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FIXED POINT RESULTS ON ORDERED METRIC SPACES AND EXISTENCE OF SOLUTIONS TO A SYSTEM OF NONLINEAR INTEGRAL EQUATIONS

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Abstract. We establish fixed point results on ordered metric spaces for a new class of contractive mappings. We use the obtained fixed point theorems in this paper to study the existence of solutions to a system of nonlinear integral equations.

1. Introduction

Fixed point theory is an important branch of modern mathematics and has always been a major theoretical tool in fields such as differential equations, topology, functional analysis, ...

The investigation of the existence of fixed points on ordered metric spaces was first considered by Turinici in [21]. Ran and Reurings in [17] extended the Banach contraction principle to the setting of ordered metric spaces, and applied their result to the study of positive definite solutions to a class of matrix equations. This study was extended in [14], where certain uniqueness and existence results for ordinary differential equations were considered. For other fixed point results on ordered metric spaces, we refer the reader to [1, 5, 6, 7, 8, 9, 13, 15, 16, 18] and the references therein. Bhaskar and Lakshmikantham [4] initiated and proved some new coupled fixed point results for mixed monotone and contraction mappings in partially ordered metric spaces. The obtained results in [4] were extended and generalized by many authors, for more details, we refer the reader to [2, 3, 10, 11, 12, 19, 20] and the references therein.

In this paper, we establish fixed point and coupled fixed point theorems on the setting of ordered metric spaces for a new class of contractive mappings. The presented theorems extend and generalize several fixed point results existing in the literature, in particular the obtained results in [4, 17]. Moreover, we apply our main results in this paper to discuss the existence of solutions to a class of systems of nonlinear integral equations.

2. Main results

At first, we need the following concepts.

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Definition 2.1. Let $(X, \preceq)$ be a partially ordered set and $A : X \to X$. We say that A is nondecreasing (with respect to $\preceq$) if

$$x, y \in X, \quad x \preceq y \Longrightarrow Ax \preceq Ay.$$

Definition 2.2. Let $(X, \preceq)$ be a partially ordered set and $A : X \times X \to X$. We say that A has the mixed monotone property if

$$(x, y), (u, v) \in X \times X, \quad x \preceq u,\, y \succeq v \Longrightarrow A(x, y) \preceq A(u, v).$$

Definition 2.3. Let $(X, \preceq)$ be a partially ordered set and $A : X \times X \to X$. We say that $(x, y) \in X \times X$ is a coupled fixed point of A if

$$x = A(x, y) \text{ and } y = A(y, x).$$

Definition 2.4. Let $(X, \preceq)$ be a partially ordered set endowed with a metric d. We say that $(X, \preceq, d)$ is regular if for every sequence $\{u_n\} \subset X$ satisfying

(i) $\lim_{n\to\infty} d(u_n, u) = 0$, for some $u \in X$;
(ii) $u_n \preceq u_{n+1}$ for all n,

we have $u_n \preceq u$ for all n.

Our first result is the following fixed point theorem.

Theorem 2.5. *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X, d) is a complete metric space. We suppose that $(X, \preceq, d)$ is regular. Let $A : X \to X$ be a nondecreasing mapping that satisfies: for any $0 < a < b < \infty$, there exists $0 < k(a, b) < 1$ such that*

$$x, y \in X, \quad x \succ y, \quad a \leq d(x, y) \leq b \Longrightarrow d(Ax, Ay) \leq k(a, b)d(x, y). \tag{2.1}$$

Suppose also that there exists $x_0 \in X$ such that $x_0 \preceq Ax_0$. Then A has a fixed point, that is, there exists $x^* \in X$ such that $Ax^* = x^*$.

Proof. We make the proof in three steps.

Step I. We shall prove that for all $r > 0$, there exists $z \in X$ with $z \preceq Az$ such that

$$x \succeq z, \quad d(x, z) \leq r \Longrightarrow d(Ax, z) \leq r.$$

In fact, suppose that there exists some $r > 0$ such that, for any $z \in X$ with $z \preceq Az$, there exists $x \succeq z$ satisfying

$$d(x, z) \leq r, \quad d(Ax, z) > r. \tag{2.2}$$

Let $z \in X$ with $z \preceq Az$. We distinguish three cases.
Case 1. $d(x, z) = 0$, that is, $x = z$. In this case, from (2.2), we have

$$d(z, Az) = d(Ax, z) > r > r/2.$$

Case 2. $0 < d(x, z) \leq d(x, z) \leq r/2$. From (2.1), we have

$$d(Ax, Az) \leq k(d(x, z), r/2)d(x, z) < d(x, z) \leq r/2,$$

which implies that

$$d(Ax, Az) < r/2. \tag{2.3}$$

On the other hand, we have

$$d(z, Az) \geq d(z, Ax) - d(Ax, Az). \tag{2.4}$$

Using (2.2), (2.3) and (2.4), we get that

$$d(z, Az) > r/2.$$

Case 3. $r/2 < d(x, z) \leq r$. From (2.1), we have

$$d(Ax, Az) \leq k(r/2, r)d(x, z) \leq k(r/2, r)r. \tag{2.5}$$

Thanks to (2.2), (2.4) and (2.5), we deduce that

$$d(z, Az) > r - k(r/2, r)r.$$

Now, in all cases, for all $z \in X$ with $z \preceq Az$, we have

$$d(z, Az) > \min\{r/2, r - k(r/2, r)r\} = a > 0. \tag{2.6}$$

Denote $b = d(x_0, Ax_0) > 0$ (obviously, if $x_0 = Ax_0$, the proof is finished). From (2.6) and the hypothesis on x_0, we have

$$Ax_0 \succ x_0, \quad a < d(x_0, Ax_0) \leq b.$$

Using (2.1), we get that

$$d(Ax_0, A^2x_0) \leq k(a, b)d(x_0, Ax_0). \tag{2.7}$$

Since A is nondecreasing, we have $A^2x_0 = A(Ax_0) \succeq Ax_0$. Similarly, we can suppose that $A^2x_0 \succ Ax_0$, otherwise, Ax_0 will be a fixed point of A, and the proof is finished. Using (2.6), we have

$$a < d(Ax_0, A^2x_0).$$

Since $0 < k(a, b) < 1$, from (2.7), we have

$$d(Ax_0, A^2x_0) < b.$$

Thus, we have

$$A(Ax_0) \succ Ax_0, \quad a < d(Ax_0, A^2x_0) < b.$$

Again, applying (2.1), we get that

$$d(A^2x_0, A^3x_0) \leq k(a, b)d(Ax_0, A^2x_0).$$

Using (2.7), we get that

$$d(A^2x_0, A^3x_0) \leq [k(a, b)]^2 d(x_0, Ax_0).$$

Thus, by induction, we have

$$d(A^nx_0, A^{n+1}x_0) \leq [k(a, b)]^n d(x_0, Ax_0), \quad \text{for all} \quad n \geq 0.$$

Therefore (since $0 < k(a, b) < 1$), we can take a positive integer n sufficiently large such that

$$d(x_n, Ax_n) < a,$$

where $x_n = A^nx_0$, which contradicts (2.6).

Thus, we proved that for all $r > 0$, there exists $z \in X$ with $z \preceq Az$ such that

$$x \succeq z, \quad d(x, z) \leq r \Longrightarrow d(Ax, z) \leq r,$$

that is (since A is nondecreasing and $z \preceq Az$),

$$x \succeq z, \quad d(x,z) \leq r \Longrightarrow Ax \succeq z, \quad d(Ax,z) \leq r.$$

Step II. Construction of a Cauchy sequence in the complete metric space (X,d). For all $r>0$ and $z \in X$, consider the set

$$\Delta(r,z) := \{x \in X \,|\, x \succeq z,\ d(x,z) \leq r\}.$$

Clearly, $\Delta(r,z) \neq \emptyset$ ($z \in \Delta(r,z)$). From Step I, we have

$$\forall\, r>0,\ \exists\, z \in X \,|\, z \preceq Az,\ A(\Delta(r,z)) \subseteq \Delta(r,z). \tag{2.8}$$

Using (2.8), for $r=1$, there exists $z_1 \in X$ with $z_1 \preceq Az_1$ such that $A(\Delta(1,z_1)) \subseteq \Delta(1,z_1) := \Delta_1$. Then, the mapping $A:\Delta_1 \to \Delta_1$ is well defined. Using (2.8), for $r=\frac{1}{2}$, there exists $z_2 \in \Delta_1$ with $z_2 \preceq Az_2$ such that $A(\Delta(\frac{1}{2},z_2)\cap\Delta_1) \subseteq \Delta(\frac{1}{2},z_2)\cap\Delta_1 := \Delta_2$. Then, we can define the mapping $A:\Delta_2 \to \Delta_2$. Again, using (2.8), for $r=\frac{1}{3}$, there exists $z_3 \in \Delta_2$ with $z_3 \preceq Az_3$ such that $A(\Delta(\frac{1}{3},z_3)\cap\Delta_2) \subseteq \Delta(\frac{1}{3},z_3)\cap\Delta_2 := \Delta_3$. Inductively, there exists $z_{n+1} \in \Delta_n$ with $z_{n+1} \preceq Az_{n+1}$ such that $A(\Delta(1/(n+1),z_{n+1})\cap\Delta_n) \subseteq \Delta(1/(n+1),z_{n+1})\cap\Delta_n := \Delta_{n+1}$ ($n=1,2,3,\dots$). Clearly, we have

$$\Delta_{n+1} \subseteq \Delta_n\,(n=1,2,3,\dots), \quad d(z_{n+m},z_n) \leq \frac{1}{n}\,(n,m=1,2,3,\dots). \tag{2.9}$$

This implies that $\{z_n\}$ is a Cauchy sequence in the complete metric space (X,d).

Step III. Existence of a fixed point. Since (X,d) is complete and $\{z_n\}$ is a Cauchy sequence in X, there exists $x^* \in X$ such that $z_n \to x^*$ as $n \to \infty$.
We shall prove that

$$x^* \in \Delta_n, \text{ for all } n \in \mathbb{N}. \tag{2.10}$$

Since $z_{n+1} \in \Delta_n$ for all $n \in \mathbb{N}$, we have

$$z_1 \preceq z_2 \preceq \cdots \preceq z_n \preceq z_{n+1} \preceq \cdots$$

From the regularity of $(X,\preceq,d)$, we obtain that

$$z_n \preceq x^*, \text{ for all } n \in \mathbb{N}.$$

From (2.9), we have

$$d(z_{n+1},z_1) \leq 1, \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we get that

$$d(x^*,z_1) \leq 1.$$

Thus we proved that $x^* \in \Delta_1$.
Again, for $n=2$, we have

$$\Delta_2 = \Delta_1 \cap \Delta\Big(\frac{1}{2},z_2\Big) = \Big\{x \in \Delta_1 \,|\, x \succeq z_2,\ d(x,z_2) \leq \frac{1}{2}\Big\}.$$

We know that $x^* \in \Delta_1$ and $x^* \succeq z_2$. From (2.9), we have

$$z_{n+1} \in \Delta_n \subseteq \Delta_2, \text{ for all } n \geq 2.$$

Then we have

$$d(z_{n+1}, z_2) \leq \frac{1}{2}, \text{ for all } n \geq 2.$$

Letting $n \to \infty$, we get that

$$d(x^*, z_2) \leq \frac{1}{2}.$$

Thus we proved that $x^* \in \Delta_2$. Continuing this process, by induction, we can show that (2.10) holds.

On the other hand, for all $n \in \mathbb{N}$, $A(\Delta_n) \subseteq \Delta_n$ and $x^* \in \Delta_n$, that is,

$$x^* \in \Delta_n, \quad Ax^* \in \Delta_n, \text{ for all } n \in \mathbb{N}.$$

This implies that

$$d(x^*, z_n) \leq \frac{1}{n}, \quad d(Ax^*, z_n) \leq \frac{1}{n}, \text{ for all } n \in \mathbb{N}.$$

Then we have

$$d(x^*, Ax^*) \leq d(x^*, z_n) + d(Ax^*, z_n) \leq \frac{2}{n}, \text{ for all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we obtain that $d(x^*, Ax^*) = 0$, that is, $x^* = Ax^*$. Thus we proved that x^* is a fixed point of A. □

Now, we give an example to illustrate our obtained result given by Theorem 2.5.

Example 2.6. Let $X = [0, \infty)$ endowed with the metric d defined by: for all $x, y \in X$,

$$d(x,y) = \begin{cases} \max\{x,y\} & \text{if} \quad x \neq y, \\ 0 & \text{if} \quad x = y. \end{cases}$$

It is not difficult to show that (X, d) is a complete metric space. Define the mapping $A : X \to X$ by

$$Ax = \begin{cases} \dfrac{x}{2} & \text{if} \quad 0 \leq x \leq 1, \\[2ex] \dfrac{3}{4} & \text{if} \quad x > 1. \end{cases}$$

Let $a, b \in \mathbb{R}$ such that $0 < a < b$. We distinguish three cases.

Case 1. $b \leq 1$.
Let $(x, y) \in X \times X$ such that $a \leq d(x, y) \leq b$ and $x > y$. This means that $a \leq x \leq b \leq 1$. Thus we have

$$d(Ax, Ay) = d\Big(\frac{x}{2}, \frac{y}{2}\Big) = \frac{x}{2} \leq \frac{1}{2}\max\{x,y\} = \frac{1}{2}d(x,y).$$

Taking $k(a, b) = \frac{1}{2}$, we obtain that $d(Ax, Ay) \leq k(a, b)d(x, y)$.

Case 2. $a > 1$.
Let $(x, y) \in X \times X$ such that $a \leq d(x, y) \leq b$ and $x > y$. This means that

$1 < a \leq x \leq b$.
If $y > 1$, then we have

$$d(Ax, Ay) = d\Big(\frac{3}{4}, \frac{3}{4}\Big) = 0 \leq kd(x, y),$$

for any $k \in (0, 1)$.
If $y \leq 1$, we have

$$d(Ax, Ay) = d\Big(\frac{3}{4}, \frac{y}{2}\Big) = \max\left\{\frac{3}{4}, \frac{y}{2}\right\} = \frac{3}{4} \leq \frac{3}{4}x = \frac{3}{4}\max\{x, y\}.$$

Taking $k(a, b) = \frac{3}{4}$, we obtain that $d(Ax, Ay) \leq k(a, b)d(x, y)$.

Case 3. $a \leq 1$ and $b > 1$.
Let $(x, y) \in X \times X$ such that $a \leq d(x, y) \leq b$ and $x > y$. This means that $a \leq x \leq b$.
If $x \leq 1$, we have

$$d(Ax, Ay) = \max\left\{\frac{x}{2}, \frac{y}{2}\right\} = \frac{x}{2} \leq \frac{1}{2}d(x, y).$$

Taking $k(a, b) = \frac{1}{2}$, we obtain that $d(Ax, Ay) \leq k(a, b)d(x, y)$.
If $1 < y < x$, we have

$$d(Ax, Ay) = d\Big(\frac{3}{4}, \frac{3}{4}\Big) = 0 \leq kd(x, y),$$

for any $k \in (0, 1)$.
If $y \leq 1 < x$, we have

$$d(Ax, Ay) = \max\left\{\frac{3}{4}, \frac{y}{2}\right\} = \frac{3}{4} \leq \frac{3}{4}x = \frac{3}{4}\max\{x, y\}.$$

Taking $k(a, b) = \frac{3}{4}$, we obtain that $d(Ax, Ay) \leq k(a, b)d(x, y)$.

Thus, in all cases, for all $0 < a < b < \infty$, there exists $k(a, b) \in (0, 1)$ such that

$$x, y \in X, \quad x > y, \quad a \leq d(x, y) \leq b \Longrightarrow d(Ax, Ay) \leq k(a, b)d(x, y).$$

Now, we shall prove that $(X, \leq, d)$ is regular. Let $\{u_n\}$ be a sequence in X such that

(i) $\lim_{n\to\infty} d(u_n, u) = 0$, for some $u \in X$;
(ii) $u_n \leq u_{n+1}$ for all n.

We have to prove that

$$u_n \leq u, \text{ for all } n. \tag{2.11}$$

Let $n \in \mathbb{N}$ be fixed. We have

$$u_n \leq u_{n+p}, \text{ for all } p \in \mathbb{N}.$$

If for some p, we have $u_{n+p} = u$, then from the above inequality, we get (2.11). Now, suppose that $u_{n+p} \neq u$ for all p. In this case, we have

$$0 \leq \max\{u_n, u\} \leq \max\{u_{n+p}, u\} = d(u_{n+p}, u) \to 0 \text{ as } p \to \infty.$$

This implies that $\max\{u_n, u\} = 0$, that is, $u_n = u$. Thus we proved that (2.11) holds, and the regularity of $(X, \leq, d)$ is proved.

Finally, for $x_0 = 0$, we have $x_0 \leq Ax_0$. Now, all the required hypotheses of Theorem 2.5 are satisfied, we deduce the existence of a fixed point of A. In this case $x^* = 0$ is the unique fixed point of A.

Now, we shall prove the following coupled fixed point result.

Theorem 2.7. *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X, d) is a complete metric space. We suppose that $(X, \preceq, d)$ satisfies the following properties:*

(i) *if a nondecreasing sequence $\{x_n\} \to x \in X$, then $x_n \preceq x$ for all n;*
(ii) *if a decreasing sequence $\{y_n\} \to y \in X$, then $y_n \succeq y$ for all n.*

Let $B : X \times X \to X$ be a mapping having the mixed monotone property. Suppose that for any $0 < a < b < \infty$, there exists $0 < k(a, b) < 1$ such that for any $(x, y), (u, v) \in X \times X$,

$$\begin{aligned}(2.12)\quad & x \succeq u,\ y \preceq v, \quad a \leq \frac{d(x, u) + d(y, v)}{2} \leq b \\ & \Longrightarrow d(B(x, y), B(u, v)) \leq \frac{k(a, b)}{2}[d(x, u) + d(y, v)].\end{aligned}$$

Suppose also that there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq B(x_0, y_0)$ and $y_0 \succeq B(y_0, x_0)$. Then B has a coupled fixed point, that is, there exists $(x^*, y^*) \in X \times X$ such that $B(x^*, y^*) = x^*$ and $B(y^*, x^*) = y^*$.

Proof. We endow the product set $Y = X \times X$ with the metric η defined by

$$\eta((x, y), (u, v)) = \frac{d(x, u) + d(y, v)}{2}, \text{ for all } (x, y), (u, v) \in Y.$$

Since (X, d) is complete, it is clear that (Y, η) is also a complete metric space. We endow Y with the partial order $\ll$ defined by

$$(x, y), (u, v) \in Y, \quad (x, y) \ll (u, v) \Longleftrightarrow x \preceq u, \quad y \succeq v.$$

From conditions (i) and (ii), it is clear that $(Y, \ll, \eta)$ is regular.

Now, define the self-mapping $A : Y \to Y$ by

$$A(x, y) = (B(x, y), B(y, x)), \text{ for all } (x, y) \in Y.$$

It follows from the mixed monotone property of B that A is a nondecreasing mapping with respect to $\ll$. Moreover, we have $(x_0, y_0) \ll A(x_0, y_0)$.

From (2.12), we can show that for any $0 < a < b < \infty$, there exists $0 < k(a, b) < 1$ such that

$$\begin{aligned}& (x, y), (u, v) \in Y,\ (x, y) \gg (u, v),\ a \leq \eta((x, y), (u, v)) \leq b \\ & \Longrightarrow \eta(A(x, y), A(u, v)) \leq k(a, b)\eta((x, y), (u, v)).\end{aligned}$$

Now, Applying Theorem 2.5, we obtain that A has a fixed point $(x^*, y^*) \in Y$, that is,

$$(x^*, y^*) = A(x^*, y^*) = (B(x^*, y^*), B(y^*, x^*)),$$

which implies that (x^*, y^*) is a coupled fixed point of B. $\square$

Theorem 2.8. *Let $(X, \preceq)$ be a partially ordered set. Suppose that there exists a metric d on X such that (X, d) is a complete metric space. We suppose that $(X, \preceq, d)$ satisfies the following properties:*

(i) *if a nondecreasing sequence $\{x_n\} \to x \in X$, then $x_n \preceq x$ for all n;*
(ii) *if a decreasing sequence $\{y_n\} \to y \in X$, then $y_n \succeq y$ for all n.*

Let $B : X \times X \to X$ be a mapping having the mixed monotone property. Suppose that for any $0 < a < b < \infty$, there exists $0 < k(a, b) < 1$ such that for any $(x, y), (u, v) \in X \times X$,

$$\begin{aligned} x \succeq u,\ y \preceq v,\ a \leq \max\{d(x, u), d(y, v)\} \leq b \\ \Longrightarrow d(B(x, y), B(u, v)) \leq k(a, b) \max\{d(x, u), d(y, v)\}. \end{aligned} \tag{2.13}$$

Suppose also that there exists $(x_0, y_0) \in X \times X$ such that $x_0 \preceq B(x_0, y_0)$ and $y_0 \succeq B(y_0, x_0)$. Then B has a coupled fixed point.

Proof. It is similar to the proof of Theorem 2.7 by considering the metric η on Y defined by

$$\eta((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}, \text{ for all } (x, y), (u, v) \in Y.$$

$\square$

Remark 2.9.

- Theorem 2.5 is a generalization of Theorem 2.1 in [17].
- Theorems 2.7 and 2.8 are generalizations of Theorem 2.2 in [4].

3. Existence of solutions to a system of nonlinear integral equations

In this section, we discuss the existence of solutions to the following system of integral equations given by

$$g(t) + \int_0^1 G(s, t) f(s, u_1(s), u_2(s))\, ds = u_1(t); \tag{3.1}$$

$$g(t) + \int_0^1 G(s, t) f(s, u_2(s), u_1(s))\, ds = u_2(t), \tag{3.2}$$

where $g : [0, 1] \to \mathbb{R}$, $G : [0, 1] \times [0, 1] \to [0, \infty)$ and $f : [0, 1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

We consider the following assumptions:

(H1) for all $s \in [0,1]$,

$$\alpha \geq \gamma,\ \beta \leq \theta \Longrightarrow 0 \leq f(s,\alpha,\beta) - f(s,\gamma,\theta) \leq \varphi\left(\frac{(\alpha-\gamma)+(\theta-\beta)}{2}\right),$$

where $\varphi : [0,\infty) \to [0,\infty)$ is continuous nondecreasing and satisfies: for all $0 < a < b < \infty$, there exists $0 < k(a,b) < 1$ such that

$$a \leq r \leq b \Longrightarrow \varphi(r) \leq k(a,b)r;$$

(H2) $\sup_{0\leq t\leq 1} \int_0^1 G(t,s)\,ds \leq 1$;

(H3) there exist two continuous functions $u_0, v_0 : [0,1] \to \mathbb{R}$ such that for all $t \in [0,1]$,

$$\begin{aligned} u_0(t) &\leq g(t) + \int_0^1 G(s,t) f(s,u_0(s),v_0(s))\,ds \\ v_0(t) &\geq g(t) + \int_0^1 G(s,t) f(s,v_0(s),u_0(s))\,ds. \end{aligned}$$

We denote by $X = C([0,1),\mathbb{R})$ the set of real continuous functions on $[0,1]$.

We have the following result.

Theorem 3.1. *Under the assumptions (H1)-(H3), Problem (3.1)-(3.2) has at least one solution* $(x^*,y^*) \in X \times X$.

Proof. Consider the operator $B : X \times X \to X$ defined by: for all $(x,y) \in X \times X$,

$$B(x,y)(t) = g(t) + \int_0^1 G(s,t) f(s,x(s),y(s))\,ds, \text{ for all } t \in [0,1].$$

Clearly, (u,v) is a solution to (3.1)-(3.2) if and only if (u,v) is a coupled fixed point of B.

We endow X with the metric d given by

$$d(x,y) = \max_{t\in[0,1]} |x(t)-y(t)|, \quad x,y \in C([0,1],\mathbb{R}).$$

It is well known that (X,d) is a complete metric space. We endow X with the partial order $\preceq$ given by

$$x,y \in C([0,1],\mathbb{R}), \quad x \preceq y \Longleftrightarrow x(t) \leq y(t) \quad \text{for all} \quad t \in [0,1].$$

We can show easily that $(X,\preceq,d)$ is regular.

From (H1) and since G is positive, we obtain that the mapping B has the mixed monotone property.

Let $0 < a < b < \infty$. Let $(x,y),(u,v) \in X \times X$ such that

$$x \succeq u,\ y \preceq v, \quad a \leq \frac{d(x,u)+d(y,v)}{2} \leq b.$$

From (H1) and (H2), we have: for all $t \in [0,1]$,

$$|B(x,y)-B(u,v)|(t) \leq \int_0^1 G(s,t)[f(s,x(s),y(s)) - f(s,u(s),v(s))]\,ds$$

$$\begin{aligned} &\leq \int_0^1 G(s,t)\varphi\left(\frac{(x(s)-u(s))+(v(s)-y(s))}{2}\right) ds \\ &\leq \varphi\left(\frac{d(x,u)+d(y,v)}{2}\right) \\ &\leq \frac{k(a,b)}{2}[d(x,u)+d(y,v)], \end{aligned}$$

which implies that

$$d(B(x,y),B(u,v)) \leq \frac{k(a,b)}{2}[d(x,u)+d(y,v)].$$

Thus we proved that for any $0 < a < b < \infty$, there exists $0 < k(a,b) < 1$ such that for any $(x,y),(u,v) \in X \times X$,

$$x \succeq u,\ y \preceq v, \quad a \leq \frac{d(x,u)+d(y,v)}{2} \leq b$$
$$\Longrightarrow d(B(x,y),B(u,v)) \leq \frac{k(a,b)}{2}[d(x,u)+d(y,v)].$$

Finally, from (H3), we have $u_0 \preceq B(u_0,v_0)$ and $v_0 \succeq B(v_0,u_0)$.

Now, the desired result follows immediately from Theorem 2.7. □

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