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WEAK AND STRONG CONVERGENCE THEOREMS FOR SEMIGROUPS OF MAPPINGS WITHOUT CONTINUITY IN HILBERT SPACES

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Abstract. In this paper, we first introduce a broad semigroup of mappings without continuity in Hilbert spaces which contains discrete semigroups generated by generalized hybrid mappings and semigroups of nonexpansive mappings. Then, using the theory of invariant means, we prove a weak convergence theorem of Mann's type iteration for the semigroups. Next, using Halpern's type iteration, we prove a strong convergence theorem for such semigroups. Using these results, we obtain new and well-known results for semigroups of mappings without continuity in Hilbert spaces.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into itself. We denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [25] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;

(ii) $A(T) = \{z \in H : \|Tx - z\| \le \|x - z\|,\ \forall x \in C\}$.

We know from [25] that $A(T)$ is closed and convex. This property is important. Kocourek, Takahashi and Yao [14] defined a class of nonlinear mappings containing nonexpansive mappings, nonspreading mappings [15, 16] and hybrid mappings [24] in a Hilbert space. A mapping $T : C \to C$ is called *generalized hybrid* [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$; see also [17]. We call such a mapping an (α, β)-*generalized hybrid* mapping. A (1, 0)-generalized hybrid mapping is *nonexpansive* [8], i.e.,

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C.$$

It is *nonspreading* [15, 16] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is *hybrid* [24] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

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In general, nonspreading and hybrid mappings are not continuous. See, for example, [12]. We also know the concept of one-parameter nonexpansive semigroups in a Hilbert space. Let H be a Hilbert space and let C be a nonempty subset of H. Let $S = \mathbb{R}^+ = \{t \in \mathbb{R} : 0 \leq t < \infty\}$. A family $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself is called a *one-parameter nonexpansive semigroup* on C if $\mathcal{S}$ satisfies the following:

(1) $S(t+s)x = S(t)S(s)x, \quad \forall x \in C,\ t, s \in \mathbb{R}^+$;
(2) $S(0)x = x, \quad \forall x \in C$;
(3) for each $x \in C$, the mapping $t \mapsto S(t)x$ from $\mathbb{R}^+$ into C is continuous;
(2) for each $t \in \mathbb{R}^+$, $S(t)$ is nonexpansive.

Of course, $S(t)$ are continuous. Such one-parameter nonexpansive semigroups are used in the theory of nonlinear evolution equations [6]. Recently, using the concept of invariant means, Takahashi, Wong and Yao [28] introduced the concept of semigroups of mappings without continuity in Hilbert spaces which contains discrete semigroups generated by generalized hybrid mappings and semigroups of nonexpansive mappings. They proved a nonlinear mean convergence theorem of Baillon's type [4] which generalized simultaneously the mean convergence theorems [14] and [5] for generalized hybrid mappings and one-parameter nonexpansive semigroups in a Hilbert space. What kind of conditions of semigroups do we need to prove a weak convergence theorem of Mann's type [18] and a strong convergence theorem of Halpern's type [9] in a Hilbert space? This question is natural.

In this paper, using the concept of strongly asymptotically invariant nets, we first introduce a broad semigroup of mappings without continuity in Hilbert spaces which contains discrete semigroups generated by generalized hybrid mappings and semigroups of nonexpansive mappings. Then, using the theory of strongly asymptotically invariant means, we prove a weak convergence theorem of Mann's type iteration for the semigroups. Next, using Halpern's type iteration, we prove a strong convergence theorem for such semigroups. Using these results, we obtain new results and well-known results for semigroups of mappings without continuity in Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known [23] that for all $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|y\|^2 - \|x\|^2 \leq 2\langle y - x, y \rangle; \tag{2.1}$$

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.2}$$

Furthermore, we have that

$$2\langle x - y, z - w \rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.3}$$

for all $x, y, z, w \in H$. From (2.3), we have that

$$2\langle x - y, z - y \rangle - \|z-y\|^2 = \|x-y\|^2 - \|x-z\|^2 \tag{2.4}$$

for all $x, y, z \in H$. Let C be a nonempty subset of H. A mapping $T : C \to C$ is *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \quad \forall x \in C, \ y \in F(T).$$

It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Itoh and Takahashi [13].

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$ (the dual space of ℓ^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞. If μ is a Banach limit on ℓ^∞, then for $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [22]. To prove our main results, we need the following lemmas:

Lemma 2.1 (Aoyama-Kimura-Takahashi-Toyoda [1])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^\infty \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^\infty \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n = 1, 2, \dots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2 (Takahashi-Toyoda [26])**.** *Let D be a nonempty closed convex subset of a real Hilbert space H. Let P be the metric projection of H onto D and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.*

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. Let μ be an element of $C(S)^*$ (the dual space of $C(S)$). We denote by $\mu(f)$ the value of μ at $f \in C(S)$. Sometimes, we denote by $\mu_t(f(t))$ or $\mu_t f(t)$ the value $\mu(f)$. For each $s \in S$ and $f \in C(S)$, we define two functions $\ell_s f$ and $r_s f$ as follows:

$$(\ell_s f)(t) = f(st) \quad \text{and} \quad (r_s f)(t) = f(ts)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a *mean* on $C(S)$ if $\mu(e) = \|\mu\| = 1$, where $e(s) = 1$ for all $s \in S$. We know that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s), \quad \forall f \in C(S).$$

A mean μ on $C(S)$ is called *left invariant* if $\mu(\ell_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called *right invariant* if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A left and right invariant invariant mean on $C(S)$ is called an *invariant* mean on $C(S)$. If $S = \mathbb{N}$, an invariant mean on $C(S) = B(S)$ is a Banach limit on ℓ^∞. The following theorem is in [22, Theorem 1.4.5].

Theorem 2.3 ([22]). *Let S be a commutative semitopological semigroup. Then there exists an invariant mean on $C(S)$, i.e., there exists an element $\mu \in C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let S be a semitopological semigroup. For any $f \in C(S)$ and $c \in \mathbb{R}$, we write

$$f(s) \to c, \quad \text{as} \quad s \to \infty_R$$

if for each $\varepsilon > 0$ there exists an $\omega \in S$ such that

$$|f(tw) - c| < \varepsilon, \quad \forall t \in S.$$

We denote the case $f(s) \to c$, as $s \to \infty_R$ by

$$\lim_{s \to \infty_R} f(s) = c, \quad \text{or} \quad \lim_s f(s) = c.$$

When S is commutative, we also denote $s \to \infty_R$ by $s \to \infty$.

Theorem 2.4 ([22]). *Let $f \in C(S)$ and $c \in \mathbb{R}$. If*

$$f(s) \to c, \quad \text{as} \quad s \to \infty_R,$$

then $\mu(f) = c$ for all right invariant mean μ on $C(S)$.

Theorem 2.5 ([22]). *If $f \in C(S)$ fulfills*

$$f(ts) \le f(s), \quad \forall t, s \in S,$$

then

$$f(t) \to \inf_{w \in S} f(w), \quad \text{as} \quad t \to \infty_R.$$

Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a semitopological semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a family of mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a *continuous representation* of S as mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and $s \mapsto T_s x$ is continuous for each $x \in C$. We denote by $F(\mathcal{S})$ the set of common fixed points of $T_s,\ s \in S$, i.e.,

$$F(\mathcal{S}) = \cap\{F(T_s) : s \in S\}.$$

A continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings on C is called a *nonexpansive semigroup* on C if each $T_s,\ s \in S$ is nonexpansive, i.e.,

$$\|T_s x - T_s y\| \le \|x - y\|, \quad \forall x, y \in C.$$

The following definition [21] is crucial in the nonlinear ergodic theory of abstract semigroups. Let $u : S \to H$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \overline{co}\{u(s) : s \in S\}$ such that

$$\mu_s \langle u(s), y \rangle = \langle z_0, y \rangle, \quad \forall y \in H. \tag{2.5}$$

We call such z_0 the *mean vector* of u for μ. In particular, if $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as mappings on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$ and $u(s) = T_s x$ for all $s \in S$, then there exists $z_0 \in H$ such tat

$$\mu_s \langle T_s x, y\rangle = \langle z_0, y\rangle, \quad \forall y \in H.$$

We denote such z_0 by $T_\mu x$.

Motivated by Takahashi and Takeuchi [25], Atsushiba and Takahashi [3] defined the set $A(\mathcal{S})$ of all common attractive points of a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself, i.e.,

$$A(\mathcal{S}) = \cap\{A(T_s) : s \in S\}.$$

A net $\{\mu_\alpha\}$ of means on $C(S)$ is said to be *asymptotically invariant* if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(\ell_s f) \to 0 \quad \text{and} \quad \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

A net $\{\mu_\alpha\}$ of means on $C(S)$ is said to be *strongly asymptotically invariant* if for each $s \in S$,

$$\|\ell_s^* \mu_\alpha - \mu_\alpha\| \to 0 \quad \text{and} \quad \|r_s^* \mu_\alpha - \mu_\alpha\| \to 0,$$

where ℓ_s^* and r_s^* are the adjoint operators of ℓ_s and r_s, respectively. See [7] and [22] for more details. Recently, Takahashi, Wong and Yao [28] proved the following theorems:

Theorem 2.6. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Let $\{T_s x : s \in S\}$ be bounded for some $x \in C$ and let μ be a mean on $C(S)$. Suppose that*

$$\mu_s \|T_s x - T_t y\|^2 \le \mu_s \|T_s x - y\|^2, \quad \forall y \in C,\ t \in S. \tag{2.6}$$

Then $A(\mathcal{S})$ is nonempty. In addition, if C is closed and convex, then $F(\mathcal{S})$ is nonempty.

Theorem 2.7. *Let H be a Hilbert space and let C be a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself such that $A(\mathcal{S}) \neq \emptyset$. Suppose that*

$$\mu_s \|T_s x - T_t y\|^2 \le \mu_s \|T_s x - y\|^2, \quad \forall x, y \in C,\ t \in S \tag{2.7}$$

for all invariant means μ on $C(S)$. Let $\{\mu_\alpha\}$ be a net of means on $C(S)$ such that for each $f \in C(S)$ and $s \in S$, $\mu_\alpha(f) - \mu_\alpha(\ell_s f) \to 0$. Then, $\{T_{\mu_\alpha} x\}$ converges weakly to $u \in A(\mathcal{S})$, where $u = \lim_s P_{A(\mathcal{S})} T_s x$. In addition, if C is closed and convex, then $\{T_{\mu_\alpha} x\}$ converges weakly to $u \in F(\mathcal{S})$, where $u = \lim_s P_{F(\mathcal{S})} T_s x$.

3. Weak convergence theorems

In this section, we first prove a weak convergence theorem of Mann's type iteration for semigroups of mappings without continuity in a Hilbert space.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty, bounded, closed and convex subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Suppose that*

$$\limsup_\alpha \sup_{x,y\in C} (\mu_\alpha)_s(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) \le 0, \quad \forall t \in S \tag{3.1}$$

for all strongly asymptotically invariant nets $\{\mu_\alpha\}$ of means on $C(S)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e.,

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$ and $z = \lim_{n\to\infty} P_{F(\mathcal{S})} x_n$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Proof. Since S is commutative, we have from Theorem 2.3 that there exists an invariant mean on $C(S)$. Let μ be an invariant mean on $C(S)$ and put $\mu_\alpha = \mu$ in (3.1). Then, we have from (3.1) that

$$\mu_s \|T_s x - T_t y\|^2 \le \mu_s \|T_s x - y\|^2, \quad \forall x, y \in C, \ t \in S.$$

So, we have from Theorem 2.6 that $A(\mathcal{S})$ is nonempty. Let $z \in A(\mathcal{S})$. Since $\|\mu_n\| = 1$, we have that for any $n \in \mathbb{N}$,

$$\begin{aligned} \|T_{\mu_n} x_n - z\|^2 &= \langle T_{\mu_n} x_n - z, T_{\mu_n} x_n - z \rangle \\ &= (\mu_n)_t \langle T_t x_n - z, T_{\mu_n} x_n - z \rangle \\ &\le \|\mu_n\| \sup_t |\langle T_t x_n - z, T_{\mu_n} x_n - z \rangle| \\ &\le \sup_t \|T_t x_n - z\| \cdot \|T_{\mu_n} x_n - z\| \\ &\le \sup_t \|x_n - v\| \cdot \|T_{\mu_n} x_n - z\| \\ &= \|x_n - v\| \cdot \|T_{\mu_n} x_n - z\| \end{aligned} \tag{3.2}$$

and hence

$$\|T_{\mu_n} x_n - z\| \le \|x_n - z\|. \tag{3.3}$$

Using (3.3), we have that

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n - z\|^2 \\ &\le \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) \|T_{\mu_n} x_n - z\|^2 \\ &\le \alpha_n \|x_n - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 \\ &= \|x_n - z\|^2 \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} \|x_n - z\|^2$ exists and hence $\{x_n\}$ is bounded. We also have from (2.2) that

$$\|x_{n+1} - z\|^2 = \|\alpha_n x_n + (1 - \alpha_n) T_{\mu_n} x_n - z\|^2$$

$$= \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|T_{\mu_n}x_n - z\|^2 - \alpha_n(1-\alpha_n)\|T_{\mu_n}x_n - x_n\|^2$$
$$\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|T_{\mu_n}x_n - x_n\|^2$$
$$= \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|T_{\mu_n}x_n - x_n\|^2.$$

Thus, we have

$$\alpha_n(1-\alpha_n)\|T_{\mu_n}x_n - x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty}\|x_n - z\|^2$ exists and $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$, we have that

$$\|T_{\mu_n}x_n - x_n\| \to 0. \tag{3.4}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. We have from (3.4) that

$$T_{\mu_{n_i}}x_{n_i} \rightharpoonup v. \tag{3.5}$$

We have from (2.3) that for $y \in C$ and $s, t \in S$,

$$2\langle T_s x_n - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 = \|T_s x_n - T_t y\|^2 - \|T_s x_n - y\|^2.$$

Applying μ_n to both sides of the inequality, we have that

$$2(\mu_n)_s\langle T_s x_n - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 = (\mu_n)_s(\|T_s x_n - T_t y\|^2 - \|T_s x_n - y\|^2)$$

and hence

$$2\langle T_{\mu_n}x_n - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 = (\mu_n)_s(\|T_s x_n - T_t y\|^2 - \|T_s x_n - y\|^2)$$

Since $T_{\mu_{n_i}}x_{n_i} \rightharpoonup v$ and $\limsup_{i\to\infty}(\mu_{n_i})_s(\|T_s x_{n_i} - T_t y\|^2 - \|T_s x_{n_i} - y\|^2) \leq 0$, we get that

$$2\langle v - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 \leq 0.$$

Since $2\langle v - T_t y, y - T_t y\rangle - \|T_t y - y\|^2 = \|v - T_t y\|^2 - \|v - y\|^2$, we have that

$$\|v - T_t y\|^2 \leq \|v - y\|^2, \quad y \in C,\ t \in S. \tag{3.6}$$

Putting $y = v$, we have $v \in F(T_t)$. Therefore $v \in F(\mathcal{S})$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. To complete the proof, we show $v_1 = v_2$. We know that $v_1, v_2 \in F(\mathcal{S})$ and hence $\lim_{n\to\infty}\|x_n - v_1\|^2$ and $\lim_{n\to\infty}\|x_n - v_2\|^2$ exist. Put

$$a = \lim_{n\to\infty}(\|x_n - v_1\|^2 - \|x_n - v_2\|^2).$$

Note that for $n \in \mathbb{N}$,

$$\|x_n - v_1\|^2 - \|x_n - v_2\|^2 = 2\langle x_n, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2.$$

From $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$, we have

$$a = 2\langle v_1, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2; \tag{3.7}$$

$$a = 2\langle v_2, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain $0 = 2\langle v_2 - v_1, v_2 - v_1\rangle$. Thus we get $v_2 = v_1$. This implies that $\{x_n\}$ converges weakly to an element $v \in F(\mathcal{S})$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $z \in F(\mathcal{S})$ and $n \in \mathbb{N}$, we obtain from Lemma 2.2 that $\{P_{F(\mathcal{S})}x_n\}$

converges strongly to an element $p \in F(\mathcal{S})$. On the other hand, we have from the property of $P_{F(\mathcal{S})}$ that

$$\langle x_n - P_{F(\mathcal{S})}x_n, P_{F(\mathcal{S})}x_n - u\rangle \geq 0$$

for all $u \in F(\mathcal{S})$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $P_{F(\mathcal{S})}x_n \to p$, we obtain

$$\langle v - p, p - u\rangle \geq 0$$

for all $u \in F(\mathcal{S})$. Putting $u = v$, we obtain $p = v$. This means $v = \lim_{n\to\infty} P_{F(\mathcal{S})}x_n$. This completes the proof. □

Using Theorem 3.1, we obtain the following weak convergence theorem for generalized hybrid mappings in a Hilbert space.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping of C into itself such that $F(T)$ is nonempty. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $B(\mathbb{N})$. Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n, \quad \forall n \in \mathbb{N},$$

where $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then $\{x_n\}$ converges weakly to $z \in F(T)$ and $z = \lim_{n\to\infty} P_{F(T)}x_n$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

Proof. Consider $S = \{0\} \cup \mathbb{N}$ and $\mathcal{S} = \{T^k : k \in \{0\} \cup \mathbb{N}\}$ in Theorem 3.1. Since $T : C \to C$ be a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{3.9}$$

for all $x, y \in C$. Putting $x = u$ in (3.9), where $u \in F(T)$, we have that

$$\|u - Ty\|^2 \leq \|u - y\|^2$$

for all $y \in C$. This implies that T is quasi-nonexpansive. From Takahashi, Wong and Yao [27], we have that $A(T) \cap C = F(T)$ and hence $A(T) \neq \emptyset$. We also have that for all $q \in F(T)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\|T_{\mu_n}x_n - q\|^2 &= \langle T_{\mu_n}x_n - q, T_{\mu_n}x_n - q\rangle \\
&= (\mu_n)_k \langle T^k x_n - q, T_{\mu_n}x_n - q\rangle \\
&\leq \|\mu_n\| \sup_k |\langle T^k x_n - q, T_{\mu_n}x_n - q\rangle| \\
&\leq \sup_k \|T^k x_n - q\| \cdot \|T_{\mu_n}x_n - q\| \\
&\leq \sup_k \|x_n - q\| \cdot \|T_{\mu_n}x_n - q\| \\
&= \|x_n - v\| \cdot \|T_{\mu_n}x_n - q\|
\end{aligned} \tag{3.10}$$

and hence $\|T_{\mu_n}x_n - q\| \leq \|x_n - q\|$. Then, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n x_n + (1-\alpha_n)T_{\mu_n}x_n - q\| \\
&\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|T_{\mu_n}x_n - q\| \\
&\leq \alpha_n\|x_n - q\| + (1-\alpha_n)\|x_n - q\|
\end{aligned}$$

$$= \|x_n - q\|.$$

Putting $M = \{y \in C : \|y - q\| \le \|x - q\|\}$, we have that $x \in M$, $TM \subset M$ and M is bounded, closed and convex. Without loss of generality, we may assume that C is bounded. Since T is generalized hybrid, we have that for all $x, y \in C$ and $k \in \mathbb{N}$,

$$\begin{aligned}
0 &\le \beta\|T^{k+1}x - y\|^2 + (1-\beta)\|T^k x - y\|^2 \\
&\quad - \alpha\|T^{k+1}x - Ty\|^2 - (1-\alpha)\|T^k x - Ty\|^2 \\
&= \beta\big\{\|T^{k+1}x - Ty\|^2 + 2\left\langle T^{k+1}x - Ty, Ty - y\right\rangle + \|Ty - y\|^2\big\} \\
&\quad + (1-\beta)\big\{\|T^k x - Ty\|^2 + 2\left\langle T^k x - Ty, Ty - y\right\rangle + \|Ty - y\|^2\big\} \\
&\quad - \alpha\|T^{k+1}x - Ty\|^2 - (1-\alpha)\|T^k x - Ty\|^2 \\
&= \|Ty - y\|^2 + 2\left\langle \beta T^{k+1}x + (1-\beta)T^k x - Ty, Ty - y\right\rangle \\
&\quad + (\beta - \alpha)\big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\big\} \\
&= \|Ty - y\|^2 + 2\left\langle T^k x - Ty + \beta(T^{k+1}x_n - T^k x), Ty - y\right\rangle \\
&\quad + (\beta - \alpha)\big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\big\}
\end{aligned}$$

and hence

$$\begin{aligned}
&2\langle T^k x - Ty, y - Ty\rangle - \|Ty - y\|^2 \\
&\quad \le 2\beta\langle T^{k+1}x_n - T^k x, Ty - y\rangle + (\beta - \alpha)\big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\big\}.
\end{aligned}$$

On the other hand, we have from (2.3) that

$$2\left\langle T^k x - Ty, y - Ty\right\rangle - \|Ty - y\|^2 = \|T^k x - Ty\|^2 - \|T^k x - y\|^2.$$

So, we have that

$$\begin{aligned}
\|T^k x - Ty\|^2 - \|T^k x - y\|^2 &\le 2\beta\left\langle T^{k+1}x_n - T^k x, Ty - y\right\rangle \\
&\quad + (\beta - \alpha)\big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\big\}.
\end{aligned}$$

If $\{\mu_\alpha\}$ is a strongly asymptotically invariant net of means on ℓ^∞, then we have that

$$\begin{aligned}
(\mu_\alpha)_k(\|T^k x - Ty\|^2 - \|T^k x - y\|^2) &\le 2\beta(\mu_\alpha)_k\left\langle T^{k+1}x_n - T^k x, Ty - y\right\rangle \\
&\quad + (\beta - \alpha)(\mu_\alpha)_k(\big\{\|T^{k+1}x - Ty\|^2 - \|T^k x - Ty\|^2\big\}) \\
&\le \|\ell_1^*\mu_\alpha - \mu_\alpha\| \sup_{k\in\mathbb{N}} |\left\langle T^k x_n, Ty - y\right\rangle| \\
&\quad + |\beta - \alpha|\|\ell_1^*\mu_\alpha - \mu_\alpha\| \sup_{k\in\mathbb{N}} \|T^k x - Ty\|^2
\end{aligned}$$

and hence

$$\limsup_\alpha \sup_{x,y\in C} (\mu_\alpha)_k(\|T^k x - Ty\|^2 - \|T^k x - y\|^2) \le 0.$$

So, we have the desired result from Theorem 3.1. □

Using Theorem 3.1, we obtain the following weak convergence theorem for semigroups of nonexpansive mappings in a Hilbert space; see also [2].

Theorem 3.3. *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H. Let S be a commutative semitopological semigroup with identity and let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e., a sequence of means on $C(S)$ such that*

$$\lim_{n\to\infty} \|\mu_n - \ell_s^* \mu_n\| = 0, \quad \forall s \in S.$$

Define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges weakly to a point $z \in F(\mathcal{S})$ and $z = \lim_{n\to\infty} P_{F(\mathcal{S})} x_n$, where $P_{F(\mathcal{S})}$ is the metric projection of H onto $F(\mathcal{S})$.

Proof. Since $\mathcal{S} = \{T_t : t \in S\}$ is a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded for some $x \in C$, we have that $F(\mathcal{S})$ is nonempty. Let $u \in F(\mathcal{S})$ and put $M = \{y \in C : \|y - u\| \le \|x - u\|\}$. Then, we have $x \in M$, $T_t M \subset M$ for all $t \in S$, and M is bounded, closed and convex. Without loss of generality, we may assume that C is bounded. Since $\mathcal{S} = \{T_t : t \in S\}$ is a nonexpansive semigroup on C, we have that for all $x, y \in C$ and $s, t \in S$,

$$\begin{aligned} \|T_s x - T_t y\|^2 - \|T_s x - y\|^2 &= \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2 \\ &\quad + \|T_{s+t} x - T_t y\|^2 - \|T_s x - y\|^2 \\ &\le \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2 \\ &\quad + \|T_s x - y\|^2 - \|T_s x - y\|^2 \\ &= \|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2. \end{aligned}$$

If $\{\mu_\alpha\}$ is a strongly asymptotically invariant net of means on $C(S)$, then we have that

$$\begin{aligned} (\mu_\alpha)_s(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) &\le (\mu_\alpha)_s(\|T_s x - T_t y\|^2 - \|T_{s+t} x - T_t y\|^2) \\ &= (\mu_\alpha)_s \|T_s x - T_t y\|^2 - (\ell_t^* \mu_\alpha)_s \|T_s x - T_t y\|^2) \\ &\le \|\mu_\alpha - \ell_t^* \mu_\alpha\| \sup_s \|T_s x - T_t y\|^2 \end{aligned}$$

and hence

$$\limsup_\alpha \sup_{x,y\in C} (\mu_\alpha)_s(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) \le 0$$

for all $t \in S$. So, we have the desired result from Theorem 3.1. $\square$

4. Strong convergence theorems

In this section, we prove a strong convergence theorem of Halpern's type iteration for semigroups of mappings without continuity in a Hilbert space.

Theorem 4.1. *Let H be a Hilbert space and let C be a nonempty, bounded, closed and convex subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mappings of C into itself. Suppose that*

$$\limsup_{\alpha} \sup_{x,y \in C} (\mu_\alpha)_s(\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) \le 0, \quad \forall t \in S \tag{4.1}$$

for all strongly asymptotically invariant nets $\{\mu_\alpha\}$ of means on $C(S)$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e.,

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Let $u \in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $z \in F(\mathcal{S})$, where $z = P_{F(\mathcal{S})} u$.

Proof. As in the proof of Theorem 3.1, we have that $A(\mathcal{S})$ is nonempty. Since $\{T_s x : s \in S\}$ is bounded for all $x \in C$, we have from the proof of Theorem 3.1 that for any $v \in A(\mathcal{S})$ and $n \in \mathbb{N}$,

$$\|T_{\mu_n} x_n - v\| \le \|x_n - v\|. \tag{4.2}$$

So, we have from (4.2) that

$$\begin{aligned} \|x_{n+1} - v\| &= \|\alpha_n u + (1 - \alpha_n) T_{\mu_n} x_n - v\| \\ &\le \alpha_n \|u - v\| + (1 - \alpha_n) \|T_{\mu_n} x_n - v\| \\ &\le \alpha_n \|u - v\| + (1 - \alpha_n) \|x_n - v\|. \end{aligned}$$

Hence, by induction, we obtain

$$\|x_n - v\| \le \max\{\|u - v\|, \|x - v\|\}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded. We also have from (4.2) that $\{T_{\mu_n} x_n\}$ is bounded. Set $z_n = T_{\mu_n} x_n$ for all $n \in \mathbb{N}$ and let $\{z_{n_i}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_i} \rightharpoonup v$ for some $v \in C$. As in the proof of Theorem 3.1, we have that $v \in F(\mathcal{S})$. On the other hand, since $x_{n+1} - z_n = \alpha_n(u - z_n)$, $\{z_n\}$ is bounded and $\alpha_n \to 0$, we have

$$\lim_{n \to \infty} \|x_{n+1} - z_n\| = 0.$$

We show $\limsup_{n \to \infty} \langle u - Pu, x_{n+1} - Pu \rangle \le 0$. We may assume without loss of generality that there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n \to \infty} \langle u - Pu, x_{n+1} - Pu \rangle = \lim_{i \to \infty} \langle u - Pu, x_{n_i+1} - Pu \rangle$$

and $x_{n_i+1} \rightharpoonup v$, where P is the metric projection of H onto $F(\mathcal{S})$. Since $\|x_{n+1} - z_n\| \to 0$, we have $z_{n_i} \rightharpoonup v$. As the above, we have that $v \in F(\mathcal{S})$. Then, we get

$$\lim_{i \to \infty} \langle u - Pu, x_{n_i+1} - Pu \rangle = \langle u - Pu, v - Pu \rangle \le 0.$$

This implies

$$\limsup_{n \to \infty} \langle u - Pu, x_{n+1} - Pu \rangle \le 0. \tag{4.3}$$

Since $x_{n+1} - Pu = (1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)$, from (2.1) and (4.2) we have

$$\begin{aligned}\|x_{n+1} - Pu\|^2 &= \|(1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)\|^2 \\ &\le (1-\alpha_n)^2\|z_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle \\ &\le (1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle.\end{aligned}$$

Putting $s_n = \|x_n - Pu\|^2$, $\beta_n = 0$ and $\gamma_n = 2\langle u - Pu, x_{n+1} - Pu\rangle$ in Lemma 2.1, we have from $\sum_{n=1}^{\infty}\alpha_n = \infty$ and (4.3) that

$$\lim_{n\to\infty}\|x_n - Pu\| = 0.$$

This completes the proof. □

Using Theorem 4.1, we can prove the following strong convergence theorem for generalized hybrid mappings in a Hilbert space.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping of C into itself such that $F(T)$ is nonempty. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $B(\mathbb{N})$. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n)z_n, \\ z_n = T_{\mu_n}x_n \end{cases}$$

for all $n \in \mathbb{N}$, where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu, where P is the metric projection of H onto $F(T)$.

Proof. Consider $S = \{0\} \cup \mathbb{N}$ and $\mathcal{S} = \{T^k : k \in \{0\} \cup \mathbb{N}\}$ in Theorem 4.1. Since $T : C \to C$ be a generalized hybrid mapping and $F(T)$ is nonempty, T is quasi-nonexpansive. From Takahashi, Wong and Yao [27], we have that $A(T) \cap C = F(T)$ and hence $A(T) \ne \emptyset$. As in the proof of Theorem 3.2, we have that for all $q \in F(T)$ and $n \in \mathbb{N}$,

$$\|z_n - q\| \le \|x_n - q\|.$$

Then, we have

$$\begin{aligned}\|x_{n+1} - q\| &= \|\alpha_n u + (1-\alpha_n)z_n - q\| \\ &\le \alpha_n\|u - q\| + (1-\alpha_n)\|z_n - q\| \\ &\le \alpha_n\|u - q\| + (1-\alpha_n)\|x_n - q\|.\end{aligned}$$

Hence, by induction, we obtain

$$\|x_n - q\| \le \max\{\|u - q\|, \|x - q\|\}$$

for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ and $\{z_n\}$ are bounded. Without loss of generality, we may assume that C is bounded. Since T is generalized hybrid, we have from the proof of Theorem 3.2 that

$$\limsup_{\alpha}\sup_{x,y\in C}(\mu_\alpha)_k(\|T^k x - Ty\|^2 - \|T^k x - y\|^2) \le 0.$$

So, we get the desired result from Theorem 4.1. □

In particular, we obtain Hojo and Takahashi strong convergence theorem [11] from Theorem 4.2.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, ...,$ where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu, where P is the metric projection of H onto $F(T)$.

Proof. Let $S = \{0\} \cap \mathbb{N}$ in Theorem 3.2 and define

$$\mu_n(f) = \frac{1}{n} \sum_{i=0}^{n-1} f(i)$$

for all $n \in \mathbb{N}$ and $f \in B(S)$. As in the proof of [10, Theorem 5], we have that $\{\mu_n : n \in \mathbb{N}\}$ is a strongly asymptotically invariant sequence of means on $B(S)$. Furthermore, we have from [22, p. 78] that for any $x \in C$ and $n \in \mathbb{N}$,

$$T_{\mu_n} x = \frac{1}{n} \sum_{i=0}^{n-1} T^i x.$$

Therefore, we obtain Theorem 4.3 by using Theorem 4.2. □

Using Theorem 4.1, we also have a strong convergence theorem for semigroups of nonexpansive mappings in a Hilbert space; see also [20].

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty closed and convex subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $\{\mu_n\}$ be a strongly asymptotically invariant sequence of means on $C(S)$, i.e.,*

$$\|\mu_n - \ell_s^* \mu_n\| \to 0, \quad \forall s \in S.$$

Let $u \in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n u + (1-\alpha_n) T_{\mu_n} x_n, \quad \forall n \in \mathbb{N},$$

where $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $z \in F(\mathcal{S})$, where $z = P_{F(\mathcal{S})} u$.

Proof. Let $x \in C$ and $z \in F(\mathcal{S})$. Put $r = \max\{\|u - z\|, \|x - z\|\}$ and set

$$M = \{y \in C : \|y - z\| \le r\}.$$

Then M is a bounded closed convex subset of C which is T_t-invariant and contains u and x. Without loss of generality, we may assume that C is bounded. As in the proof of Theorem 3.3, we have that

$$\limsup_{\alpha} \sup_{x,y \in C} (\mu_\alpha)_s (\|T_s x - T_t y\|^2 - \|T_s x - y\|^2) \le 0$$

for all $t \in S$. So, we have the desired result from Theorem 4.1. □

In particular, we have the following strong convergence theorem from Theorem 4.4.

Theorem 4.5 (Shimizu and Takahashi [19]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\mathcal{S} = \{S(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. Let $u \in C$ and define a sequence $\{x_n\}$ in C as follows: $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} S(t)x_n dt, \quad \forall n \in \mathbb{N},$$

where $0 < \lambda_n < \infty$, $\lambda_n \to \infty$, $0 \le \alpha_n \le 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $z \in F(\mathcal{S})$, where $z = P_{F(\mathcal{S})}u$.

Proof. Let $S = \mathbb{R}^+$. For any $f \in C(\mathbb{R}^+)$, define

$$\mu_n(f) = \frac{1}{\lambda_n}\int_0^{\lambda_n} f(t)dt, \quad \forall \lambda_n \in (0,\infty).$$

Then $\{\mu_n\}$ is a strongly asymptotically invariant sequence of means on $C(\mathbb{R}^+)$; see [10, Theorem 6]. Furthermore, we have from [22, Theorem 3.5.2] that for any $x \in C$ and $\lambda_n \in (0,\infty)$,

$$T_{\mu_n}x = \frac{1}{\lambda_n}\int_0^{\lambda_n} S(t)xdt.$$

Therefore, we have the desired result from Theorem 4.4. □

References

[1] K. Aoyama, Y. Kimura, W.Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[2] S. Atsushiba and W. Takahashi, *Approximating common fixed points of nonexpansive semigroups by the Mann iteration process*, Ann. Univ. Mariae Curie-Skodowska Sect. A **51** (1997), 1–16.

[3] S. Atsushiba and W. Takahashi, *Nonlinear ergodic theorems without convexity for nonexpansive semigroups in Hilbert spaces*, J. Nonlinear Convex Anal. **14** (2013), 209–219.

[4] J.-B. Baillon, *Un theoreme de type ergodique pour les contractions non lineaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Ser. A-B **280** (1975), 1511–1514.

[5] J.-B. Baillon and H. Brezis, *Une remarque sur le comportement asymptotique des semigroupes non lineares*, Houston J. Math. **4** (1978), 1–9.

[6] F. E. Browder, *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces*, Proc. Sympos. Pure Math. 100-2, Amer. Math. Soc. Providence, R.I., 1976.

[7] M. M. Day, *Amenable semigroup*, Illinois J. Math. **1** (1957), 509–544.

[8] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[9] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[10] N. Hirano, K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal. **12** (1988), 1269–1281.

[11] M. Hojo and W. Takahashi, *Weak and strong convergence theorems for generalized hybrid mappings in Hilbert spaces*, Sci. Math. Jpn. **73** (2011), 31–40.

[12] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.) Yokohama Publishers, Yokohama, 2008, pp. 75–85.

[13] S. Itoh and W. Takahashi, *The common fixed points theory of single-valued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.

[14] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for genelalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[15] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.

[16] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.

[17] T. Maruyama, W. Takahashi and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 185–197.

[18] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506-510.

[19] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), 71–83.

[20] N. Shioji and W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive semigroups in Hilbert spaces*, Nonlinear Anal. **34** (1998), 87–99.

[21] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.

[22] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.

[23] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[24] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[25] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.

[26] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[27] W. Takahashi, N.-C. Wong and J.-C. Yao, *Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **13** (2012), 745–757.

[28] W. Takahashi, N.-C. Wong and J.-C. Yao, *Attractive point and mean convergence theorems for semigroups of mappings without continuity in Hilbert spaces*, J. Nonlinear Convex Anal., to appear.

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