B}$, where $\rightharpoonup$ denotes weak convergence. Then, by Proposition 2.4, $\bar{x}$ is the asymptotic center of $\{z_k\}_{k=1}^{\infty}$.

Next we show that $\{z_k\}_{k=1}^{\infty}$ is an approximating sequence for T. Indeed,

$$\rho(z_k, Tz_k) = \rho(x_{n_k}, Tx_{n_k}) \leq \rho(x_{n_k}, x_{n_k+1}) + \rho(x_{n_k+1}, Tx_{n_k}).$$

By Step 2, $\lim_{k\to\infty} \rho(x_{n_k+1}, x_{n_k}) = 0$. Also, using the definition of the algorithm and the hyperbolic property of the metric ρ, we see that

$$\rho(x_{n_k+1}, Tx_{n_k}) = \rho(\alpha_{n_k} u \oplus (1-\alpha_{n_k})Tx_{n_k}, Tx_{n_k}) \leq \alpha_{n_k}\rho(u, Tx_{n_k}),$$

and since, by Step 1, the sequence $\{\rho(u, Tx_{n_k})\}_{k=1}^{\infty}$ is bounded, we therefore obtain

$$\lim_{k\to\infty} \rho(z_k, Tz_k) = 0. \tag{3.4}$$

Thus $\{z_k\}_{k=1}^{\infty}$ is indeed an approximating sequence for T and applying Theorem 2.5, we conclude that its asymptotic center $\bar{x}$ is a fixed point of T. Namely, $\bar{x} \in F$. From (3.4) it also follows that $z_k - Tz_k \to 0$ as $k \to \infty$ [2, page 91]. Hence $Tz_k \rightharpoonup \bar{x}$ as $k \to \infty$.

Using the definition of the inner product in the tangent Hilbert space, we see that without loss of generality we may write, denoting the norm in H by $|\cdot|$,

$$\begin{aligned} c_k &= \mathrm{Re}\langle i([P_F(u), u]), i([P_F(u), Tz_k])\rangle \\ &= \mathrm{Re}\langle i([P_F(u), u]), \frac{\rho(P_F(u), Tz_k)}{|M_{-P_F(u)}(Tz_k)|} M_{-P_F(u)}(Tz_k)\rangle. \end{aligned}$$

If $\bar{x} = P_F(u)$, then using the weak continuity of $M_{-P_F(u)}$ [2, page 116], we have

$$\mathrm{Re}\langle i([P_F(u), u]), M_{-P_F(u)}(Tz_k)\rangle \to 0.$$

By Step 1, the sequence $\{\rho(P_F(u), Tz_k)\}_{k=1}^{\infty}$ is bounded and so if the sequence $\{1/(|M_{-P_F(u)}(Tz_k)|)\}_{k=1}^{\infty}$ is also bounded, then $\lim_{k\to\infty} c_k = 0$. Assume there exists a subsequence $\{M_{-P_F(u)}(Tz_{k_l})\}_{l=1}^{\infty}$ that tends to the origin as $l \to \infty$. Applying $M_{P_F(u)}$, we get $Tz_{k_l} \to M_{P_F(u)}(0) = P_F(u)$ as $l \to \infty$. Hence in this case $\rho(P_F(u), Tz_{k_l}) \to 0$ as $l \to \infty$, the subsequence

$$\{(M_{-P_F(u)}(Tz_{k_l}))/(|M_{-P_F(u)}(Tz_{k_l})|)\}_{l=1}^{\infty}$$

is bounded, and once again we have $\lim_{l\to\infty} c_{k_l} = 0$.

Assume now that $\bar{x} \neq P_F(u)$. In this case we see that

$$c_k = a_k \cdot b_k,$$

where

$$a_k := \frac{|M_{-P_F(u)}(\bar{x})|}{\rho(P_F(u), \bar{x})} \cdot \frac{\rho(P_F(u), Tz_k)}{|M_{-P_F(u)}(Tz_k)|}$$

and

$$b_k := \mathrm{Re}\langle i([P_F(u), u]), \frac{\rho(P_F(u), \bar{x})}{|M_{-P_F(u)}(\bar{x})|} M_{-P_F(u)}(Tz_k)\rangle.$$

Using again the weak continuity of $M_{-P_F(u)}$ along with Lemma 2.1, we see that

$$\lim_{k\to\infty} b_k = \mathrm{Re}\langle [P_F(u), u], [P_F(u), \bar{x}]\rangle \leq 0.$$

As for the sequence $\{a_k\}_{k=1}^{\infty}$, it is clearly nonnegative. We claim that it is bounded. For that we need to make sure that the sequence $\{M_{-P_F(u)}(Tz_k)\}_{k=1}^{\infty}$ is bounded away from the origin. As before, assume to the contrary that there exists a subsequence $\{M_{-P_F(u)}(Tz_{k_l})\}_{l=1}^{\infty}$ which tends to the origin as $l \to \infty$. Applying $M_{P_F(u)}$, we get $Tz_{k_l} \to M_{P_F(u)}(0) = P_F(u)$ as $l \to \infty$ and so $\bar{x} = P_F(u)$. This contradicts our assumption that $\bar{x} \neq P_F(u)$, and so we see that the sequence $\{a_k\}_{k=1}^{\infty}$ is indeed bounded. We conclude that $\lim_{k\to\infty} c_k \leq 0$, as claimed.

Step 4. $\lim_{n\to\infty} \rho(x_n, P_F(u)) = 0$.
Fix $n \geq 0$, and set $p = Tx_n$ and $q = P_F(u)$. Considering the metric triangle $\Delta(u, p, q)$ and its comparison triangle $\Delta(u', p', q')$, we have

$$\rho(u, q) = \|u' - q'\| \quad \text{and} \quad \rho(p, q) = \|p' - q'\|.$$

Let β denote the angle at q and β' its comparison angle at q'. Then by part (i) of Lemma 2.8 we have $\beta \leq \beta'$ and so $\cos\beta' \leq \cos\beta$.

Consider the point $x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)p$ on the metric segment $[u, p]$ and denote its comparison point by $x'_{n+1} = \alpha_n u' + (1 - \alpha_n)p'$. Then, by part (ii) of Lemma 2.8, we have

$$\begin{aligned}
\rho^2(x_{n+1}, P_F(u)) &\leq \|x'_{n+1} - q'\|^2 \\
&= \|\alpha_n(u' - q') + (1 - \alpha_n)(p' - q')\|^2 \\
&= \alpha_n^2\|u' - q'\|^2 + (1 - \alpha_n)^2\|p' - q'\|^2 \\
&\quad + 2\alpha_n(1 - \alpha_n)\|u' - q'\|\|p' - q'\| \cos\beta' \\
&\leq \alpha_n^2\rho^2(u, q) + (1 - \alpha_n)^2\rho^2(p, q) \\
&\quad + 2\alpha_n(1 - \alpha_n)\rho(u, q)\rho(p, q)\cos\beta
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)^2\rho^2(Tx_n, P_F(u)) \\
&\quad + \alpha_n[\alpha_n\rho^2(u, P_F(u)) + 2(1-\alpha_n)\mathrm{Re}\langle[P_F(u), u], [P_F(u), Tx_n]\rangle] \\
&\le (1-\alpha_n)\rho^2(x_n, P_F(u)) + \alpha_n b_n,
\end{aligned}
$$

where

$$b_n = \alpha_n\rho^2(u, P_F(u)) + 2(1-\alpha_n)\mathrm{Re}\langle[P_F(u), u], [P_F(u), Tx_n]\rangle.$$

By Step 3, $\limsup_{n\to\infty} b_n \le 0$. Thus we may apply Lemma 2.9 and conclude that $\lim_{n\to\infty} \rho^2(x_n, P_F(u)) = 0$. Hence the sequence $\{x_n\}_{n=0}^{\infty}$ converges in norm to $P_F(u)$ [2, page 91] and this completes the proof of Theorem 3.1.

Alternatively, we may use the CN inequality and the law of cosines [12, page 638] to obtain

$$
\begin{aligned}
\rho^2(q, \alpha_n u \oplus (1-\alpha_n)p) &\le \alpha_n\rho^2(q, u) + (1-\alpha_n)\rho^2(q, p) - \alpha_n(1-\alpha_n)\rho^2(u, p) \\
&\le \alpha_n^2\rho^2(q, u) + (1-\alpha_n)^2\rho^2(q, p) + 2\alpha_n(1-\alpha_n)\mathrm{Re}\langle[q, u], [q, p]\rangle \\
&\le (1-\alpha_n)^2\rho^2(p, q) + \alpha_n b_n \le (1-\alpha_n)\rho^2(x_n, P_F(u)) + \alpha_n b_n,
\end{aligned}
$$

as before.

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