Journal of Nonlinear and Convex Analysis
Volume 14, Number 4, 2013, 817–839

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

MULTICAST DECENTRALIZED OPTIMIZATION ALGORITHM FOR NETWORK RESOURCE ALLOCATION PROBLEMS

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Abstract. We consider a network resource allocation problem that has global information on the whole network such as the explicit forms of all users' utility functions and constraint sets and propose a novel decentralized optimization algorithm for solving it. The algorithm can be implemented through cooperation between individual users and their neighboring users, and it enables each user in a network to determine his or her own optimal resource allocation without using other users' private information such as their utility functions and constraint sets. The main result in this paper is a proof that the algorithm converges to the solution to the network resource allocation problem. We apply the algorithm to concrete network resource allocation problems and provide numerical examples for these problems.

1. Introduction

Network resource allocation [3, 30, 38] is a central issue in modern networks. The main objective of the allocation is to share the available resource among users in the network so as to maximize the sum of their utilities subject to the feasible region for allocating the resource. Such a maximization problem, called the *network resource allocation problem* (see [3], [25], [38], [43], [30], and references therein), includes future network resource allocation problems such as the channel allocation problem for a multi-carrier system [24], the utility maximization problem for a sensor network [50], the storage allocation problem for a peer-to-peer network [29], the power allocation problem for a wireless data network [36], and the bandwidth allocation problem [38]. In this paper, we present a novel *decentralized optimization algorithm* to solve the network resource allocation problem and its convergence analysis.

In the case of the power control for the uplink or downlink in a code-division multiple-access (CDMA) data network, the base station plays the role of a centralized operator that governs all the resource allocations, and hence, it can get all the user information such as the explicit forms of all users' objective functions (called *utility functions*) and feasible sets from the start. To control the power allocation in the network, the base station executes a *centralized optimization algorithm*, that requires the centralized operator to use all the user information, and transmits the powers computed by the algorithm to all users in the network. However, large-scale

2010 *Mathematics Subject Classification.* 65K05, 90C25, 90C35.

Key words and phrases. Concave function, centralized optimization problem, network resource allocation problem, decentralized optimization algorithm.

This work was supported by the Japan Society for the Promotion of Science through a Grant-in-Aid for Young Scientists (B) (23760077), and in part by the Japan Society for the Promotion of Science through a Grant-in-Aid for Scientific Research (C) (23500090).

and complex system networks tend to lack centralized operators and can change size at any time. Moreover, each user in such a network should not be able to know other users' private information such as their utility functions and feasible sets. Therefore, *decentralized mechanisms* should be used for network resource allocation instead of centralized ones that involve extra infrastructure. Decentralized mechanisms enable each user to adjust his or her own resource allocation in cooperation with other users but without using other users' private utility functions or feasible sets. Since the network resource allocation problem has global information on the whole network, it is referred to as a *centralized optimization problem*.

Many decentralized algorithms for solving the centralized optimization problem have been presented, and these fall into three classifications. (I) Decentralized optimization methods [26, 28, 35, 38] for solving the centralized optimization problem work in the case that user i's utility function depends on only user i's resource allocation, i.e., the domain of the utility function is one dimensional, and that each user has a common feasible set. These methods are based on Lagrangian duality and dual subgradient algorithms. Recently, reference [30] presented a subgradient method and provided lower and upper bounds of the approximate value that are applicable for general convex optimization problems. (II) References [4,7,10,23,30,32,33,42,44] presented consensus algorithms for solving the centralized optimization problem when each user's utility function is concave on the Euclidean space and its feasible set is equal to the whole Euclidean space. Reference [30] presented a subgradient algorithm that can optimize concave utility functions over a time-varying network topology and described the lower and upper bounds of the method's approximate sum of the utility functions. (III) References [12, 14, 19, 21] presented broadcast decentralized optimization algorithms, which require each user to communicate with other users directly, for solving the centralized optimization problem when each user's utility function is concave and its feasible set is a closed convex subset of a real Hilbert space, and proved that the algorithms converge strongly or weakly to the solution to the centralized optimization problem under the standard assumptions. Reference [36] considered the centralized optimization problem for the power control and presented a broadcast algorithm to adjust the users' transmission powers under constraints so as to maximize the sum of users' utilities.

However, we encounter two issues when the methods of (I), (II), and (III) above are used to solve the centralized optimization problem:

(a) Since the centralized optimization problems in [24], [29], [36], and [38] can be expressed as a maximization problem for the sum of the concave utility functions over a closed convex subset of a certain Euclidean space, the decentralized optimization algorithms in (I) and (II) cannot be applied to such problems directly.

(b) It would be physically impossible for all users to communicate with each other. Hence, it is difficult to apply the broadcast decentralized optimization algorithm in (III) to the the centralized optimization problems in large-scale and complex networks.

In this paper, we present a *multicast type of decentralized optimization algorithm* to resolve the above issues (a) and (b). The main advantages of this algorithm are as follows:

(A) The algorithm can be used to solve the centralized optimization problem when each user's utility function is concave and its feasible set is a closed convex subset of a real Hilbert space.

(B) The algorithm can be applied when each user directly communicates with its neighbor users, and the applications of the algorithm do not depend on the network's topology.

Each user's utility function given in [36] is modeled as a function from $\mathbb{R}^K$ into $\mathbb{R}$, where K is the total number of users participating in the network, while the domain of the utility function in [24] is a more than K-dimensional Euclidean space. Meanwhile, the domain of the utility function in [29] and [38] is a less than K-dimensional Euclidean space. We will deal with a centralized optimization problem in an infinite space so that we can consider the centralized optimization problems in [24], [29], [36], and [38] together. Thanks to advantage (A), each user that executes the algorithm can solve centralized optimization problems including those of [24], [29], [36], and [38].

Let us consider the simple networks in Figures 1 and 2 in Section 2 where each user in the network communicates with only its neighbor users within one hop. Such a network will arise depending on the users' specifications (e.g., processor speed and transmitted power) and physical constraints (e.g., interference and reflection). Under this communication assumption, since each user cannot directly communicate with other users beyond one hop, the broadcast decentralized optimization algorithms that require each user to communicate with all users directly cannot be applied to even the simple cases of Figures 1 and 2. However, advantage (B) enable the proposed algorithm to be applied to any network structure and hence it can work in the cases illustrated in the figures.

The proposed algorithm (Algorithm 2.9) embodies three ideas: The first is a decentralized method for setting the weighted parameters for each user and its neighbors (see Subsection 2.2 and Equation (2.2)). The weighted parameters are used to execute the proposed algorithm. The second is the *proximal point methods* with the *resolvents* [1, 5, 12–14] of bifunctions and monotone operators. A use of the resolvent implies that each user computes the unique maximizer of its objective function over its feasible set (see Equation (2.1)). Proximal point methods are also used to solve problems in image processing [15] and network flows [1]. The third is based on the *ergodic iteration technique* [9] for solving the monotone variational inequality problem (see Equation (2.3)). Such a technique leads to convergence of the algorithm to the solution to the centralized optimization problem.

The key contributions of this paper are (i) to propose a multicast decentralized optimization algorithm for solving the centralized optimization problem with information on the whole network; and (ii) to prove that the algorithm weakly converges to the solution to the centralized optimization problem under certain assumptions. Although there are many decentralized algorithms being used in the network field, few of them have been proven to converge to the desired solutions. The analyses presented in the literature tend to rely on computational simulations. Moreover, the literature does not seem to have any multicast type of decentralized optimization algorithm for solving a centralized optimization problem. The proposed algorithm can be modified to work in large-scale and complex networks that have the properties of incompleteness and asymmetry.

This paper is organized as follows. Section 2 describes our basic model of a network, the main problem, and the proposed algorithm, and it provides an outline of our results. Section 3 proves that the algorithm converges weakly to the solution to the problem under certain assumptions. Section 4 considers the network bandwidth allocation and network storage allocation and provides numerical examples for these problems. Section 5 concludes the paper and mentions future subjects for development of the algorithm. An extension of the algorithm to the case that the utility function of each user is nonsmooth is given in the Appendix. It also gives mathematical preliminaries on monotone operators, variational inequalities, and metric projections.

2. Outline of results

2.1. **The basic model.** Let $I := \{1, 2, \dots, K\}$ be the set of users who must compete for the network resource, and let $I(i)$ $(i \in I)$ denote the set of user i and users neighboring user i.[1] Each user's index of satisfaction in the network is represented as a *utility function*, and it is a function depending on the allocations to other users. Hence, the utility function, $\mathcal{U}^{(i)}$, of user i is modeled as the function from $\mathbb{R}^K$ into $\mathbb{R}$. Also, the domain of $\mathcal{U}^{(i)}$ is a more (or less) than K-dimensional Euclidean space (for example, the domain of the utility function for a multi-carrier system is more than K [24]). Hence, we shall assume that the domain of $\mathcal{U}^{(i)}$ is a real Hilbert space H to include the cases in [24], [29], [36], and [38]. The *feasible set*, $C^{(i)}$, of user i is a subset of H, and the set, $C := \bigcap_{i \in I} C^{(i)}$ $(\neq \emptyset)$, is called the feasible region for allocating the resource. We need to make the following assumptions about the network structure:

Assumption 2.1.

(A1) Each user's utility function, $\mathcal{U}^{(i)} \colon H \to \mathbb{R}$ $(i \in I)$, is strictly concave and continuously Fréchet differentiable. The explicit form of $\mathcal{U}^{(i)}$ is its own private information; that is, other users cannot know the explicit form of $\mathcal{U}^{(i)}$.

(A2) Each user's feasible set, $C^{(i)}$, is a nonempty, bounded, closed convex subset of H. The explicit form of $C^{(i)}$ is its own private information.

We will assume the network has the following properties:

Assumption 2.2.

(A3) $C := \bigcap_{i \in I} C^{(i)} \neq \emptyset$.

(A4) Each user can communicate with neighbor users.[2]

Let us consider the following *network resource allocation problem*, called the *centralized optimization problem*, with information on the whole network (see [3], [25], [38], [43], [30], and references therein):

[1] For example, $I(2)$ in Figure 2 is $\{1, 2, 4, 5\}$.

[2] User 2 in Figure 2 can communicates with users $1, 4$, and 5 directly, while it cannot communicate with users $3, 6$, and 7 directly.

Problem 2.3 (Centralized optimization problem)**.** Under Assumptions 2.1 and 2.2,

$$\text{maximize } \sum_{i \in I} \mathcal{U}^{(i)}(x) \text{ subject to } x \in C := \bigcap_{i \in I} C^{(i)},$$

where one assumes

$$\text{(A5) } \operatorname*{Argmax}_{x \in C} \sum_{i \in I} \mathcal{U}^{(i)}(x) := \left\{ x^* \in C \colon \sum_{i \in I} \mathcal{U}^{(i)}(x^*) = \max_{x \in C} \sum_{i \in I} \mathcal{U}^{(i)}(x) \right\} \neq \emptyset.$$

Assumption (A1) implies that the operator, $-\sum_{i \in I} \nabla \mathcal{U}^{(i)} \colon H \to H$, is strictly monotone. Moreover, Assumptions (A2) and (A3) imply that $C := \bigcap_{i \in I} C^{(i)} (\subset H)$ is nonempty, closed, and convex. Accordingly, from the closedness of C, Proposition 7.1 (ii), and Relation (7.1) (see the Appendix), Assumption (A5) is true when at least one of $C^{(i)}$s is compact. Under Assumptions (A1), (A2), (A3), and (A5), Proposition 7.1 (iii) guarantees the following:

Proposition 2.4. *Problem 2.3 has a unique solution; that is,* $\{x^*\} = \operatorname{Argmax}_{x \in C} \sum_{i \in I} \mathcal{U}^{(i)}(x)$.

2.2. **Weighted parameters for each user and its neighbors.** Before presenting the algorithm for solving Problem 2.3, we need some preliminaries. The proposed algorithm will require user i ($i \in I$) to use an (ω_{ij}) ($j \in I(i)$) satisfying the following conditions:

Assumption 2.5. (A6) User i ($i \in I$) initially has $\omega_{ij} \in [0, 1)$ ($j \in I$) satisfying $\sum_{j \in I(i)} \omega_{ij} = 1$ and $\omega_{ij} = 0$ ($j \notin I(i)$). Moreover, for each $j \in I$, it is assumed that $\omega_{kj} \in [0, 1)$ ($k \in I$) satisfies $\sum_{k \in I(j)} \omega_{kj} = 1$ and $\omega_{kj} = 0$ ($k \notin I(j)$).

Assumption (A6) implies that the sum of elements of each row and column of the matrix, $\Omega := [\omega_{ij}]_{i,j \in I}$, is equal to 1. Since there is no user that knows the whole network structure, each user must determine its own weighted parameters on the basis of Assumption (A4) in cooperation with its neighbor users. The following are examples of decentralized methods for setting the weighted parameters of simple networks (Note that the decentralized methods for determining Ω for the cases of Figures 1 and 3 are not unique):

Example 2.6. Consider the network structure in Figure 1. Since users 1, 2, and 3 initially have $I(1) := \{1, 2\}$ (i.e., $\omega_{1j} = 0$ ($j \notin I(1)$)), $I(2) := \{1, 2, 3\}$, and $I(3) := \{2, 3\}$ (i.e., $\omega_{3j} = 0$ ($j \notin I(3)$)), respectively, user 2 finds $\Omega := \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \cdots \\ \omega_{21} & \omega_{22} & \omega_{23} & \cdots \\ \omega_{31} & \omega_{32} & \omega_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} = \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & \omega_{23} \\ 0 & \omega_{32} & \omega_{33} \end{pmatrix}$ by cooperating with its neighbor users. When user 2 sets $\omega_{21} = \omega_{22} = \omega_{23} := \frac{1}{3}$, user 2 has the matrix, $\Omega = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$. User 2 then transmits $(\omega_{1j})_{j \in I}$ and $(\omega_{3j})_{j \in I}$ to users 1 and 3, respectively.

Example 2.7. Consider the case of Figure 2. By cooperating with its neighbor users, user 2 can get $\Omega_1 := [\omega_{ij}]_{i=1,2,4,5, j \in \mathbb{N} \setminus \{0\}} = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & 0 & 0 & 0 \cdots \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 \cdots \\ 0 & \omega_{42} & 0 & \omega_{44} & 0 & 0 \cdots \\ 0 & \omega_{52} & 0 & 0 & \omega_{55} & 0 \cdots \end{pmatrix} =$

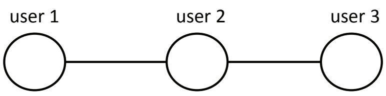

FIGURE 1. Network with three users

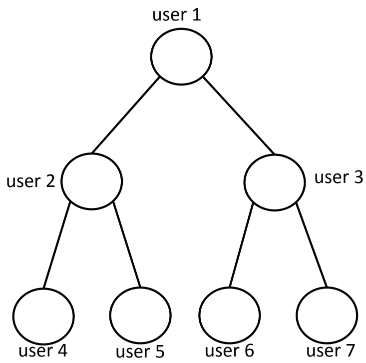

FIGURE 2. Network with seven users

$\begin{pmatrix} \omega_{11} & \frac{1}{4} & \omega_{13} & 0 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & \cdots \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & \cdots \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 & \cdots \end{pmatrix}$ and transmit it to users $1, 4$, and 5. On the other hand, user 3 has $\Omega_2 := [\omega_{ij}]_{i=1,3,6,7, j \in \mathbb{N}\setminus\{0\}} = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & 0 & 0 & 0 & 0 & 0 & \ldots \\ \omega_{31} & 0 & \omega_{33} & 0 & 0 & \omega_{36} & \omega_{37} & 0 & \ldots \\ 0 & 0 & \omega_{63} & 0 & 0 & \omega_{66} & 0 & 0 & \ldots \\ 0 & 0 & \omega_{73} & 0 & 0 & 0 & \omega_{77} & 0 & \ldots \end{pmatrix}$. Hence, user 1 can find

$$\Omega := [\omega_{ij}]_{i,j \in I} = \begin{pmatrix} \omega_{11} & \frac{1}{4} & \omega_{13} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \omega_{31} & 0 & \omega_{33} & 0 & 0 & \omega_{36} & \omega_{37} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \omega_{63} & 0 & 0 & \omega_{66} & 0 \\ 0 & 0 & \omega_{73} & 0 & 0 & 0 & \omega_{77} \end{pmatrix} = \begin{pmatrix} \frac{2}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

That is, each user can get its own weighted parameters satisfying Assumption (A6).

2.3. Multicast decentralized optimization algorithm for solving centralized optimization problem.

Now we shall present the multicast decentralized optimization algorithm that enables each user to determine its own optimal resource allocation in cooperation with its neighbor users. First, we assume that all users in the network start with the following common information:

Assumption 2.8.

(A7) All users initially have the step size, $(\alpha_n)_{n\in\mathbb{N}} \subset (0, 1]$, satisfying the following conditions:[3]

$$\text{(C1) } \alpha_{n+1} \leq \alpha_n \ (n \in \mathbb{N}), \ \text{(C2) } \lim_{n\to\infty} \alpha_n = 0, \ \text{(C3) } \sum_{n=0}^{\infty} \alpha_n = \infty.$$

The algorithm for solving Problem 2.3 under Assumptions (A1)-(A7) is as follows.

Algorithm 2.9 (Multicast decentralized optimization algorithm)**.**

Step 0. User i chooses $\bar{x}_0^{(i)} \in H$ arbitrarily, and let $n := 0$.

[3]Examples of $(\alpha_n)_{n\in\mathbb{N}}$ satisfying Conditions (C1), (C2), and (C3) are $\alpha_n := \frac{1}{(n+1)^\rho}$ $(\rho \in (0, 1])$.

Step 1. User i computes $x_0^{(i)} \in H$ by $\{x_0^{(i)}\} = \mathrm{Argmax}_{x \in C^{(i)}}[\mathcal{U}^{(i)}(x) - \frac{1}{2\alpha_0}\|x - \bar{x}_0^{(i)}\|^2]$ and transmits this point to the neighbor users. User i computes $y_0^{(i)} \in H$ by $y_0^{(i)} := \sum_{j \in I(i)} \omega_{ij} x_0^{(j)}$, where $(\omega_{ij})_{j \in I(i)}$ is the weighted parameters in Assumption (A6).

Step 2. Given $y_n^{(i)} \in H$, user i computes $x_{n+1}^{(i)} \in H$ by

$$\left\{x_{n+1}^{(i)}\right\} = \operatorname*{Argmax}_{x \in C^{(i)}} \left[\mathcal{U}^{(i)}(x) - \frac{1}{2\alpha_{n+1}}\left\|x - y_n^{(i)}\right\|^2\right] \tag{2.1}$$

and transmits this point to the neighbor users. User i computes $y_{n+1}^{(i)} \in H$ by

$$y_{n+1}^{(i)} := \sum_{j \in I(i)} \omega_{ij} x_{n+1}^{(j)}. \tag{2.2}$$

User i then computes $z_{n+1}^{(i)} \in H$ by calculating

$$z_{n+1}^{(i)} := \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \alpha_k x_k^{(i)}. \tag{2.3}$$

Put $n := n+1$, and go to *Step* 2.

The point, $x_{n+1}^{(i)}$ $(i \in I, n \in \mathbb{N})$, in Equation (2.1) is the unique maximizer of the strictly concave function, $\mathcal{U}^{(i)} - \frac{1}{2\alpha_{n+1}}\| \cdot - y_n^{(i)}\|^2$, over $C^{(i)}$. This point is called the *resolvent* [1,5,12–14] of $-\alpha_{n+1}\nabla\mathcal{U}^{(i)}$ at $y_n^{(i)}$. The possibility of calculating $x_{n+1}^{(i)}$ $(i \in I, n \in \mathbb{N})$ depends on the forms of $\mathcal{U}^{(i)}$ and $C^{(i)}$. If $H = \mathbb{R}^L$ $(L \geq 1)$, this problem can be solved by using convex optimization techniques such as projection methods and interior-point methods [8, Chapter III, 10 and 11], [31, Chapters 15-19]. We will provide examples of approximately calculating $x_{n+1}^{(i)}$ in Equation (2.1) in Subsections 4.1 and 4.2. Some important examples in which $x_{n+1}^{(i)}$ can be explicitly solved are given in [15, Subsection 2.6]. From Assumption (A6), we can see that $y_{n+1}^{(i)}$ $(i \in I, n \in \mathbb{N})$ in Equation (2.2) is given by the convex combination of $x_{n+1}^{(j)}$s $(j \in I(i))$. $z_{n+1}^{(i)}$ $(i \in I, n \in \mathbb{N})$ in Equation (2.3) is the mean of $(x_k^{(i)})_{k=1}^{n+1}$. The idea of using a mean sequence is based on the ergodic algorithm [9] for solving variational inequality problems for monotone operators in a real Hilbert space.

2.4. **Convergence analysis on proposed algorithm.** The following theorem constitutes the convergence analysis of Algorithm 2.9.

Theorem 2.10. *If, for each $i, j \in I$, the sequence,*[4]

$$\left(\frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \left\|y_k^{(i)} - y_k^{(j)}\right\|^2\right)_{n \in \mathbb{N}}, \text{ is bounded,} \tag{2.4}$$

[4]For examples satisfying Condition (2.4), see Section 4.

the sequence, $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$*, in Algorithm 2.9 converges weakly*[5] *to a unique solution to Problem 2.3.*

3. Lemmas and proof of Theorem 2.10

The proof of Theorem 2.10 is divided into five steps: Lemmas 3.1, 3.2, 3.3, and 3.4, and Proof of Theorem 2.10. For the notations in this section, see the Appendix.

Lemma 3.1. *Let* $y \in H$ *and* $r > 0$*, and suppose that* $f\colon H \to \mathbb{R}$ *is convex and Fréchet differentiable and that* D $(\subset H)$ *is nonempty, closed, and convex. Then,* $\bar{x} \in D$ *satisfies the equation,*

$$\{\bar{x}\} = \operatorname*{Argmin}_{x\in D} \Big[f(x) + \frac{1}{2r}\|x-y\|^2\Big], \tag{3.1}$$

if and only if $\bar{x} \in D$ *satisfies* $\bar{x} = P_D(y - r\nabla f(\bar{x}))$*, where* P_D *stands for the metric projection onto* D (see Subsection 7.3) *and* ∇f *is the gradient of* f*.*

Proof. Since $\bar{x} \in D$ in Equation (3.1) is a minimizer of the convex function, $g(\cdot) := f(\cdot) + \frac{1}{2r}\|\cdot - y\|^2$, over D $(\subset H)$, from Relation (7.1), we find that $\bar{x} \in D$ satisfies Equation (3.1) if and only if $\bar{x} \in \mathrm{VI}(D, \nabla g)$; that is, for all $x \in D$, $0 \le \langle x - \bar{x}, \nabla g(\bar{x})\rangle = \langle x - \bar{x}, \nabla f(\bar{x}) + \frac{1}{r}(\bar{x} - y)\rangle$. This is equivalent to $0 \le \langle x - \bar{x}, \bar{x} - (y - r\nabla f(\bar{x}))\rangle$ $(x \in D)$. Proposition 7.2 (i) guarantees that $\bar{x} \in D$ satisfies $\bar{x} = P_D(y - r\nabla f(\bar{x}))$. □

Lemma 3.2. *Let* $(x_n^{(i)})_{n\in\mathbb{N}}$*,* $(y_n^{(i)})_{n\in\mathbb{N}}$*, and* $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ *be sequences generated by Algorithm 2.9 and define* $A^{(i)} := -\nabla\mathcal{U}^{(i)}$ $(i \in I)$*. Then,*

(i) $(x_n^{(i)})_{n\in\mathbb{N}}$*,* $(y_n^{(i)})_{n\in\mathbb{N}}$*, and* $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$ *are bounded;*

(ii) *For all* $n \in \mathbb{N}$ *and for all* $y \in C$*,*

$$-\frac{\sum_{i\in I}\left\|x_0^{(i)} - y\right\|^2}{\sum_{k=0}^n \alpha_{k+1}} \le 2\sum_{i\in I}\left\langle y - z_{n+1}^{(i)}, A^{(i)}(y)\right\rangle - \sum_{i\in I}\frac{1}{\sum_{k=0}^n \alpha_{k+1}}\sum_{k=0}^n \left\|x_{k+1}^{(i)} - y_k^{(i)}\right\|^2;$$

(iii) *For each* $i \in I$*,*

$$\Big(\frac{1}{\sum_{k=0}^n \alpha_{k+1}}\sum_{k=0}^n \left\|x_{k+1}^{(i)} - y_k^{(i)}\right\|^2\Big)_{n\in\mathbb{N}} \text{ is bounded.}$$

Proof. (i) Equation (2.1) and the boundedness of $C^{(i)}$ $(i \in I)$ guarantee that $(x_n^{(i)})_{n\in\mathbb{N}} \subset C^{(i)}$ $(i \in I)$ is bounded. $(y_n^{(i)})_{n\in\mathbb{N}}$ is bounded from Equation (2.2) and the boundedness of $(x_n^{(i)})_{n\in\mathbb{N}}$. Moreover, the convexity of $C^{(i)}$ $(i \in I)$, and Equation (2.3) ensure that $(z_{n+1}^{(i)})_{n\in\mathbb{N}} \subset C^{(i)}$ $(i \in I)$. Accordingly, the boundedness of $C^{(i)}$ $(i \in I)$ implies that $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ is bounded.

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[5] $(z_n)_{n\in\mathbb{N}}$ $(\subset H)$ is said to converge weakly to $x^* \in H$ if, for any $y \in H$, $\lim_{n\to\infty}\langle z_n, y\rangle = \langle x^*, y\rangle$. When H is finite dimensional, the weak convergence of $(z_n)_{n\in\mathbb{N}}$ to x^* is coincident with the convergence of $(z_n)_{n\in\mathbb{N}}$ to x^* in the sense of the norm.

(ii) Equation (2.2), Assumption (A6), and the convexity of $\|\cdot\|^2$ guarantee that, for all $i \in I$, for all $n \in \mathbb{N}$, and for all $y \in C$,

$$\left\|y_n^{(i)} - y\right\|^2 = \Big\| \sum_{j \in I(i)} \omega_{ij} x_n^{(j)} - y \Big\|^2 = \Big\| \sum_{j \in I(i)} \omega_{ij} \left(x_n^{(j)} - y \right) \Big\|^2 \leq \sum_{j \in I(i)} \omega_{ij} \left\| x_n^{(j)} - y \right\|^2.$$

Summing this inequality over all i ensures that

$$\sum_{i \in I} \left\| y_n^{(i)} - y \right\|^2 \leq \sum_{i \in I} \sum_{j \in I(i)} \omega_{ij} \left\| x_n^{(j)} - y \right\|^2.$$

Moreover, Assumption (A6) guarantees that

$$\sum_{i \in I} \sum_{j \in I(i)} \omega_{ij} \left\| x_n^{(j)} - y \right\|^2 = \sum_{i \in I} \sum_{j \in I(i)} \omega_{ji} \left\| x_n^{(i)} - y \right\|^2 = \sum_{i \in I} \left\| x_n^{(i)} - y \right\|^2,$$

which implies that, for all $n \in \mathbb{N}$ and for all $y \in C$,

$$\sum_{i \in I} \left\| y_n^{(i)} - y \right\|^2 \leq \sum_{i \in I} \left\| x_n^{(i)} - y \right\|^2. \tag{3.2}$$

Choose any $i \in I$. Proposition 7.2 (ii) and (iii), Equation (2.1), and Lemma 3.1 guarantee that, for all $y \in C := \bigcap_{j \in I} C^{(j)} = \bigcap_{j \in I} \mathrm{Fix}(P_{C^{(j)}}) \subset \mathrm{Fix}(P_{C^{(i)}})$ and for all $k \in \mathbb{N}$,

$$\begin{aligned} \left\| x_{k+1}^{(i)} - y \right\|^2 &= \left\| P_{C^{(i)}} \left(y_k^{(i)} - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) \right) - P_{C^{(i)}}(y) \right\|^2 \\ &\leq \left\langle y_k^{(i)} - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) - y, x_{k+1}^{(i)} - y \right\rangle. \end{aligned}$$

By using the equality, $\langle x, y \rangle = \frac{1}{2}(\|x\|^2 + \|y\|^2 - \|x-y\|^2)$ $(x, y \in H)$, we find that

$$\begin{aligned} \left\| x_{k+1}^{(i)} - y \right\|^2 &\leq \left\langle y_k^{(i)} - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) - y, x_{k+1}^{(i)} - y \right\rangle \\ &= \frac{1}{2} \Big\{ \left\| y_k^{(i)} - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) - y \right\|^2 + \left\| x_{k+1}^{(i)} - y \right\|^2 \\ &\quad - \left\| \left(y_k^{(i)} - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) - y \right) - \left(x_{k+1}^{(i)} - y \right) \right\|^2 \Big\}, \end{aligned}$$

which means that

$$\begin{aligned} \left\| x_{k+1}^{(i)} - y \right\|^2 &\leq \left\| \left(y_k^{(i)} - y \right) - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) \right\|^2 \\ &\quad - \left\| \left(y_k^{(i)} - x_{k+1}^{(i)} \right) - \alpha_{k+1} A^{(i)} \left(x_{k+1}^{(i)} \right) \right\|^2 \\ &= \left\| y_k^{(i)} - y \right\|^2 - 2\alpha_{k+1} \left\langle y_k^{(i)} - y, A^{(i)} \left(x_{k+1}^{(i)} \right) \right\rangle - \left\| y_k^{(i)} - x_{k+1}^{(i)} \right\|^2 \\ &\quad - 2\alpha_{k+1} \left\langle x_{k+1}^{(i)} - y_k^{(i)}, A^{(i)} \left(x_{k+1}^{(i)} \right) \right\rangle \\ &= \left\| y_k^{(i)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(i)}, A^{(i)} \left(x_{k+1}^{(i)} \right) \right\rangle - \left\| y_k^{(i)} - x_{k+1}^{(i)} \right\|^2. \end{aligned}$$

The monotonicity of $A^{(i)}$ $(:= -\nabla \mathcal{U}^{(i)})$ (see Subsection 7.1) guarantees that

$$\left\| x_{k+1}^{(i)} - y \right\|^2 \leq \left\| y_k^{(i)} - y \right\|^2 + 2\alpha_{k+1} \left\langle y - x_{k+1}^{(i)}, A^{(i)}(y) \right\rangle - \left\| x_{k+1}^{(i)} - y_k^{(i)} \right\|^2.$$

Summing this inequality over all i and Inequality (3.2) ensure that, for all $y \in C$ and for all $k \in \mathbb{N}$,

$$\sum_{i\in I}\left\|x_{k+1}^{(i)}-y\right\|^2 \leq \sum_{i\in I}\left\|x_k^{(i)}-y\right\|^2 - \sum_{i\in I}\left\|x_{k+1}^{(i)}-y_k^{(i)}\right\|^2 + 2\alpha_{k+1}\sum_{i\in I}\left\langle y-x_{k+1}^{(i)}, A^{(i)}(y)\right\rangle.$$

By summing this inequality from $k=0$ to $k=n$ ($n\in\mathbb{N}$), one gets

$$\sum_{i\in I}\left\|x_{n+1}^{(i)}-y\right\|^2 \leq \sum_{i\in I}\left\|x_0^{(i)}-y\right\|^2 + 2\sum_{k=0}^{n}\alpha_{k+1}\sum_{i\in I}\left\langle y-x_{k+1}^{(i)}, A^{(i)}(y)\right\rangle - \sum_{k=0}^{n}\sum_{i\in I}\left\|x_{k+1}^{(i)}-y_k^{(i)}\right\|^2.$$

Therefore, we find that, for all $y\in C$ and for all $n\in\mathbb{N}$,

$$-\sum_{i\in I}\left\|x_0^{(i)}-y\right\|^2 \leq -\sum_{i\in I}\sum_{k=0}^{n}\left\|x_{k+1}^{(i)}-y_k^{(i)}\right\|^2 + 2\sum_{i\in I}\left\langle \sum_{k=0}^{n}\alpha_{k+1}y - \sum_{k=0}^{n}\alpha_{k+1}x_{k+1}^{(i)}, A^{(i)}(y)\right\rangle,$$

and hence,

$$-\frac{\sum_{i\in I}\left\|x_0^{(i)}-y\right\|^2}{\sum_{k=0}^{n}\alpha_{k+1}} \leq -\sum_{i\in I}\frac{1}{\sum_{k=0}^{n}\alpha_{k+1}}\sum_{k=0}^{n}\left\|x_{k+1}^{(i)}-y_k^{(i)}\right\|^2 + 2\sum_{i\in I}\left\langle y-z_{n+1}^{(i)}, A^{(i)}(y)\right\rangle.$$

(iii) Condition (C3) and the boundedness of $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ $(i\in I)$ imply that the sequences, $(\frac{\sum_{i\in I}\|x_0^{(i)}-y\|^2}{\sum_{k=0}^{n}\alpha_{k+1}})_{n\in\mathbb{N}}$ and $(\sum_{i\in I}\langle y-z_{n+1}^{(i)}, A^{(i)}(y)\rangle)_{n\in\mathbb{N}}$, are bounded for all $y\in C$. Moreover, Lemma 3.2 (ii) implies that, for all $n\in\mathbb{N}$ and for all $y\in C$,

$$\sum_{i\in I}\frac{1}{\sum_{k=0}^{n}\alpha_{k+1}}\sum_{k=0}^{n}\left\|x_{k+1}^{(i)}-y_k^{(i)}\right\|^2 \leq \frac{\sum_{i\in I}\left\|x_0^{(i)}-y\right\|^2}{\sum_{k=0}^{n}\alpha_{k+1}} + 2\sum_{i\in I}\left\langle y-z_{n+1}^{(i)}, A^{(i)}(y)\right\rangle.$$

Therefore, $(\frac{\sum_{k=0}^{n}\|x_{k+1}^{(i)}-y_k^{(i)}\|^2}{\sum_{k=0}^{n}\alpha_{k+1}})_{n\in\mathbb{N}}$ $(i\in I)$ is bounded. □

Lemma 3.2 leads us to the following:

Lemma 3.3. *Let $(y_n^{(i)})_{n\in\mathbb{N}}$ and $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ be the sequences generated by Algorithm 2.9 and define a sequence, $(w_{n+1}^{(i)})_{n\in\mathbb{N}}$ $(i\in I)$, by, for all $n\in\mathbb{N}$,*

$$w_{n+1}^{(i)} := \frac{1}{\sum_{k=1}^{n+1}\alpha_k}\sum_{k=1}^{n+1}\alpha_k y_{k-1}^{(i)}. \tag{3.3}$$

Then, $\lim_{n\to\infty} \|z_{n+1}^{(i)} - w_{n+1}^{(i)}\| = 0$ *for each* $i \in I$*. Moreover, if Condition* (2.4) *is satisfied,* $\lim_{n\to\infty} \|w_{n+1}^{(i)} - w_{n+1}^{(j)}\| = 0$ *for all* $i, j \in I$*.*

Proof. Choose any $i \in I$. From Lemma 3.2 (i) and (iii), there exist $M_1, M_2 > 0$ such that, for all $n \in \mathbb{N}$, $\|x_{n+1}^{(i)} - y_n^{(i)}\|^2 \leq M_1$ and $\frac{\sum_{k=0}^{n} \|x_{k+1}^{(i)} - y_k^{(i)}\|^2}{\sum_{k=0}^{n} \alpha_{k+1}} \leq M_2$. Choose $\varepsilon > 0$ arbitrarily. Then, Condition (C2) guarantees that $m_1(\varepsilon) \in \mathbb{N}$ exists such that $\alpha_n \leq \varepsilon$ for all $n \geq m_1(\varepsilon)$. Moreover, Condition (C3) ensures that, for $m_1(\varepsilon)$, there exists $m_2(m_1(\varepsilon)) \in \mathbb{N}$ such that

$$\frac{1}{\sum_{k=1}^{m_2} \alpha_k} \sum_{k=1}^{m_1} \alpha_k \left\| x_k^{(i)} - y_{k-1}^{(i)} \right\|^2 \leq \frac{M_1}{\sum_{k=1}^{m_2} \alpha_k} \sum_{k=1}^{m_1} \alpha_k \leq \varepsilon.$$

Hence, we find that, for all $n \geq m_2$,

$$\frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{m_1} \alpha_k \left\| x_k^{(i)} - y_{k-1}^{(i)} \right\|^2 \leq \varepsilon. \tag{3.4}$$

Equations (2.3) and (3.3), Inequality (3.4), the convexity of $\|\cdot\|^2$, and Condition (C1) thus imply that, for all $n \geq n_0 := \max\{m_1, m_2\}$,

$$\begin{aligned}
\left\| z_{n+1}^{(i)} - w_{n+1}^{(i)} \right\|^2 &= \left\| \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \alpha_k \left(x_k^{(i)} - y_{k-1}^{(i)} \right) \right\|^2 \\
&\leq \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \alpha_k \left\| x_k^{(i)} - y_{k-1}^{(i)} \right\|^2 \\
&= \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{m_1} \alpha_k \left\| x_k^{(i)} - y_{k-1}^{(i)} \right\|^2 \\
&\quad + \frac{1}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=m_1+1}^{n+1} \alpha_k \left\| x_k^{(i)} - y_{k-1}^{(i)} \right\|^2 \\
&\leq \varepsilon + \frac{\alpha_{m_1+1}}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=m_1+1}^{n+1} \left\| x_k^{(i)} - y_{k-1}^{(i)} \right\|^2 \\
&\leq \varepsilon + \frac{\varepsilon}{\sum_{k=1}^{n+1} \alpha_k} \sum_{k=1}^{n+1} \left\| x_k^{(i)} - y_{k-1}^{(i)} \right\|^2 \\
&\leq (1 + M_2)\varepsilon,
\end{aligned} \tag{3.5}$$

which implies that $\lim_{n\to\infty} \|z_{n+1}^{(i)} - w_{n+1}^{(i)}\| = 0$ for all $i \in I$.

Choose $i, j \in I$ and suppose that Condition (2.4) is satisfied. Then, $M_3, M_4 > 0$ exist such that, for all $n \in \mathbb{N}$, $\|y_n^{(i)} - y_n^{(j)}\|^2 \leq M_3$ and $\frac{\sum_{k=0}^{n} \|y_k^{(i)} - y_k^{(j)}\|^2}{\sum_{k=0}^{n} \alpha_{k+1}} \leq M_4$. The same calculation in Inequality (3.5) guarantees that, for all $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$\left\| w_{n+1}^{(i)} - w_{n+1}^{(j)} \right\|^2 \leq (1 + M_4)\varepsilon;$$

that is, $\lim_{n\to\infty} \|w_{n+1}^{(i)} - w_{n+1}^{(j)}\| = 0$ for all $i, j \in I$. □

The next lemma follows from Lemmas 3.2 and 3.3:

Lemma 3.4. *Suppose that Condition* (2.4) *is satisfied. Then, for all* $i \in I$, $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ *generated by Algorithm 2.9 has a subsequence converging weakly to a point in* $\operatorname{Argmax}_{x\in C} \sum_{i\in I} \mathcal{U}^{(i)}(x)$.

Proof. Choose an $i \in I$. From the boundedness of $(z_n^{(i)})_{n\in\mathbb{N}\setminus\{0\}}$ in Lemma 3.2 (i), there exist a subsequence, $(z_{n_l}^{(i)})_{l\in\mathbb{N}}$, of $(z_n^{(i)})_{n\in\mathbb{N}\setminus\{0\}}$ and a point, $z_*^{(i)} \in H$, such that $(z_{n_l}^{(i)})_{l\in\mathbb{N}}$ converges weakly to $z_*^{(i)}$. First, we shall show that $z_*^{(i)} \in C$. The closedness and convexity of $C^{(i)} (\subset H)$ and $(z_{n_l}^{(i)})_{l\in\mathbb{N}} \subset C^{(i)}$ guarantee that $z_*^{(i)} \in C^{(i)}$. Lemma 3.3 then ensures that $(w_{n_l}^{(i)})_{l\in\mathbb{N}}$ converges weakly to $z_*^{(i)}$. Choose $j \in I\setminus\{i\}$ arbitrarily. Then, from Lemma 3.3 and the weak convergence of $(w_{n_l}^{(i)})_{l\in\mathbb{N}}$ to $z_*^{(i)}$, we find that $(w_{n_l}^{(j)})_{l\in\mathbb{N}}$ converges weakly to $z_*^{(i)}$. By using Lemma 3.3 again, $(z_{n_l}^{(j)})_{l\in\mathbb{N}}$ converges weakly to $z_*^{(i)} \in C^{(i)}$. Moreover, the closedness and convexity of $C^{(j)}$ and $(z_{n_l}^{(j)})_{l\in\mathbb{N}} \subset C^{(j)}$ guarantee that $z_*^{(i)} \in C^{(j)}$. Therefore, $z_*^{(i)} \in C^{(i)} \cap \bigcap_{j\in I\setminus\{i\}} C^{(j)} =: C$.

Next, we shall show that $z_*^{(i)} \in \operatorname{Argmax}_{x\in C} \sum_{i\in I} \mathcal{U}^{(i)}(x)$. Lemma 3.2 (ii) ensures that, for all $y \in C$ and for all $l \in \mathbb{N}$,

$$-\frac{\sum_{j\in I} \left\| x_0^{(j)} - y \right\|^2}{\sum_{k=0}^{n_l-1} \alpha_{k+1}} \leq 2 \sum_{j\in I} \left\langle y - z_{n_l}^{(j)}, A^{(j)}(y) \right\rangle.$$

The weak convergence of $(z_{n_l}^{(j)})_{l\in\mathbb{N}}$ ($j \in I$) to $z_*^{(i)} \in C$ and Condition (C3) guarantee that, for all $y \in C$,

$$0 \leq 2 \sum_{j\in I} \left\langle y - z_*^{(i)}, A^{(j)}(y) \right\rangle = 2\left\langle y - z_*^{(i)}, \sum_{j\in I} A^{(j)}(y) \right\rangle.$$

From the monotonicity and hemicontinuity of $\sum_{i\in I} A^{(i)}$ (see Subsection 7.1) and Proposition 7.1 (i), we find that

$$0 \leq \left\langle y - z_*^{(i)}, \sum_{j\in I} A^{(j)}\left(z_*^{(i)}\right) \right\rangle \text{ for all } y \in C.$$

Therefore, from Relation (7.1), we have that $z_*^{(i)} \in \mathrm{VI}(C, \sum_{j\in I} A^{(j)}) = \operatorname{Argmax}_{x\in C} \sum_{i\in I} \mathcal{U}^{(i)}(x)$. This completes the proof of Lemma 3.4. □

Now we are in a position to prove Theorem 2.10 by using Lemma 3.4:
Proof of Theorem 2.10. From Proposition 2.4 and Lemma 3.4, for each $i \in I$, there exists a subsequence, $(z_{n_l}^{(i)})_{l\in\mathbb{N}}$, of $(z_n^{(i)})_{n\in\mathbb{N}\setminus\{0\}}$ such that it converges weakly to the unique point, $x^* \in \operatorname{Argmax}_{x\in C} \sum_{i\in I} \mathcal{U}^{(i)}(x)$. Take another subsequence, $(z_{n_m}^{(i)})_{m\in\mathbb{N}}$, of $(z_n^{(i)})_{n\in\mathbb{N}}$. Then, Lemma 3.4 ensures that $(z_{n_m}^{(i)})_{m\in\mathbb{N}}$ also converges weakly to x^*. Since, for each $i \in I$, any subsequence of $(z_n^{(i)})_{n\in\mathbb{N}}$ converges weakly to the same

point, we conclude that $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ converges weakly to the solution to Problem 2.3 for all $i \in I$. □

4. Application to network resource allocation problem

Let us look at typical numerical examples to see how Algorithm 2.9 works for different network topologies. The computer used in the experiment had an Intel Boxed Core i7 i7-870 2.93 GHz 8M CPU and 8 GB of memory. The language was MATLAB 7.9.

4.1. **Numerical examples for bandwidth allocation.**

The objective of utility-based bandwidth allocation is to share the available bandwidth among traffic sources so as to maximize the overall utility subject to the capacity constraints [25, 38]. In this subsection, we apply Algorithm 2.9 to the network bandwidth allocation problem in [30, Subsection 10.2.5] on a simple network that consists of two links and three sources, as shown in Figure 3:

Problem 4.1 (Network bandwidth allocation problem)**.**

$$\text{Maximize } \sum_{i\in I} \mathcal{U}^{(i)}(\boldsymbol{x}) := \sum_{i \in I} \sqrt{x_i}$$
$$\text{subject to } \boldsymbol{x} \in C := \Big\{ (x_1, x_2, x_3)^T \in \mathbb{R}_+^3 \colon x_1 + x_2 \leq c_1, x_1 + x_3 \leq c_2 \Big\},$$

where c_l $(l = 1, 2)$ stands for the capacity of link l and let $c_1 := 1$ and $c_2 := 2$.

The utility function of source i of which the explicit form is its own private information is defined by the strictly concave, continuously differentiable function, $\mathcal{U}^{(i)}(\boldsymbol{x}) := \sqrt{x_i}$ $(\boldsymbol{x} := (x_1, x_2, x_3)^T \in \mathbb{R}_+^3)$. The feasible region, C, for allocating the bandwidth is a nonempty, compact, convex subset of $\mathbb{R}^3$, and we assume that sources i commonly have $C^{(i)} := C$. The optimal solution of Problem 4.1 is $x^* := (0.2686, 0.7314, 1.7314)^T$. We assume that three sources find the optimal bandwidth pair by executing Algorithm 2.9, that, for all $n \in \mathbb{N}$, source 2 can get $x_n^{(i)}$ $(i = 1, 2, 3)$ in Equation (2.1), and sources 1 and 3 can get $x_n^{(i)}$ $(i = 1, 2)$ and $x_n^{(i)}$ $(i = 2, 3)$, respectively, and that, given $n \in \mathbb{N}$, firstly, source 1 transmits $x_n^{(1)}$ to source 2, secondly, source 2 transmits $x_n^{(2)}$ to sources 1 and 3, and thirdly, source 3 transmits $x_n^{(3)}$ to source 2. We also assume that source i can compute $y_n^{(i)}$, $x_{n+1}^{(i)}$, and $z_{n+1}^{(i)}$ as soon as it gets all the points, $x_n^{(j)}$s $(j \in I(i))$, transmitted from the neighbor sources. We further assume that each source has the weighted parameters in Example 2.6 and $\alpha_n := \frac{1}{\sqrt{n+1}}$ $(n \in \mathbb{N})$ satisfying Assumption 2.8.

The point, $x_{n+1}^{(i)} \in H$ $(i \in I, n \in \mathbb{N})$, in Equation (2.1) can be approximately calculated by using the algorithm in [47], as follows: for each $i \in I$ and for each

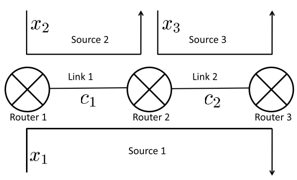

Figure 3: Network with two links and three sources with $c_1 := 1$ and $c_2 := 2$

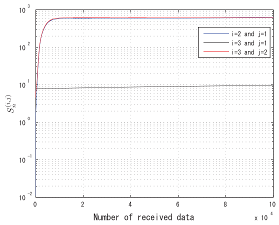

Figure 4: Behavior of $S_n^{(i,j)} := \frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \|y_k^{(i)} - y_k^{(j)}\|^2$ $(n \in \mathbb{N}, i, j \in I)$

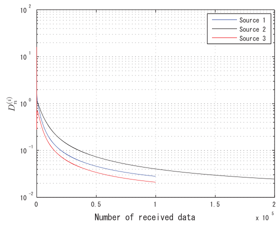

Figure 5: Behavior of $D_n^{(i)} := \|z_n^{(i)} - x^*\|^2$ $(i \in I)$

$n \in \mathbb{N}$, choose an initial point, $x_{n+1,0}^{(i)} := y_n^{(i)} \in H$, and compute

(4.1)
$$\begin{cases} x_{n+1,m+1}^{(i)} := T\Big(x_{n+1,m}^{(i)} - \lambda\Big[-\nabla \mathcal{U}^{(i)}\Big(x_{n+1,m}^{(i)}\Big) + \dfrac{1}{\alpha_{n+1}}\Big(x_{n+1,m}^{(i)} - y_n^{(i)}\Big)\Big]\Big) \\ \quad (m = 0, \ldots, M-1), \\ x_{n+1}^{(i)} := x_{n+1,M}^{(i)}, \end{cases}$$

where T is a nonexpansive mapping with $\mathrm{Fix}(T) = C$ (see Subsection 7.1), $\lambda > 0$, and $M > 0$. In this case, we used $T := P_{\mathbb{R}_+^3} P_{C_1} P_{C_2}$ satisfying $\mathrm{Fix}(T) = C = \mathbb{R}_+^3 \cap C_1 \cap C_2 \neq \emptyset$, $\lambda := 0.01$, and $M = 3$, where $C_1 := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1 + x_2 \leq 1\}$ and $C_2 := \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : x_1 + x_3 \leq 2\}$.

The behaviors of $(\frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \|y_k^{(i)} - y_k^{(j)}\|^2)_{n \in \mathbb{N}}$ $(i, j \in I)$ versus the number of received data are presented in Figure 4. This figure shows that, for each $i, j \in I$, $(\frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \|y_k^{(i)} - y_k^{(j)}\|^2)_{n \in \mathbb{N}}$ is stable and $\frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \|y_k^{(i)} - y_k^{(j)}\|^2 < 700$ for all $n \in \mathbb{N}$, which means that Condition (2.4) in Theorem 2.10 is satisfied. Figure 5 describes the behavior of $D_n^{(i)} := \|z_n^{(i)} - x^*\|^2$ $(i \in I, n \in \mathbb{N})$ versus the number of received data. We can see that all $D_n^{(i)}$s converge to 0 at the same convergence rate; that is, Algorithm 2.9 converges to the solution to Problem 4.1, as promised by Theorem 2.10.

4.2. **Numerical examples for storage allocation.** Here, we consider a utility-based maximization storage allocation problem [29] for a peer-to-peer (P2P) storage system network (Figure 2) in which each node supplies a storage capacity, y, which is then shared with other nodes, and demands a storage capacity, x, which is to be used for storing its own data. Storage capacities, x and y, are referred to as the demand and supply storage capacities. The utility function of node i in a P2P storage system network is defined as follows: for all $\boldsymbol{x} := (x_1, x_2, \ldots, x_7)^T, \boldsymbol{y} := (y_1, y_2, \ldots, y_7)^T \in \mathbb{R}^7$,

$$\mathcal{U}^{(i)}(\boldsymbol{x}, \boldsymbol{y}) := V^{(i)}(x_i) - P^{(i)}(y_i), \tag{4.2}$$

where

$$V^{(i)}\Big(x_i\Big) := \frac{1}{b^{(i)}}\Big(-\frac{x_i^2}{2} + b^{(i)}p_{\max}^{(i)}x_i\Big),\ P^{(i)}\Big(y_i\Big) := \frac{1}{a^{(i)}}\frac{y_i^2}{2} + p_{\min}^{(i)}y_i,$$

$p_{\min}^{(i)}$ and $p_{\max}^{(i)}$ respectively represent the minimum value of the unit price p^o such that node i sells some of its own disk space and the maximum value of the unit price p^s such that it buys some storage space, and $a^{(i)}$ and $b^{(i)}$ respectively correspond to the increase in sold capacity with the unit price p^o ($\geq p_{\min}^{(i)}$) and the decrease in bought storage space with the unit price p^s ($\leq p_{\max}^{(i)}$). These parameters, $a^{(i)}, b^{(i)}, p_{\min}^{(i)}$, and $p_{\max}^{(i)}$, for node i are its own private information; that is, other users cannot know them.

In the P2P storage system network considered here, since each node expects to have sufficient demand storage capacity within its supply storage capacity to store its own data, the utility maximization is constrained by the condition, $y \geq x$. Therefore, the solution to the problem of maximizing the overall utility of the nodes under the condition, $\sum_{i\in I} y_i \geq \sum_{\in I} x_i$, should yield reasonable pairs of demand and supply storage capacities for all nodes in the network. Hence, our objective is to solve the following maximization problem [29, Subsection III. A]:

Problem 4.2 (Network storage allocation problem)**.**

$$\text{Maximize } \sum_{i\in I}\mathcal{U}^{(i)}(\boldsymbol{x},\boldsymbol{y}) := \sum_{i\in I}\Big[V^{(i)}(x_i) - P^{(i)}(y_i)\Big]$$

$$\text{subject to } (\boldsymbol{x},\boldsymbol{y}) \in C := \Big\{((x_i)_{i\in I}^T,(y_i)_{i\in I}^T) \in \mathbb{R}_+^7 \times \mathbb{R}_+^7 \colon \sum_{i\in I} y_i \geq \sum_{i\in I} x_i\Big\}.$$

We can see that the explicit form of $\mathcal{U}^{(i)}(\boldsymbol{x},\boldsymbol{y}) := V^{(i)}(x_i) - P^{(i)}(y_i)$ ($(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{R}^7 \times \mathbb{R}^7$) is its own private information because only node i knows $a^{(i)}, b^{(i)}, p_{\min}^{(i)}$, and $p_{\max}^{(i)}$. Moreover, $\mathcal{U}^{(i)}$ has strict concavity and continuous differentiability. The feasible region, C, for allocating the storages is a nonempty, compact, convex subset of $\mathbb{R}^7$, and we assume that nodes i commonly have $C^{(i)} := C$. In the same way as in Subsection 4.1, we assume that, given $n \in \mathbb{N}$, node i can transmit $x_n^{(i)}$ to node j ($j \in I(i)$) in the order from node 1 to node 7, that node i can compute $y_n^{(i)}$, $x_{n+1}^{(i)}$, and $z_{n+1}^{(i)}$ as soon as it gets all $x_n^{(j)}$s ($j \in I(i)$), and that each node has its weighted parameter in Example 2.7 and $\alpha_{n+1} := \frac{1}{\sqrt{n+1}}$ ($n \in \mathbb{N}$). We used $a^{(i)}, b^{(i)} \in [0.0001, 1]$, as given by uniform random numbers, $p_{\min}^{(i)} := 0.0001$, $p_{\max}^{(i)} := 1$ ($i \in I$). We used $T := P_{\mathbb{R}_+^7} P_{\bar{C}}$ ($\bar{C} := \{((x_i)_{i\in I}^T,(y_i)_{i\in I}^T) \in \mathbb{R}^7 \times \mathbb{R}^7 \colon \sum_{i\in I} y_i \geq \sum_{i\in I} x_i\}$) with $\mathrm{Fix}(T) = \mathbb{R}_+^7 \cap \bar{C} = C \neq \emptyset$ (see also Subsection 7.1), $\lambda := 0.001$, and $M := 10$ to compute the approximate point of $x_{n+1}^{(i)}$ ($i \in I, n \in \mathbb{N}$) in Equation (4.1).

Figures 6–8 show the behaviors of $(\frac{1}{\sum_{k=0}^n \alpha_{k+1}} \sum_{k=0}^n \|y_k^{(i)} - y_k^{(j)}\|^2)_{n\in\mathbb{N}}$ ($i, j \in I$) versus the number of received data. These figures indicate that, for each $i, j \in I$, $(\frac{1}{\sum_{k=0}^n \alpha_{k+1}} \sum_{k=0}^n \|y_k^{(i)} - y_k^{(j)}\|^2)_{n\in\mathbb{N}}$ is stable and $\frac{1}{\sum_{k=0}^n \alpha_{k+1}} \sum_{k=0}^n \|y_k^{(i)} - y_k^{(j)}\|^2 < 10^4$ for all $n \in \mathbb{N}$, which means that Condition (2.4) in Theorem 2.10 is satisfied. Therefore, Theorem 2.10 guarantees that Algorithm 2.9 converges to the solution to

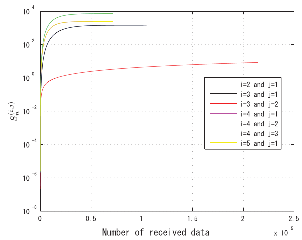

Figure 6: Behavior of $S_n^{(i,j)} := \frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \|y_k^{(i)} - y_k^{(j)}\|^2$ $(n \in \mathbb{N}, i = 2, 3, 4, 5, j = 1, 2, 3)$

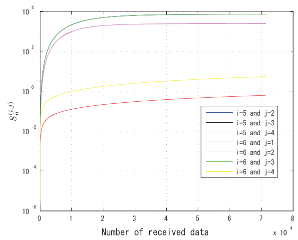

Figure 7: Behavior of $S_n^{(i,j)} := \frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \|y_k^{(i)} - y_k^{(j)}\|^2$ $(n \in \mathbb{N}, i = 5, 6, j = 1, 2, 3, 4)$

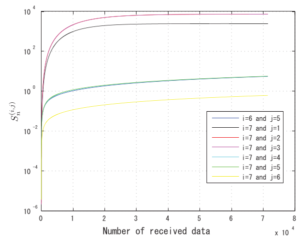

Figure 8: Behavior of $S_n^{(i,j)} := \frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \|y_k^{(i)} - y_k^{(j)}\|^2$ $(n \in \mathbb{N}, i = 6, 7, j = 1, 2, 3, 4, 5, 6)$

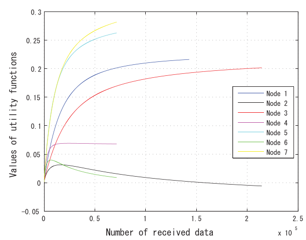

Figure 9: Behavior of the utility function of each node

Problem 4.2. Figure 9 plots the behavior of the utility function of each node versus the number of received data. We can see that the utilities of nodes 1, 3, 5, and 7 increase and the utility of node 4 becomes stable after the received data exceed 0.2×10^5, while the utilities of nodes 2 and 6 decrease after the received data exceed 0.2×10^5.

5. Conclusion and future work

We presented a multicast decentralized optimization algorithm for solving the centralized optimization problem with information on the whole network. The proposed algorithm enables each user to set his or her own optimal resource allocation in cooperation with neighbor users and is practical from the viewpoint of network scalability. We also presented a convergence analysis of the algorithm. The analysis ensures that the algorithm converges weakly to the solution to the problem under certain assumptions. To demonstrate the effectiveness of the algorithm, we applied it to concrete network resource allocation problems and provided some numerical examples.

In the future, we should consider developing decentralized resource allocation algorithms to resolve the problems listed below.

- We must discuss practical implementations to which the theoretical approach given in Section 2 cannot be applied. For example, we need to consider a situation where users can move into and out of the network.

- To resolve many practical resource allocation problems, we need to devise a multicast decentralized algorithm that works when each user's utility function is non-concave (for example, the signal-to-interference-plus-noise ratio (SINR), which is used to evaluate the performance of each user in a wireless network, is not concave) and do a convergence analysis on it. Centralized algorithms for solving non-convex optimization problems were presented in [20, 37].
- It would be good to have a decentralized optimization algorithm that works even when the intersection of all users' feasible sets is empty. There are centralized algorithms [11, 20, 22, 46] for solving the optimization problem with an infeasible constraint set by using fixed point theory for nonexpansive mappings. Hence, application of fixed point theory would be a way to devise such an algorithm.
- The existing decentralized optimization algorithms require the cooperation of all users. However, there would likely be some users in a network who would not want to cooperate with other users. For such cases, we need a distributed control mechanism that enables each user to determine its own optimal resource allocation independently.

Acknowledgments

The author thanks Professors W. Takahashi, T. Takine, M. Tsuru, and M. Uchida for their discussions on the algorithm. The author also thanks Dr. M. Yamagishi for his discussion on the numerical experiment.

6. Appendix–extension of Algorithm 2.9 to nonsmooth utility functions

Let us consider the network resource allocation problem for the case such that H is finite dimensional and that

(A1)' The utility function, $\mathcal{U}^{(i)} \colon \mathbb{R}^L \to \mathbb{R}$ $(i \in I)$, is strictly concave[6] (and is not always differentiable).

The following holds under Assumptions (A1)', (A2), (A3), (A4), (A6), and (A7):

Theorem 6.1. *Algorithm 2.9 converges to a unique solution to Problem 2.3 if Condition* (2.4) *is satisfied.*

Proof. As in the proof of Lemma 3.2, we can prove the boundedness of $(x_n^{(i)})_{n\in\mathbb{N}}$, $(y_n^{(i)})_{n\in\mathbb{N}}$, and $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ $(i \in I)$, and Inequality (3.2). Choose an $i \in I$. Since $x_{n+1}^{(i)} \in C^{(i)}$ $(n \in \mathbb{N})$ in Equation (2.1) is the minimizer of a convex function, $-\mathcal{U}^{(i)}(\cdot) + \frac{1}{2\alpha_{n+1}}\| \cdot - y_n^{(i)}\|^2$, over $C^{(i)}$, the following variational inequality holds true from Relation (7.1): for all $n \in \mathbb{N}$, there exists $u_{n+1}^{(i)} \in \partial(-\mathcal{U}^{(i)})(x_{n+1}^{(i)})$ such that, for all $y \in C \subset C^{(i)}$,

$$\left\langle y - x_{n+1}^{(i)}, u_{n+1}^{(i)} + \frac{1}{\alpha_{n+1}}\left(x_{n+1}^{(i)} - y_n^{(i)}\right)\right\rangle \geq 0,$$

[6] Any concave function on the Euclidean space is continuous [6, Theorem 4.1.3].

which means that

$$\left\langle y - x_{n+1}^{(i)}, u_{n+1}^{(i)} \right\rangle \geq \left\langle y - x_{n+1}^{(i)}, \frac{1}{\alpha_{n+1}} \left(y_n^{(i)} - x_{n+1}^{(i)} \right) \right\rangle$$
$$= \frac{1}{2\alpha_{n+1}} \left\{ \left\| y - x_{n+1}^{(i)} \right\|^2 + \left\| y_n^{(i)} - x_{n+1}^{(i)} \right\|^2 - \left\| y - y_n^{(i)} \right\|^2 \right\}.$$

This inequality and the subdifferentiability of $-\mathcal{U}^{(i)}$ at $x_{k+1}^{(i)}$ (see Subsection 7.1) guarantee that, for all $y \in C$ and for all $k \in \mathbb{N}$,

$$\mathcal{U}^{(i)}\left(x_{k+1}^{(i)} \right) \geq \mathcal{U}^{(i)}(y) + \left\langle y - x_{k+1}^{(i)}, u_{k+1}^{(i)} \right\rangle$$
$$\geq \mathcal{U}^{(i)}(y) + \frac{1}{2\alpha_{k+1}} \left\{ \left\| y - x_{k+1}^{(i)} \right\|^2 + \left\| y_k^{(i)} - x_{k+1}^{(i)} \right\|^2 - \left\| y - y_k^{(i)} \right\|^2 \right\}.$$

Summing this inequality over all i and Inequality (3.2) imply that, for all $y \in C$ and for all $k \in \mathbb{N}$,

$$\sum_{i \in I} \mathcal{U}^{(i)}\left(x_{k+1}^{(i)} \right)$$
$$\geq \sum_{i \in I} \mathcal{U}^{(i)}(y) + \frac{1}{2\alpha_{k+1}} \left\{ \sum_{i \in I} \left\| y_k^{(i)} - x_{k+1}^{(i)} \right\|^2 + \sum_{i \in I} \left(\left\| x_{k+1}^{(i)} - y \right\|^2 - \left\| x_k^{(i)} - y \right\|^2 \right) \right\}.$$

By summing this inequality from $k = 0$ to $k = n$ ($n \in \mathbb{N}$), we have

$$2 \sum_{i \in I} \sum_{k=0}^{n} \alpha_{k+1} \mathcal{U}^{(i)}\left(x_{k+1}^{(i)} \right)$$
$$\geq 2 \sum_{k=0}^{n} \alpha_{k+1} \sum_{i \in I} \mathcal{U}^{(i)}(y) + \sum_{i \in I} \sum_{k=0}^{n} \left\| y_k^{(i)} - x_{k+1}^{(i)} \right\|^2 + \sum_{i \in I} \left(\left\| x_{n+1}^{(i)} - y \right\|^2 - \left\| x_0^{(i)} - y \right\|^2 \right).$$

On the other hand, the concavity of $\mathcal{U}^{(i)}$ and Equation (2.3) ensure that, for all $i \in I$ and for all $n \in \mathbb{N}$,

$$\frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \alpha_{k+1} \mathcal{U}^{(i)}\left(x_{k+1}^{(i)} \right) \leq \mathcal{U}^{(i)}\left(z_{n+1}^{(i)} \right).$$

Hence, we find that, for all $y \in C$ and for all $n \in \mathbb{N}$,

$$2 \sum_{i \in I} \mathcal{U}^{(i)}\left(z_{n+1}^{(i)} \right) \tag{6.1}$$
$$\geq 2 \sum_{i \in I} \mathcal{U}^{(i)}(y) - \frac{\sum_{i \in I} \left\| x_0^{(i)} - y \right\|^2}{\sum_{k=0}^{n} \alpha_{k+1}} + \sum_{i \in I} \frac{1}{\sum_{k=0}^{n} \alpha_{k+1}} \sum_{k=0}^{n} \left\| y_k^{(i)} - x_{k+1}^{(i)} \right\|^2.$$

The compactness of $C^{(i)}$ ($i \in I$), $(z_{n+1}^{(i)})_{n \in \mathbb{N}} \subset C^{(i)}$, and the continuity of $\mathcal{U}^{(i)}$ ($i \in I$) guarantee the boundedness of $\mathcal{U}^{(i)}(z_{n+1}^{(i)})$ ($i \in I, n \in \mathbb{N}$). Therefore, Inequality (6.1) and Condition (C3) imply that $\left(\frac{\sum_{k=0}^{n} \|x_{k+1}^{(i)} - y_k^{(i)}\|^2}{\sum_{k=0}^{n} \alpha_{k+1}}\right)_{n \in \mathbb{N}}$ ($i \in I$) is bounded.

Choose an $i \in I$. As in the proofs of Lemmas 3.3 and 3.4, we can prove that a subsequence, $(z_{n_l}^{(i)})_{l\in\mathbb{N}}$, of $(z_n^{(i)})_{n\in\mathbb{N}}$ and a point, $z_*^{(i)} \in C$, exist such that $(z_{n_l}^{(i)})_{l\in\mathbb{N}}$ converges to $z_*^{(i)}$. Moreover, from Inequality (6.1), we find that, for all $y \in C$ and for all $l \in \mathbb{N}$,

$$2\sum_{j\in I}\mathcal{U}^{(j)}\left(z_{n_l}^{(j)}\right) \geq 2\sum_{j\in I}\mathcal{U}^{(j)}(y) - \frac{\sum_{j\in I}\left\|x_0^{(j)} - y\right\|^2}{\sum_{k=0}^{n_l-1}\alpha_{k+1}}.$$

The continuity of $\mathcal{U}^{(j)}$ $(j \in I)$, the convergence of $(z_{n_l}^{(j)})_{l\in\mathbb{N}}$ $(j \in I)$ to $z_*^{(i)}$ (see the proof of Lemma 3.4), and Condition (C3) guarantee that, for all $y \in C$,

$$\sum_{j\in I}\mathcal{U}^{(j)}\left(z_*^{(i)}\right) \geq \sum_{j\in I}\mathcal{U}^{(j)}(y);$$

that is, $z_*^{(i)} \in \mathrm{Argmax}_{x\in C}\sum_{j\in I}\mathcal{U}^{(j)}(x)$. Since $\mathrm{Argmax}_{x\in C}\sum_{j\in I}\mathcal{U}^{(j)}(x)$ consists of one point, $(z_{n+1}^{(i)})_{n\in\mathbb{N}}$ converges to the maximizer of $\sum_{i\in I}\mathcal{U}^{(i)}$ over $C := \bigcap_{i\in I} C^{(i)}$. □

7. Appendix–mathematical preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|$, and let $\mathbb{N}$ be the set of zero and all positive integers, that is, $\mathbb{N} := \{0, 1, 2, \ldots\}$. Let $\mathbb{R}^N$ and $\mathbb{R}_+^N$ denote an N-dimensional Euclidean space and $\{(x_1, x_2, \ldots, x_N)^T \in \mathbb{R}^N : x_i \geq 0\ (i = 1, 2, \ldots, N)\}$, respectively.

7.1. **Monotonicity, continuity, and nonexpansivity.** A set-valued operator, $A\colon H \to 2^H$, is said to be *monotone* [48, Definition 32.2 (c)] if, for all $(x, u), (y, v) \in \mathrm{G}(A) := \{(z, w) \in H \times H : w \in A(z)\}$, $\langle x - y, u - v\rangle \geq 0$. $A\colon H \to H$ is referred to as a *strictly monotone* operator [48, Definition 25.2 (ii)] if, for all $x, y \in H$ with $x \neq y$, $\langle x - y, A(x) - A(y)\rangle > 0$. A monotone operator, $A\colon H \to 2^H$, is said to be *maximal* [48, Definition 32.2 (b), (d)] if $\mathrm{G}(A)$ is not properly contained in $\mathrm{G}(B)$ of any monotone operator $B\colon H \to 2^H$. Suppose that $f\colon H \to \mathbb{R}$ is convex and lower semicontinuous.[7] Then, the *subdifferential*, $\partial f\colon H \to 2^H$, of f is defined as follows [34, Part V]: for all $x \in H$, $\partial f(x) := \{z \in H : f(y) \geq f(x) + \langle y - x, z\rangle\ (y \in H)\}$, and it satisfies the maximal monotonicity condition [34, Corollary 31.5.2], [49, Theorem 47.F (1)], [40, Theorem 4.6.6]. If f is convex and continuous at $x \in H$, the condition, $\partial f(x) \neq \emptyset$, holds true [49, Theorem 47.A (2)]. Moreover, if f is convex and Gâteaux differentiable on H, the condition, $\partial f(x) = \{\nabla f(x)\}$, holds for all $x \in H$, where $\nabla f\colon H \to H$ stands for the gradient of f [48, Proposition 32.13 (a)].

$A\colon H \to H$ is said to be *hemicontinuous* [40, p.204], [48, Definition 27.14] if, for any $x, y \in H$, a mapping, $g\colon [0, 1] \to H$, defined by $g(t) := A(tx + (1 - t)y)$ $(t \in [0, 1])$ is continuous, where H has a weak topology. Any single-valued, monotone, hemicontinuous operator satisfies the maximality condition [48, Proposition 32.7]. $A\colon H \to H$ is referred to as a *Lipschitz continuous* (L-Lipschitz continuous) operator [18, Subsection 1.1], [48, Definition 27.14] if $L > 0$ exists such that $\|A(x) - A(y)\| \leq$

[7] $f\colon H \to \mathbb{R}$ is said to be *lower semicontinuous* on H if, for any $a \in \mathbb{R}$, the set, $\{x \in H : f(x) \leq a\}$, is closed.

$L\|x-y\|$ for all $x, y \in H$. $A\colon H \to H$ is called a *nonexpansive* mapping [18, Subsection 1.1], [17, Section 3] when it is 1-Lipschitz continuous; that is, $\|A(x) - A(y)\| \leq \|x-y\|$ for all $x, y \in H$. The *fixed point set* of a (nonexpansive) mapping, A, is denoted by $\mathrm{Fix}(A) := \{x \in H \colon A(x) = x\}$.

7.2. **Variational inequality problem for monotone operators.** The variational inequality problem [27, Chapter III], [49, Chapters 54-57], [16, Chapter II], [6, Subsection 8.3] for a monotone operator, $A\colon H \to 2^H$, over a nonempty, closed convex set, D $(\subset H)$, is to

$$\begin{aligned}&\text{find } x^* \in \mathrm{VI}(D, A)\\ &\qquad := \left\{x^* \in D \colon u^* \in A(x^*) \text{ exists such that } \left\langle y - x^*, u^* \right\rangle \geq 0 \ (y \in D)\right\}.\end{aligned}$$

When A is single-valued, $\mathrm{VI}(D, A)$ is equal to $\{x^* \in D \colon \langle y - x^*, A(x^*)\rangle \geq 0 \ (y \in D)\}$. The following theorem characterizes the solution set of the variational inequality problem and proves the existence of a point in the set (Proposition 7.1 (iii) can be readily proved by using strict monotonicity):

Proposition 7.1. *Let D $(\subset H)$ be a nonempty, closed convex set, and let $A\colon H \to H$ be monotone and hemicontinuous. Then,*

(i) [40, Lemma 7.1.7] $\mathrm{VI}(D, A) = \left\{x^* \in D \colon \langle y - x^*, A(y)\rangle \geq 0 \ (y \in D)\right\}$*;*
(ii) [6, Theorem 8.3.6], [40, Theorem 7.1.8] $\mathrm{VI}(D, A) \neq \emptyset$ *if D is compact;*
(iii) *there exists a unique point in* $\mathrm{VI}(D, A)$ *if $A\colon H \to H$ is strictly monotone and if* $\mathrm{VI}(D, A) \neq \emptyset$.

Suppose that $f\colon H \to \mathbb{R}$ is convex and lower semicontinuous and that D $(\subset H)$ is a nonempty, closed convex set. Then, the set of minimizers of f over D is coincident with the solution set of the variational inequality problem for ∂f over D [49, Theorem 47.C (1)], [6, Subsection 8.3]; that is,

$$\mathrm{VI}(D, \partial f) = \operatorname*{Argmin}_{x \in D} f(x). \tag{7.1}$$

The variational inequality problem also includes many nonlinear problems such as the fixed point problem for a nonexpansive mapping [41, Theorem 7.7.2], the complementarity problem for a monotone operator [41, Problem 7.7.2], and so on.

7.3. **Metric projections onto closed convex sets.** Let D $(\subset H)$ be nonempty, closed, and convex. A mapping that assigns every point, $x \in H$, to its unique nearest point in D is called a *metric projection* [2, Facts 1.5], [39, Equation (2.3-13)], [40, p.56] onto D and is denoted by P_D; that is, $P_D(x) \in D$ and $\|x - P_D(x)\| = \inf_{y \in D} \|x-y\|$. The metric projection, P_D, satisfies the following conditions:

Proposition 7.2.

(i) [2, Facts 1.5 (ii)], [40, Lemma 3.1.3] *Let $x \in H$. Then, $\bar{x} = P_D(x)$ if and only if $\bar{x} \in D$ and $\langle \bar{x} - x, y - \bar{x}\rangle \geq 0$ for all $y \in D$.*
(ii) *The fixed point set of P_D is coincident with D; that is,* $\mathrm{Fix}(P_D) := \{x \in H \colon P_D(x) = x\} = D$.

(iii) [17, Equation (12.5)], [2, Facts 1.5 (i)], [39, Theorem 2.4-1 (ii)], [40, Proof (i) of Theorem 3.1.4] *P_D satisfies the firm nonexpansivity condition; that is,* $\|P_D(x) - P_D(y)\|^2 \leq \langle x - y, P_D(x) - P_D(y)\rangle$ *for all* $x, y \in H$.

From Proposition 7.2 (iii) and the Cauchy-Schwarz inequality, we find that $\|P_D(x) - P_D(y)\| \leq \|x - y\|$ $(x, y \in H)$; that is, P_D is nonexpansive. If D is a linear variety, a closed ball, a closed cone, or a closed polytope, the explicit form of P_D is known, which implies that P_D can be explicitly calculated [45].

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Manuscript received February 15, 2013
revised April 28, 2013

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