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EXTENDED FARKAS'S LEMMAS AND STRONG LAGRANGE DUALITIES FOR DC INFINITE PROGRAMMING

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Abstract. We consider the DC (difference of two convex functions) optimization problem

$$(P)\qquad \begin{array}{ll} \text{Minimize} & f(x)-g(x), \\ \text{s. t.} & f_t(x)-g_t(x)\le 0,\ t\in T, \\ & x\in C, \end{array}$$

where T is an arbitrary (possibly infinite) index set, C is a nonempty convex subset of a locally convex (Hausdorff topological vector) space X and $f, g, f_t, g_t : X \to \mathbb{R}\cup\{+\infty\}$, $t \in T$, are proper convex functions. By using properties of the epigraph of conjugate functions, we introduce some new constraint qualifications and obtain some complete characterizations for the (stable) Farkas lemmas and for the (weak/strong/stable) Lagrange dualities.

1. Introduction

Let X be a real locally convex Hausdorff topological vector space, whose dual space X^* is endowed with the weak*-topologies $W^*(X^*, X)$. Let T be an arbitrary (possibly infinite) index set, C be a nonempty convex subset of X and let $h, h_t : X \to \overline{\mathbb{R}} := \mathbb{R}\cup\{+\infty\}$, $t \in T$, be proper convex functions. Consider the following optimization problem (cf. [5, 8, 9, 17, 18, 14, 15, 24, 27, 28, 29, 30, 31, 32])

$$(\mathcal{P})\qquad \begin{array}{ll} \text{Minimize} & h(x), \\ \text{s. t.} & h_t(x)\le 0,\ t\in T, \\ & x\in C \end{array}$$

and its Lagrange dual problem

$$(\mathcal{D})\qquad \max_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in C}\{h(x)+\sum_{t\in T}\lambda_t h_t(x)\},$$

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where

$$\mathbb{R}_+^{(T)} := \{(\lambda_t) \in \mathbb{R}^T : \lambda_t \geq 0 \text{ for each } t \in T \text{ and only finitely many } \lambda_t \neq 0\}.$$

The optimal values of problems $(\mathcal{P})$ and $(\mathcal{D})$ are denoted by $v(\mathcal{P})$ and $v(\mathcal{D})$ respectively.

It is well-known that the so-called weak Lagrange duality holds between problems $(\mathcal{P})$ and $(\mathcal{D})$, i.e., $v(\mathcal{P}) \geq v(\mathcal{D})$, but a duality gap may occur, i.e., we may have $v(\mathcal{P}) > v(\mathcal{D})$. A challenge in convex analysis is to find sufficient conditions which guarantee the strong Lagrange duality, i.e. the situation when $v(\mathcal{P}) = v(\mathcal{D})$ and the dual problem $(\mathcal{D})$ has at least an optimal solution. In the case when the involved functions are proper convex functions, several interiority-type conditions were given in order to preclude the existence of such a duality gap in different settings (see, for instance, [36, Theorem 2.9.3]). Taking inspiration from the works due to Burachik and Jeyakumar [6, 7], some authors approached the strong Lagrange duality by using some epigraph properties for conjugate functions of the involved functions h^* and $h_t^*, t \in T$; see, for instance, [8, 9, 14].

Recent interests are focused on the DC (difference of two convex functions) optimization problem, that is, the involved functions h and/or h_t in problem $(\mathcal{P})$ are DC functions. As pointed in [10], problems of DC programming are highly important from both viewpoints of optimization theory and applications, and they have been extensively studied in the literature, see for example [1, 4, 10, 11, 12, 13, 16, 19, 33, 34, 35] and the references therein.

Inspired by the works mentioned above, we continue to study the optimization problem $(\mathcal{P})$ but with $h := f - g$ and $h_t := f_t - g_t, t \in T$, being DC functions, that is, the DC problem defined by

$$(1.1) \qquad (P) \quad \begin{array}{ll} \text{Minimize} & f(x) - g(x), \\ \text{s. t.} & f_t(x) - g_t(x) \leq 0, \ t \in T, \\ & x \in C, \end{array}$$

where $f, g, f_t, g_t : X \to \overline{\mathbb{R}}, \ t \in T$, are proper convex functions. Throughout this paper, we assume that

$$(1.2) \qquad \emptyset \neq A := \{x \in C : f_t(x) - g_t(x) \leq 0 \quad \text{for each} \quad t \in T\}.$$

Here and throughout the whole paper, following [36, page 39], we adapt the convention that $(+\infty) + (-\infty) = (+\infty) - (+\infty) = +\infty$, $0 \cdot (+\infty) = +\infty$ and $0 \cdot (-\infty) = 0$. Then, for any two proper convex functions $h_1, h_2 : X \to \overline{\mathbb{R}}$, we have that

$$(1.3) \qquad h_1(x) - h_2(x) := \begin{cases} h_1(x) - h_2(x) \in \mathbb{R}, & x \in \operatorname{dom} h_1 \cap \operatorname{dom} h_2, \\ -\infty, & x \in \operatorname{dom} h_1 \setminus \operatorname{dom} h_2, \\ +\infty, & x \notin \operatorname{dom} h_1; \end{cases}$$

hence,

$$(1.4) \qquad h_1 - h_2 \text{ is proper} \iff \operatorname{dom} h_1 \subseteq \operatorname{dom} h_2.$$

In the case when $g, g_t, t \in T$, are lower semicontinuous (lsc in brief), one can use the equalities $g = g^{**}$ and $g_t = g_t^{**}$, $t \in T$, to deduce that, for each $\lambda \in \mathbb{R}_+^{(T)}$,

(1.5)
$$\begin{aligned} &\inf_{x\in C}\{f(x) - g(x) + \textstyle\sum_{t\in T}\lambda_t(f_t(x) - g_t(x))\} \\ = \ &\inf_{(u^*,v^*)\in H^*}\{g^*(u^*) + \textstyle\sum_{t\in T}\lambda_t g_t^*(v_t^*) - (f + \delta_C + \sum_{t\in T}\lambda_t f_t)^*(u^* + \sum_{t\in T}\lambda_t v_t^*)\},\end{aligned}$$

where $H^* := \operatorname{dom} g^* \times \prod_{t\in T} \operatorname{dom} g_t^*$ and φ^{**} denotes the second conjugate of a convex function φ on X. Thus the dual problem $(\mathcal{D})$ with $h := f_1 - f_2$ and $h_t := f_t - g_t$, $t \in T$, is reduced to the following one:

$$(1.6) \qquad (D) \qquad \sup_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{w^*\in H^*} L(w^*, \lambda),$$

where the Lagrange function $L : H^* \times \mathbb{R}_+^{(T)} \to \overline{\mathbb{R}}$ for (1.1) is defined by

$$(1.7) \qquad L(w^*, \lambda) := g^*(u^*) + \sum_{t\in T}\lambda_t g_t^*(v_t^*) - \big(f + \delta_C + \sum_{t\in T}\lambda_t f_t\big)^*\big(u^* + \sum_{t\in T}\lambda_t v_t^*\big)$$

for any $(w^*, \lambda) \in H^* \times \mathbb{R}_+^{(T)}$ with $w^* = (u^*, (v_t^*)) \in H^*$ and $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$. Another interesting and extensively studied issue related to problem $(\mathcal{P})$ is to find sufficient conditions ensuring the Farkas rule (cf. [14]), that is, for each $\alpha \in \mathbb{R}$,

$$(1.8) \quad [h(x) \geq \alpha,\ \forall x \in C, h_t(x) \leq 0, t \in T] \Leftrightarrow \\ [\exists\lambda \in \mathbb{R}_+^{(T)} \text{ s.t. } h(x) + \sum_{t\in T}\lambda_t h_t(x) \geq \alpha,\ \forall x \in C].$$

Specializing in the case when $h := f_1 - f_2$ and each $h_t := f_t - g_t$ with g, g_t, $t \in T$, being lsc, we get from equality (1.5) that (1.8) is equivalent to the following one:

$$(1.9) \ [f(x) - g(x) \geq \alpha, \forall x \in A] \Longleftrightarrow [(\exists\lambda \in \mathbb{R}_+^{(T)})(\forall w^* \in H^*) \quad \text{s.t.} \quad L(w^*, \lambda) \geq \alpha].$$

However, without assuming the lower semicontinuity of g and g_t, equality (1.5) does not necessarily hold. Thus, $(\mathcal{D})$ and (D), (1.8) and (1.9) are, in general, not equivalent; see Example 5.2 in Section 5.

Our main aim in the present paper is focused on two aspects: One is about the strong Lagrange duality, that is, one seeks conditions to characterize the strong Lagrange dualities between (P) and (D); and the other is about the Farkas lemma, that is, we look for conditions to ensure (1.9) holds. To the best of our knowledge, not many results are known to provide characterizations for the strong Lagrange duality or for the Farkas lemma for DC optimization problems, except the works in [4, 33] where, some sufficient conditions in terms of the interiority are provided for DC optimization problems with finite constraints but different formulations of the dual problem via the standard convexification technique; while, in [12, 13], the epigraph closure conditions are used to establish the Fenchel-Lagrange duality and the extended Farkas lemma for the conical optimization problem with DC objective function, but also for the dual problem defined via the convexification technique.

Unlike the convex case, the weak Lagrange duality between (P) and (D) does not necessarily hold, in general, as showed in Example 5.2. In the present paper we will use the epigraph technique, the powerful tool in convex programming (see [6, 7, 17, 20, 21, 4, 5, 11, 10, 12, 13, 14, 26, 30, 31]), to provide complete characterizations for the (weak/strong/strong stable) Lagrangian dualities between (P) and (D) and for the Farkas rules and the stable Farkas rules. In general, we do not impose any topological assumption on C or on f, g, f_t and g_t, that is, C is not necessarily closed and $f, g, f_t, g_t, t \in T$, are not necessarily lsc. Most of results obtained in the present paper seem new and are proper extensions of the results in [14] in the special case when $f_2 = g_2 = 0$. In [14], the authors established the strong Lagrange duality between $(\mathcal{P})$ and $(\mathcal{D})$ and the Farkas lemmas. As we noted earlier, in general, we do not have the equivalences between $(\mathcal{D})$ and (D), and between (1.8) and (1.9) in the case when g and g_t are not lsc. In particular, our dual problem and regularity conditions introduced here are defined in terms of conjugates of the convex functions f, g, f_t and g_t rather than of the DC functions $f - g$ and $f_t - g_t$, which are different from the consideration in [14] for the general (not necessarily convex) case.

The paper is organized as follows. The next section contains some necessary notations and preliminary results. In Section 3, some new constraint qualifications are introduced and studied. Complete characterizations for the (stable) Farkas lemmas and for the (weak/strong/stable) Lagrange dualities are obtained in Sections 4 and 5, respectively. Applications to conical programming problem are given in Section 6.

2. Notations and preliminaries

The notations used in this paper are standard (cf. [36]). In particular, we assume throughout the whole paper that X is a real locally convex space and let X^* denote the dual space of X. For $x \in X$ and $x^* \in X^*$, we write $\langle x^*, x\rangle$ for the value of x^* at x, that is, $\langle x^*, x\rangle := x^*(x)$. Let Z be a set in X. The closure of Z is denoted by $\operatorname{cl} Z$. If $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak*-closure of W. For the whole paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w^*(X^*, X)$ and the usual Euclidean topology.

The indicator function δ_Z and the support function σ_Z of the nonempty set Z are respectively defined by

$$\delta_Z(x) := \begin{cases} 0, & x \in Z, \\ +\infty, & \text{otherwise}, \end{cases}$$

and

$$\sigma_Z(x^*) := \sup_{x \in Z} \langle x^*, x\rangle \quad \text{for each } x^* \in X^*.$$

Let f be a proper function defined on X. The effective domain, the conjugate function and the epigraph of f are denoted by $\operatorname{dom} f$, f^* and $\operatorname{epi} f$ respectively; they are defined by

$$\operatorname{dom} f := \{x \in X : \ f(x) < +\infty\},$$

$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : \ x \in X\} \quad \text{for each } x^* \in X^*,$$

and

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : \ f(x) \le r\}.$$

It is well known and easy to verify that $\operatorname{epi} f^*$ is weak*-closed. The closure of f is denoted by $\operatorname{cl} f$, which is defined by

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f).$$

Then (cf. [36, Theorems 2.3.1]),

$$f^* = (\mathrm{cl}\, f)^*. \tag{2.1}$$

By [36, Theorem 2.3.4], if $\mathrm{cl}\, f$ is proper and convex, then the following equality holds:

$$f^{**} = \mathrm{cl}\, f. \tag{2.2}$$

By definition, the Young-Fenchel inequality below holds:

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle \text{ for each pair } (x, x^*) \in X \times X^*. \tag{2.3}$$

Furthermore, if g, h are proper functions, then

$$\mathrm{epi}\, g^* + \mathrm{epi}\, h^* \subseteq \mathrm{epi}\, (g + h)^*, \tag{2.4}$$

and

$$g \leq h \Rightarrow g^* \geq h^* \Leftrightarrow \mathrm{epi}\, g^* \subseteq \mathrm{epi}\, h^*. \tag{2.5}$$

Moreover, if g is convex and lsc on $\mathrm{dom}\, h$, then the same argument for the proof of [16, Lemma 3.1] shows that

$$\mathrm{epi}(h - g)^* = \bigcap_{u^* \in \mathrm{dom} g^*} (\mathrm{epi} h^* - (u^*, g^*(u^*))). \tag{2.6}$$

The following lemma is a direct consequence of the definitions of a conjugate function and an epigraph. In particular, statements (i) and (ii) were used in [36, Theorem 2.13(i)] and [30, equation (2.5)], respectively.

Lemma 2.1. *Let I be an index set and let $\{f_i : i \in I\}$ be a family of proper convex functions. Then the following statements hold.*

(i) $\mathrm{epi}\, (\sup_{i \in I} f_i) = \cap_{i \in I} \mathrm{epi}\, f_i$.

(ii) $(\inf_{i \in I} f_i)^* = \sup_{i \in I} f_i^*$; *consequently,* $\mathrm{epi}\, (\inf_{i \in I} f_i)^* = \cap_{i \in I} \mathrm{epi}\, f_i^*$.

We end this section with a remark that an element $p \in X^*$ can be naturally regarded as a function on X in such a way that

$$p(x) := \langle p, x \rangle \quad \text{for each } x \in X. \tag{2.7}$$

Thus the following facts are clear for any $a \in \mathbb{R}$ and any function $h : X \to \overline{\mathbb{R}}$:

$$(h + p + a)^*(x^*) = h^*(x^* - p) - a \quad \text{for each} \quad x^* \in X^*, \tag{2.8}$$

$$\mathrm{epi}(h + p + a)^* = \mathrm{epi} h^* + (p, -a). \tag{2.9}$$

3. New constraint qualifications

Let X be a real locally convex Hausdorff vector space, and $C \subseteq X$ be a convex set. Let T be an index set and let $f, g, f_t, g_t, t \in T$, be proper convex functions such that $f - g$ and $f_t - g_t, t \in T$, are proper functions such that $A \cap \mathrm{dom}\, (f - g) \neq \emptyset$, where, as before, A is the solution set of the following system:

$$x \in C; \; f_t(x) - g_t(x) \leq 0 \text{ for each } t \in T. \tag{3.1}$$

Then, by (1.4), we have that

$$\emptyset \neq \mathrm{dom}\, f \subseteq \mathrm{dom}\, g \quad \text{and} \quad \emptyset \neq \mathrm{dom}\, f_t \subseteq \mathrm{dom}\, g_t. \tag{3.2}$$

Following [25], we use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda = (\lambda_t)$ with only finitely many $\lambda_t \neq 0$, and let $\mathbb{R}_+^{(T)}$ denote the nonnegative cone in $\mathbb{R}^{(T)}$, that is

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t) \in \mathbb{R}^{(T)} : \ \lambda_t \geq 0 \quad \text{for each} \quad t \in T\}.$$

For simplicity, we denote

$$H^* := \mathrm{dom} g^* \times \prod_{t\in T} \mathrm{dom} g_t^*.$$

To make the dual problem considered here well-defined, we further assume that $\mathrm{cl}\, g$ and $\mathrm{cl}\, g_t, t \in T$, are proper. Then $H^* \neq \emptyset$. For the whole paper, any elements $\lambda \in \mathbb{R}_+^{(T)}$ and $v^* \in \prod_{t\in T} \mathrm{dom} g_t^*$ are understood as $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$ and $v^* = (v_t^*) \in \prod_{t\in T} \mathrm{dom} g_t^*$, respectively. Let K denote the following characteristic set defined by

(3.3)

$$K := \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \Big(\bigcap_{(u^*,v^*)\in H^*} \big(\mathrm{epi}\, (f + \delta_C + \sum_{t\in T} \lambda_t f_t)^* - (u^*, g^*(u^*)) - \sum_{t\in T} \lambda_t (v_t^*, g_t^*(v_t^*))\big)\Big).$$

The functions $\bar{h}$ and $\bar{h}_t, t \in T$, which play a bridging role for our study, are defined respectively by

$$\bar{h} := f - \mathrm{cl}\, g \quad \text{and} \quad \bar{h}_t := f_t - \mathrm{cl}\, g_t \quad \text{for each} \quad t \in T. \tag{3.4}$$

Then

$$h := f - g \leq \bar{h} \quad \text{and} \quad h_t := f_t - g_t \leq \bar{h}_t \quad \text{for each} \quad t \in T. \tag{3.5}$$

Further, by (3.2), $\bar{h}$ and $\bar{h}_t, t \in T$, are proper.

Lemma 3.1. *The following equality holds:*

$$K = \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \mathrm{epi}\, (\bar{h} + \delta_C + \sum_{t\in T} \lambda_t \bar{h}_t)^*. \tag{3.6}$$

Proof. Let $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$. Since $\mathrm{cl}\, g$ and $\mathrm{cl}\, g_t, t \in T$, are proper lsc convex functions, it follows from (2.2) that

$$\mathrm{cl}\, g = g^{**} \quad \text{and} \quad \mathrm{cl}\, g_t = g_t^{**} \quad \text{for each} \quad t \in T. \tag{3.7}$$

Hence, by (3.4) and the definition of conjugate function, we have that

$$\begin{aligned} & \bar{h} + \delta_C + \textstyle\sum_{t\in T} \lambda_t \bar{h}_t \\ = \ & \inf_{(u^*,v^*)\in H^*} (f + \delta_C + \textstyle\sum_{t\in T} \lambda_t f_t - u^* - \sum_{t\in T} \lambda_t v_t^* + g^*(u^*) + \sum_{t\in T} \lambda_t g_t^*(v_t^*)). \end{aligned}$$

Moreover, by (2.9), one has that

$$\begin{aligned} & \mathrm{epi}(f + \delta_C + \textstyle\sum_{t\in T} \lambda_t f_t - u^* - \sum_{t\in T} \lambda_t v_t^* + g^*(u^*) + \sum_{t\in T} \lambda_t g_t^*(v_t^*))^* \\ = \ & \mathrm{epi}\, (f + \delta_C + \textstyle\sum_{t\in T} \lambda_t f_t)^* - (u^*, g^*(u^*)) - \sum_{t\in T} \lambda_t (v_t^*, g_t^*(v_t^*)). \end{aligned}$$

Thus applying Lemma 2.1, we conclude that

$$\begin{aligned} & \mathrm{epi}\, (\bar{h} + \delta_C + \textstyle\sum_{t\in T} \lambda_t \bar{h}_t)^* \\ = \ & \mathrm{epi}\big(\sup_{(u^*,v^*)\in H^*} \big(f + \delta_C + \textstyle\sum_{t\in T} \lambda_t f_t - \langle u^*, \cdot\rangle \\ & - \textstyle\sum_{t\in T} \lambda_t \langle v_t^*, \cdot\rangle + g^*(u^*) + \sum_{t\in T} \lambda_t g_t^*(v_t^*)\big)^*\big) \\ = \ & \textstyle\bigcap_{(u^*,v^*)\in H^*} \big(\mathrm{epi}\, (f + \delta_C + \sum_{t\in T} \lambda_t f_t)^* - (u^*, g^*(u^*)) - \sum_{t\in T} \lambda_t (v_t^*, g_t^*(v_t^*))\big), \end{aligned}$$

and (3.6) is established. □

Let $A^{\rm cl}$ denote the solution set of the system $\{x \in C; \bar{h}_t(x) \le 0, t \in T\}$, that is

$$A^{\rm cl} := \{x \in C : \bar{h}_t(x) \le 0 \quad \text{for each} \quad t \in T\}. \tag{3.8}$$

Then, $A^{\rm cl} \subseteq A$ (cf. (3.5)) and, by [14, equation (3.5)], one has that

$$K = \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \operatorname{epi}(\bar{h} + \delta_C + \sum_{t \in T} \lambda_t \bar{h}_t)^* \subseteq \operatorname{epi}(\bar{h} + \delta_{A^{\rm cl}})^*. \tag{3.9}$$

Thus, in the case when g and $g_t, t \in T$, are lsc, then $A^{\rm cl} = A$ and $\bar{h} = f - g$; hence the following inclusion holds:

$$K \subseteq \operatorname{epi}(f - g + \delta_A)^*. \tag{3.10}$$

The following example shows that (3.10) does not hold in general.

Example 3.2. Let $X = C := \mathbb{R}$. Define $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$ respectively by $f = f_1 := \delta_{(-\infty,0]}$, $g_1 := 0$ and for each $x \in \mathbb{R}$,

$$g(x) := \begin{cases} 0 & x < 0, \\ 1 & x = 0, \\ +\infty & x > 0. \end{cases}$$

Then f, g, f_1 and g_1 are proper convex functions. Consider the system (3.1) with $T := \{1\}$. Then one sees that

$$A = \{x \in \mathbb{R} : f_1(x) - g_1(x) \le 0\} = (-\infty, 0].$$

It is easy to see that for each $x \in \mathbb{R}$,

$$(f - g + \delta_A)(x) = \begin{cases} 0 & x < 0, \\ -1 & x = 0, \\ +\infty & x < 0, \end{cases}$$

and for each $x^* \in \mathbb{R}$,

$$(f - g + \delta_A)^*(x^*) = \begin{cases} 1 & x^* \ge 0, \\ +\infty & x^* < 0. \end{cases}$$

Hence,

$$\operatorname{epi}(f - g + \delta_A)^* = [0, +\infty) \times [1, +\infty).$$

Moreover, since $g_1^* = \delta_{\{0\}}$, $g^* = \delta_{[0,+\infty)}$ and $(f + \lambda f_1)^* = \delta_{[0,+\infty)}$ for each $\lambda \ge 0$, it follows that

$$K = \bigcup_{\lambda \ge 0} (\bigcap_{u^* \in [0,+\infty)} (\operatorname{epi}(f + \lambda f_1)^* - (u^*, g^*(u^*)))) = [0, +\infty) \times [0, +\infty).$$

Therefore, $K \nsubseteq \operatorname{epi}(f - g + \delta_A)^*$.

Moreover, it is well known that, even in the case when $g = 0$ and each $g_t = 0$, the converse inclusion of (3.10) is not true, in general. Considering the possible inclusions between $\operatorname{epi}(f - g + \delta_A)^*$ and K, we introduce the following definition.

Definition 3.3. Let $p \in X^*$. The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ is said to satisfy

(a) the conical weak epigraph hull property at p (denote by p-$(WEHP)$) if

$$\operatorname{epi}(f - g + \delta_A)^* \cap (\{p\} \times \mathbb{R}) = K \cap (\{p\} \times \mathbb{R}); \tag{3.11}$$

(b) the conical semi weak epigraph hull property at p (denote by p-$(SWEHP)$) if

$$\operatorname{epi}(f - g + \delta_A)^* \cap (\{p\} \times \mathbb{R}) \supseteq K \cap (\{p\} \times \mathbb{R}); \tag{3.12}$$

(c) the conical asymptotic weak epigraph hull property at p (denote by p-$(AWEHP)$) if

$$\operatorname{epi}(f - g + \delta_A)^* \cap (\{p\} \times \mathbb{R}) = \operatorname{cl}[K \cap (\{p\} \times \mathbb{R})]; \tag{3.13}$$

(d) the conical $(WEHP)$ (resp. the conical $(SWEHP)$, the conical $(AWEHP)$) if it satisfies the conical p-$(WEHP)$ (resp. the conical p-$(SWEHP)$, the conical p-$(AWEHP)$) at each $p \in X^*$.

Remark 3.4. (a) Let $p \in X^*$. Then the following equivalences/implications hold by definition:

(3.14)

the conical p-$(WEHP) \Longrightarrow$ the conical p-$(AWEHP) \Longrightarrow$ the conical p-$(SWEHP)$;

$$\text{the conical } (SWEHP) \Longleftrightarrow \operatorname{epi}(f - g + \delta_A)^* \supseteq K; \tag{3.15}$$

$$\text{the conical } (WEHP) \Longleftrightarrow \operatorname{epi}(f - g + \delta_A)^* = K. \tag{3.16}$$

Moreover, the following equivalence holds (see Proposition 5.7 in Section 5):

$$\text{the conical } (AWEHP) \Longleftrightarrow [\operatorname{epi}(f - g + \delta_A)^* = \operatorname{cl}K \text{ and } p \mapsto v(D_p) \text{ is upper semicontinuous}]. \tag{3.17}$$

(b) By (3.9), if g and $g_t, t \in T$, are lsc, then the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical $(SWEHP)$. The converse is not true, in general, as showed by Example 3.5 below.

(d) Recall from [14] that the family $\{\delta_C; h_t : t \in T\}$ satisfies the conical $(WEHP)_h$ if

$$\operatorname{epi}(h + \delta_A)^* = \bigcup_{\lambda \in R_+^{(T)}} \operatorname{epi}\big(h + \delta_C + \sum_{t \in T} \lambda_t h_t\big)^*. \tag{3.18}$$

Therefore, if g and $g_t, t \in T$, are lsc, then the conical $(WEHP)$ for the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ is equivalent to the conical $(WEHP)_h$ for the family $\{\delta_C; h_t : t \in T\}$ because of (3.6) and the lower semicontinuity of g and g_t.

Example 3.5. Let $X = C =: \mathbb{R}$ and $T := \{1\}$. Define $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$ respectively by $f = f_1 := \delta_{[0,+\infty)}, g_1 := 0$ and

$$g(x) := \begin{cases} \frac{1}{x} & x > 0, x \neq 1, \\ 2 & x = 1, \\ 1 & x = 0, \\ +\infty & x < 0 \end{cases} \quad \text{for each} \quad x \in \mathbb{R}.$$

Then, for each $\lambda \geq 0$,

$$(f - \operatorname{cl}g + \lambda(f_1 - g_1))(x) = \begin{cases} -\frac{1}{x} & x > 0, \\ -1 & x = 0, \\ +\infty & x < 0 \end{cases} \quad \text{for each} \quad x \in \mathbb{R}.$$

It follows that for each $\lambda \geq 0$,

$$\mathrm{cl}(f - \mathrm{cl}g + \lambda(f_1 - g_1))(x) = \begin{cases} -\frac{1}{x} & x > 0, \\ -\infty & x = 0, \\ +\infty & x < 0 \end{cases} \quad \text{for each} \quad x \in \mathbb{R}.$$

This implies that $K = \cup_{\lambda \geq 0} \mathrm{epi}(f - \mathrm{cl}g + \lambda(f_1 - g_1))^* = \emptyset$. Hence, $K \subseteq \mathrm{epi}(f - g + \delta_A)^*$ and the conical $(SWEHP)$ holds by (3.15). However, g is not lsc at $x = 1$.

The following proposition establishes the relationship between the conical $(SWEHP)$ (resp. the conical $(AWEHP)$, the conical $(WEHP)$) and the conical 0-$(SWEHP)$ (resp. the conical 0-$(AWEHP)$, the conical 0-$(WEHP)$).

Proposition 3.6. *The family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* $(SWEHP)$ *(resp. the conical* $(AWEHP)$*, the conical* $(WEHP)$ *) if and only if, for each* $p \in X^*$*, the family* $\{f - p, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* 0-$(SWEHP)$ *(resp. the conical* 0-$(AWEHP)$*, the conical* 0-$(WEHP)$*).*

Proof. Let $p \in X^*$ and let $K(p)$ be the set defined by

$$K(p) := \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \Big(\bigcap_{(u^*, v^*) \in H^*} \big(\mathrm{epi}\,(f - p(\cdot) + \delta_C + \sum_{t \in T} \lambda_t f_t)^* - (u^*, g^*(u^*)) - \sum_{t \in T} \lambda_t (v_t^*, g_t^*(v_t^*))\big)\Big).$$

Then, $K(p) = K + (-p, 0)$ by (2.9) and so

$$K(p) \cap (\{0\} \times \mathbb{R}) = K \cap (\{p\} \times \mathbb{R}) + (-p, 0). \tag{3.19}$$

Moreover, using (2.9), we conclude that

$$\mathrm{epi}\,(f - p - g + \delta_A)^* \cap (\{0\} \times \mathbb{R}) = \mathrm{epi}\,(f - g + \delta_A)^* \cap (\{p\} \times \mathbb{R}) + (-p, 0). \tag{3.20}$$

Thus the conclusion follows from the definition and the proof is complete. ☐

4. Farkas Lemmas for DC programming

Throughout this section, the notations f, g, C, f_t, g_t, $t \in T$, A and K are as explained at Section 3. In this section, we consider the Farkas lemma and the stable Farkas lemma for DC optimization problem (1.1). Recall that the Lagrange function L is defined by (1.7). We say that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the Farkas rule if, for each $\alpha \in \mathbb{R}$,

(4.1)

$$[f(x) - g(x) \geq \alpha,\ \forall x \in A] \Longleftrightarrow [(\exists \lambda \in \mathbb{R}_+^{(T)})(\forall w^* \in H^*) \quad \text{s.t.} \quad L(w^*, \lambda) \geq \alpha],$$

and the stable Farkas rule if, for each $p \in X^*$, the family $\{f - p, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the Farkas rule.

Remark 4.1. In the case when g and $g_t, t \in T$, are lsc, the Farkas rule for the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$, thanks to (1.5), is reduced to (1.8) for the family $\{h, \delta_C; h_t : t \in T\}$ studied in [14] .

The following lemma is useful for our study in the present paper.

Lemma 4.2. *Let $r \in \mathbb{R}$. Then the following statements hold.*

(i) $(0, r) \in \operatorname{epi}(f - g + \delta_A)^*$ *if and only if* $v(P) \geq -r$.

(ii) $(0, r) \in K$ *if and only if there exists* $\lambda \in \mathbb{R}_+^{(T)}$ *such that for each* $w^* = (u^*, v^*) \in H^*$, *one has that*

$$L(w^*, \lambda) \geq -r. \tag{4.2}$$

Proof. (i) By the definition of the conjugate function, one has

$$v(P) = -(f - g + \delta_A)^*(0).$$

Hence, the result is clear.

(ii)Let $(0, r) \in K$. Then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that for each $(u^*, v^*) \in H^*$, one has

$$(0, r) \in \operatorname{epi}(f + \delta_C + \sum_{t \in T} \lambda_t f_t)^* - (u^*, g^*(u^*)) - \sum_{t \in T} \lambda_t (v_t^*, g_t^*(v_t^*)). \tag{4.3}$$

Let $(u^*, v^*) \in H^*$. Then there exists $(x^*, r_1) \in \operatorname{epi}(f + \delta_C + \sum_{t \in T} \lambda_t f_t)^*$ such that

$$x^* - u^* - \sum_{t \in T} \lambda_t v_t^* = 0 \tag{4.4}$$

and

$$r_1 - g^*(u^*) - \sum_{t \in T} \lambda_t g_t^*(v_t^*) = r. \tag{4.5}$$

Since $(f + \delta_C + \sum_{t \in T} \lambda_t f_t)^*(x^*) \leq r_1$, it follows from (4.4) and (4.5) that

$$\begin{aligned} L(w^*, \lambda) &= g^*(u^*) + \sum_{t \in T} \lambda_t g_t^*(v_t^*) - (f + \delta_C + \sum_{t \in T} \lambda_t f_t)^*(x^*) \\ &\geq g^*(u^*) + \sum_{t \in T} \lambda_t g_t^*(v_t^*) - r_1 \\ &= -r. \end{aligned}$$

Conversely, suppose that there exists $\overline{\lambda} \in \mathbb{R}_+^{(T)}$ such that for each $w^* = (u^*, v^*) \in H^*$, (4.2) holds. Let $(u^*, v^*) \in H^*$. Then

$$g^*(u^*) + \sum_{t \in T} \overline{\lambda}_t g_t^*(v_t^*) - (f + \delta_C + \sum_{t \in T} \overline{\lambda}_t f_t)^*(u^* + \sum_{t \in T} \overline{\lambda}_t v_t^*) \geq -r,$$

that is

$$(f + \delta_C + \sum_{t \in T} \overline{\lambda}_t f_t)^*(u^* + \sum_{t \in T} \overline{\lambda}_t v_t^*) \leq r + g^*(u^*) + \sum_{t \in T} \overline{\lambda}_t g_t^*(v_t^*).$$

This means that $(z^*, s) \in \operatorname{epi}(f + \delta_C + \sum_{t \in T} \overline{\lambda}_t f_t)^*$, where,

$$z^* := u^* + \sum_{t \in T} \overline{\lambda}_t v_t^* \quad \text{and} \quad s := r + g^*(u^*) + \sum_{t \in T} \overline{\lambda}_t g_t^*(v_t^*).$$

Hence,

$$\begin{aligned} (0, r) &= (z^*, s) - (u^*, g^*(u^*)) - \textstyle\sum_{t \in T} \overline{\lambda}_t (v_t^*, g_t^*(v_t^*)) \\ &\in \operatorname{epi}(f + \delta_C + \textstyle\sum_{t \in T} \overline{\lambda}_t f_t)^* - (u^*, g^*(u^*)) - \textstyle\sum_{t \in T} \overline{\lambda}_t (v_t^*, g_t^*(v_t^*)). \end{aligned} \tag{4.6}$$

Since (4.6) holds for each $(u^*, v^*) \in H^*$, it follows that

$$(0, r) \in \bigcap_{(u^*,v^*)\in H^*} (\operatorname{epi}(f + \delta_C + \sum_{t\in T} \bar{\lambda}_t f_t)^* - (u^*, g^*(u^*)) - \sum_{t\in T} \bar{\lambda}_t (v_t^*, g_t^*(v_t^*)))$$

and hence $(0, r) \in K$. The proof is complete. □

Theorem 4.3. *The following statements are equivalent.*

(i) *The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical 0-$(WEHP)$.*

(ii) *For each $\alpha \in \mathbb{R}$,*

$$(0, -\alpha) \in \operatorname{epi}(f - g + \delta_A)^* \iff (0, -\alpha) \in K. \tag{4.7}$$

(iii) *The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the Farkas rule.*

Proof. It is evident that (i)⇔(ii). By Lemma 4.2(i), the condition stated in the left-hand side of (4.1) and that of (4.7) are equivalent. The corresponding assertion regarding the right-hand side is also valid by Lemma 4.2(ii). Therefore (4.1) and (4.7) are equivalent, and so (ii)⇔(iii). □

Using Theorem 4.3 and Proposition 3.6, we get straightforwardly the following global version of Theorem 4.3, which gives a complete characterization for the stable Farkas rule to hold.

Theorem 4.4. *The following statements are equivalent.*

(i) *The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical $(WEHP)$.*

(ii) *For each $p \in X^*$ and each $\alpha \in \mathbb{R}$,*

$$(p, -\alpha) \in \operatorname{epi}(f - g + \delta_A)^* \iff (p, -\alpha) \in K. \tag{4.8}$$

(iii) *The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the stable Farkas rule.*

5. Lagrange dualities for DC programming

Let $p \in X^*$ and consider the following DC optimization problem:

$$(P_p) \qquad \begin{array}{ll} \text{Minimize} & f(x) - g(x) - \langle p, x\rangle, \\ \text{s. t.} & f_t(x) - g_t(x) \le 0, \ t \in T, \\ & x \in C \end{array} \tag{5.1}$$

and its dual problem defined by

$$(D_p) \qquad \sup_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{w^*\in H^*} L_p(w^*, \lambda), \tag{5.2}$$

where the Lagrange function $L_p : H^* \times \mathbb{R}_+^{(T)} \to \overline{\mathbb{R}}$ is defined by

$$L_p(w^*, \lambda) := g^*(u^*) + \sum_{t\in T} \lambda_t g_t^*(v_t^*) - \big(f - p + \delta_C + \sum_{t\in T} \lambda_t f_t\big)^*\big(u^* + \sum_{t\in T} \lambda_t v_t^*\big) \tag{5.3}$$

for any $(w^*, \lambda) \in H^* \times \mathbb{R}_+^{(T)}$ with $w^* = (u^*, v^*) \in H^*$ and $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$. In particular, in the case when $p = 0$, problem (P_p) as well as its dual problem (D_p) are reduced to the problem (P) and problem (D) respectively. Let $v(P_p)$ and $v(D_p)$ denote the optimal values of (P_p) and (D_p), respectively.

Definition 5.1. We say that

(a) the weak Lagrange duality holds (between (P) and (D)) if $v(D) \leq v(P)$;

(b) the Lagrange duality holds (between (P) and (D)) if $v(D) = v(P)$;

(c) the strong Lagrange duality holds (between (P) and (D)) if $v(P) = v(D)$ and the problem (D) has an optimal solution;

(d) the stable weak Lagrange duality (resp. the stable Lagrange duality, the stable strong Lagrange duality) holds if the weak Lagrange duality (resp. the Lagrange duality, the strong Lagrange duality) between (P_p) and (D_p) holds for each $p \in X^*$.

This section is devoted to the study of the weak/stong and stable weak/strong Lagrange dualities between (P) and (D). We first note by the following example that the weak Lagrange duality does not necessarily hold in general.

Example 5.2. Let X, C, T and $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$ be defined as Example 3.2. Then f, g, f_1 and g_1 are proper convex functions and $A = (-\infty, 0]$. Thus,

$$v(P) = \inf_{x \in (-\infty, 0]} \{f(x) - g(x)\} = -1.$$

Clearly, $f^* = g^* = f_1^* = \delta_{[0,+\infty)}$ and $g_1^* = \delta_{\{0\}}$. Hence,

$$v(D) = \sup_{\lambda \geq 0} \inf_{u^* \geq 0, v^* = 0} \{g^*(u^*) + g_1^*(v^*) - (f + \lambda f_1)^*(u^* + v^*)\} = 0.$$

This implies that $v(P) < v(D)$. Consequently, the weak Lagrange duality does not hold. Moreover, since

$$v(\mathcal{D}) = \max_{\lambda \geq 0} \inf_{x \in \mathbb{R}} \{f(x) - g(x) + \lambda(f_1(x) - g_1(x))\} = -1,$$

it follows that the problem $(\mathcal{D})$ and (D) are not equivalent. Furthermore, it is easy to see that (1.8) and (1.9) are not equivalent.

The following theorem shows that the conical 0-$(SWEHP)$ is a sufficient and necessary condition for the weak Lagrange duality holds.

Theorem 5.3. (i) *The weak Lagrange duality holds if and only if the family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* 0-$(SWEHP)$.

(ii) *The stable weak Lagrange duality holds if and only if the family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* $(SWEHP)$.

Proof. Assertion (ii) is a global version of assertion (i). Hence, by Proposition 3.6, we only need to prove assertion (i). To do this, suppose that the weak Lagrange duality holds, that is, $v(D) \leq v(P)$. Let $(0, r) \in K$. Then, by Lemma 4.2(ii) and the definition of $v(D)$, one has that $v(D) \geq -r$ and so, $v(P) \geq -r$, which implies that $(0, r) \in \operatorname{epi}(f - g + \delta_A)^*$, thanks to Lemma 4.2(i). Hence,

$$K \cap (\{0\} \times \mathbb{R}) \subseteq \operatorname{epi}(f - g + \delta_A)^* \cap (\{0\} \times \mathbb{R}), \tag{5.4}$$

that is, the conical 0-$(SWEHP)$ holds.

Conversely, suppose that the conical 0-$(SWEHP)$ holds. Then (5.4) holds. To show $v(D) \leq v(P)$, suppose on the contrary that $v(P) < v(D)$. Then there exists $r \in \mathbb{R}$ such that $v(P) < -r < v(D)$. Thus, by the definition of $v(D)$, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that for each $(u^*, v^*) \in H^*$, $L(w^*, \lambda) \geq -r$ holds. Hence, by Lemma 4.2(ii), we have $(0, r) \in K$, and $(0, r) \in \operatorname{epi}(f - g + \delta_A)^*$ by (5.4). This together with

Lemma 4.2(i) implies that $v(P) \geq -r$, which contradicts $v(P) < -r$. Consequently, $v(P) \geq v(D)$ and the proof is complete. □

The following theorem provides some complete characterizations for the Lagrange duality and the stable Lagrange duality .

Theorem 5.4. (i) *The Lagrange duality holds if and only if the family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* 0-$(AWEHP)$.

(ii) *The stable Lagrange duality holds if and only if the family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* $(AWEHP)$.

Proof. As noted for the proof for Theorem 5.3, it is sufficient to prove assertion (i). To do this, suppose that the Lagrange duality holds. Then, by Theorem 5.3, the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical 0-$(SWEHP)$, that is, (5.4) holds. Since $\operatorname{epi}(f - g + \delta_A)^* \cap (\{0\} \times \mathbb{R})$ is w^*-closed, it follows that

$$\operatorname{cl}[K \cap (\{0\} \times \mathbb{R})] \subseteq \operatorname{epi}(f - g + \delta_A)^* \cap (\{0\} \times \mathbb{R}). \tag{5.5}$$

To verify the converse inclusion, let $(0, r) \in \operatorname{epi}(f - g + \delta_A)^*$. By Lemma 4.2(i), $v(P) \geq -r$ and so $v(D) = v(P) \geq -r$. Let $\epsilon > 0$. Then, there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that for each $w^* = (u^*, v^*) \in H^*$,

$$L(w^*, \lambda) \geq -r - \epsilon,$$

which implies that $(0, r + \epsilon) \in K \cap (\{0\} \times \mathbb{R})$ thanks to Lemma 4.2(ii). Hence, $(0, r) \in \operatorname{cl}[K \cap (\{0\} \times \mathbb{R})]$, which shows the converse inclusion of (5.5). Hence, the conical 0-$(AWEHP)$ holds.

Conversely, suppose that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical 0-$(AWEHP)$. Then by Remark 3.4, the conical 0-$(SWEHP)$ holds. Hence, by Theorem 5.3Extended Farkas's lemmas and strong Lagrange dualities for DC infinite programming, we have that $v(D) \leq v(P)$. To show the converse inequality, suppose on the contrary that $v(D) < v(P)$. Then there exists $r \in \mathbb{R}$ such that $v(D) < -r < v(P)$. Thus, $(0, r) \in \operatorname{epi}(f - g + \delta_A)^*$, thanks to Lemma 4.2(i). Since the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical 0-$(AWEHP)$, it follows that $(0, r) \in \operatorname{cl}[K \cap (\{0\} \times \mathbb{R})]$. Therefore, there exists a net $\{(0, r_\tau)\} \subseteq K$ such that $(0, r_\tau) \to (0, r)$. Hence, by Lemma 4.2(ii), there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that for each $w^* = (u^*, v^*) \in H^*$,

$$L(w^*, \lambda) \geq -r_\tau \to -r.$$

This together with the definition of $v(D)$ implies that $v(D) \geq -r$, which contradicts $v(D) < -r$. Hence, $v(D) = v(P)$ and the proof is complete. □

The following theorem characterizes the strong Lagrange dualities in term of the conical 0-$(WEHP)$ and the conical $(WEHP)$.

Theorem 5.5. (i) *The strong Lagrange duality holds if and only if the family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* 0-$(WEHP)$.

(ii) *The stable strong Lagrange duality holds if and only if the family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* $(WEHP)$.

Proof. As noted for the proof for Theorem 5.3, it is sufficient to prove assertion (i). To do this, suppose that the strong Lagrange duality holds. Then, by Theorem 5.3, we have (5.4) holds and so, we only need to verify the set on the right side of (5.4) is contained in the set on the left side. To do this, let $(0, r) \in \operatorname{epi}(f - g + \delta_A)^*$. Then, by Lemma 4.2(i), we have $v(P) \geq -r$. Hence, by the strong Lagrange duality, $v(D) = v(P)$ and there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that (4.2) holds for each $(u^*, v^*) \in H^*$. This together with Lemma 4.2(ii) implies that $(0, r) \in K$. Hence, $\operatorname{epi}(f - g + \delta_A)^* \cap (\{0\} \times \mathbb{R}) \subseteq K \cap (\{0\} \times \mathbb{R})$.

Conversely, suppose that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical 0-$(WEHP)$. Then, by Theorem 5.3, $v(D) \leq v(P)$. Thus, to show the strong Lagrange duality, it suffices to show $v(P) \leq v(D)$ and (D) has an optimal solution. Note that the conclusion holds trivially if $v(P) = -\infty$. Below we only consider the case when $r = -v(P) \in \mathbb{R}$. By Lemma 4.2(i), $(0, r) \in \operatorname{epi}(f - g + \delta_A)^*$ and so $(0, r) \in K$, thanks to the assumed conical 0-$(WEHP)$. Thus, by Lemma 4.2(ii) and the definition of $v(D)$, we have that $v(D) \geq -r$ and there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that (4.2) holds for each $(u^*, v^*) \in H^*$. This implies that $v(D) = v(P)$ and λ is an optimal solution of the problem (D). Hence, the strong Lagrange duality holds and the proof is complete. □

For the following proposition on characterization of the $(AWEHP)$, we introduce the upper semicontinuous hull of a proper function on X. Let $\phi : X \to \overline{\mathbb{R}}$ be a proper function. The upper semicontinuous hull of ϕ is defined by

$$\limsup_{y \to x} \phi(y) := \inf_{V \in \mathcal{N}(x)} \sup_{y \in V} \phi(y) \quad \text{for each } x \in X,$$

where $\mathcal{N}(x)$ denotes the set of the neighborhoods of x. Clearly, one has by definition that

$$\limsup_{y \to x} \phi(y) \geq \phi(x) \quad \text{for each } x \in X. \tag{5.6}$$

Recall that ϕ is said to be upper semicontinuous (usc in brief) at $x_0 \in X$ if $\limsup_{x \to x_0} \phi(x) = \phi(x_0)$, and usc if ϕ is usc at each point of its domain.

Lemma 5.6. *The function $p \mapsto v(P_p)$ is usc on X^*.*

Proof. Let $p_0 \in X^*$. By (5.6), we only need to show that $\limsup_{p \to p_0} v(P_p) \leq v(P_{p_0})$. Suppose on the contrary that $\limsup_{p \to p_0} v(P_p) > v(P_{p_0})$. Then there exists $r \in \mathbb{R}$ such that $\limsup_{p \to p_0} v(P_p) > r > v(P_{p_0})$. Hence, for each $V \in \mathcal{N}(p_0)$, there exists $p_V \in V$ such that

$$v(P_{p_0}) < r \leq v(P_{p_V}).$$

Let V be fixed. Then, for each $x \in X$,

$$f(x) - g(x) + \delta_A(x) - \langle p_V, x \rangle \geq r.$$

Since $\{p_V : V \in \mathcal{N}(p_0)\} \subseteq X^*$, it follows that for each $x \in X$,

$$f(x) - g(x) + \delta_A(x) - \langle p_0, x \rangle \geq \limsup_{p_V \to p_0} \{f(x) - g(x) + \delta_A(x) - \langle p_V, x \rangle\} \geq r.$$

Hence, $v(P_{p_0}) \geq r$, which is a contradiction. The proof is complete. □

Below we give a characterization for the conical $(AWEHP)$.

Proposition 5.7. *The family* $\{f, g, \delta_C; f_t, g_t : t \in T\}$ *satisfies the conical* $(AWEHP)$ *if and only if the function* $p \mapsto v(D_p)$ *is usc on* X^* *and the following equality holds:*

$$\text{epi}\,(f - g + \delta_A)^* = \text{cl}\,K. \tag{5.7}$$

Proof. Suppose that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical $(AWEHP)$. Then, by Theorem 5.4, we have that for each $p \in X^*$, $v(P_p) = v(D_p)$ holds, which implies that the function $p \mapsto v(D_p)$ is usc (note by Lemma 5.6 that the function $p \mapsto v(P_p)$ is usc) and hence

$$v(P_p) = \limsup_{q \to p} v(D_q) \quad \text{for each} \quad p \in X^*. \tag{5.8}$$

Below we show that (5.7) holds. By the assumed conical $(AWEHP)$, we have that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the conical $(SWEHP)$ by (3.14). Since $\text{epi}\,(f - g + \delta_A)^*$ is weak*-closed, it follows that

$$\text{cl}\,K \subseteq \text{epi}\,(f - g + \delta_A)^*. \tag{5.9}$$

To verify the converse inclusion, let $(p, r) \in \text{epi}\,(f - g + \delta_A)^*$. Then $v(P_p) \geq -r$ and by (5.8), we have that

$$\limsup_{q \to p} v(D_q) = v(P_p) \geq -r.$$

Hence, for each $V \in \mathcal{N}(p)$, there exists $p_V \in V$ such that $v(D_{p_V}) \geq -r$. Let $\epsilon > 0$ and let V be fixed. Then there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that for each $w^* = (u^*, v^*) \in H^*$,

$$L_{p_V}(w^*, \lambda) \geq -r - \epsilon,$$

which implies that $(p_V, r + \epsilon) \in K$, thanks to Lemma 4.2(ii). Thus, $(p, r) \in \text{cl}\,K$, which shows the converse inclusion of (5.9). Hence, (5.7) holds.

Conversely, suppose that (5.7) holds and the function $p \mapsto v(D_p)$ is usc. To show the conical $(AWEHP)$, by Theorem 5.4, it suffices to show that (5.8) holds (note that the function $p \mapsto v(D_p)$ is usc). Note by (5.7) that $K \subseteq \text{epi}\,(f - g + \delta_A)^*$ and hence the conical $(SWEHP)$ holds by Remark 3.4(a). Let $p \in X^*$. Then, by Theorem 5.3(ii), one has that $v(D_p) \leq v(P_p)$ and so, $\limsup_{q \to p} v(D_q) \leq \limsup_{q \to p} v(P_q)$. Note by Lemma 5.6 that the function $q \mapsto v(P_q)$ is usc. It follows that

$$\limsup_{q \to p} v(D_q) \leq v(P_p). \tag{5.10}$$

To show the converse inequality, let $-r < v(P_p)$. Then, by Lemma 4.2(i) and (2.8), $(p, r) \in \text{epi}\,(f - g + \delta_A)^*$ and $(p, r) \in \text{cl}\,K$ by (5.7). Therefore, $(V \times (r - \delta, r + \delta)) \cap K \neq \emptyset$ for each $V \in \mathcal{N}(p)$ and each $\delta > 0$. Fix $V \in \mathcal{N}(p)$, $\delta > 0$ and choose $(p_V, r_\delta) \in (V \times (r - \delta, r + \delta)) \cap K$. Then, applying Lemma 4.2(ii), we have that $v(D_{p_V}) \geq -r_\delta \geq -r - \delta$, and consequently,

$$\inf_{V \in \mathcal{N}(p)} \sup_{p \in V} v(D_p) \geq \inf_{V \in \mathcal{N}(p)} v(D_{p_V}) \geq \sup_{\delta > 0}(-r - \delta) \geq -r.$$

This shows that $\limsup_{q \to p} v(D_q) \geq -r$, and so $\limsup_{q \to p} v(D_q) \geq v(P_p)$ as $-r < v(P_p)$ is arbitrary. Thus $\limsup_{q \to p} v(D_q) = v(P_p)$ is proved by (5.10). The proof is complete. ∎

6. Applications to conic programming

Throughout this section, let X, Y be locally convex spaces, $C \subseteq X$ be a nonempty convex set. Let $S \subseteq Y$ be a closed convex cone. Define an order on Y by saying that $y \leq_S x$ if $y - x \in -S$. We attach a greatest element ∞ with respect to $\leq_S$ and denote $Y^\bullet := Y \cup \{+\infty\}$. The following operations are defined on $Y^\bullet$: for any $y \in Y$, $y + \infty = \infty + y = \infty$ and $t\infty = \infty$ for any $t \geq 0$. Let $f, g : X \to \overline{\mathbb{R}}$ be proper convex functions and $G : X \to Y^\bullet$ be S-convex in the sense that for every $u, v \in X$ and every $t \in [0, 1]$,

$$G(tu + (1-t)v) \leq_S tG(u) + (1-t)G(v)$$

(see [4, 5, 21]). Consider the following DC conic programming problem

$$(6.1) \qquad (P(S)) \qquad \begin{array}{ll} \text{Minimize} & f(x) - g(x), \\ \text{s. t.} & x \in C,\ G(x) \in -S. \end{array}$$

Problem (6.1) has been studied in [12, 13] and also studied in [2, 5, 3, 14, 15, 20, 22, 21, 24, 23] for the special case when $g = 0$. As before, we use A to denote the solution set of the following system

$$(6.2) \qquad x \in C;\ G(x) \in -S,$$

and assume that $A \cap \operatorname{dom}(f - g) \neq \emptyset$. Following [4, 14, 31], we define for each $\lambda \in S^\oplus$,

$$(6.3) \qquad (\lambda G)(x) := \begin{cases} \langle \lambda, G(x) \rangle & \text{if } x \in \operatorname{dom} G, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that G is S-convex if and only if $(\lambda G)(\cdot) : X \to \overline{\mathbb{R}}$ is a convex function for each $\lambda \in S^\oplus$. Thus, problem (6.1) can be viewed as an example of (1.1) by setting

$$(6.4) \qquad T := S^\oplus,\ f_t := tG,\ g_t := 0 \quad \text{for each } t \in T,$$

and the approaches in previous sections are applicable. In particular, one can check easily that the corresponding dual problem and Farkas rule are respectively reduced to

$$(6.5) \qquad (D(S)) \quad \sup_{\lambda \in S^\oplus} \inf_{u^* \in \operatorname{dom} g^*} \{g^*(u^*) - (f + \delta_C + \lambda G)^*(u^*)\}$$

and for each $\alpha \in \mathbb{R}$,

$$[f(x) - g(x) \geq \alpha,\ \forall x \in A] \iff$$
$$[(\exists \lambda \in S^\oplus)(\forall u^* \in \operatorname{dom} g^*) \text{ s.t. } g^*(u^*) - (f + \delta_C + \lambda G)^*(u^*) \geq \alpha].$$

Moreover, the corresponding characteristic set K defined by (3.3) for (6.4) can ba expressed as

$$(6.6) \qquad K = \bigcup_{\lambda \in S^\oplus} \Big(\bigcap_{u^* \in \operatorname{dom} g^*} \big(\operatorname{epi}(f + \delta_C + \lambda G)^* - (u^*, g^*(u^*))\big)\Big).$$

Thus, Theorems 4.3, 4.4, 5.3, 5.4 and 5.5 are applicable to establishing the corresponding results on Farkas lemmas and Lagrange dualities for DC conic programming problem (6.1). In particular, we make the following definition.

Definition 6.1. The family $\{\delta_C; G\}$ is said to satisfy the $C(f, g; A)$ if

$$\operatorname{epi}(f - g + \delta_A)^* = K. \tag{6.7}$$

Remark 6.2. Let $\phi := f - g$ be a proper DC function. Recall that the family $\{\delta_C; G\}$ satisfies the $C(\phi, A)$ if

$$C(\phi, A) \qquad \operatorname{epi}(\phi + \delta_A)^* = \bigcup_{\lambda \in S^{\oplus}} \operatorname{epi}(\phi + \delta_C + \lambda G)^*, \tag{6.8}$$

which was introduced in [14] (in the case when $g = 0$) for studying the Farkas lemma and the strong Lagrange duality between $(P(S))$ and $(D(S))$. In the case when g is lsc, one sees by (3.6) (applied to $\{\lambda G, 0, S^{\oplus}\}$ in place of $\{f_t, 0, T\}$) that

$$K = \bigcup_{\lambda \in S^{\oplus}} \operatorname{epi}(h + \delta_C + \lambda G)^*;$$

hence the $C(f - g, A)$ and the $C(f, g; A)$ coincide for the family $\{\delta_C; G\}$. However, it is not true, in general, as shown in Example 6.4 below.

The main theorem of this section is as follows. In particular, the equivalences (i)⇔(iii)⇔(iv) (in the special case when $g = 0$) in Theorem 6.3 below were given in [14, Theorem 6.7] and (i)⇔(iii) was proved also in [3, Theorem 2] under the assumptions that f is lsc, $g = 0$, C is closed and G is S-epi-closed, that is $\operatorname{epi}_S(G) := \{(x, y) \in X \times Y : y \in G(x) + S\}$ is closed.

Theorem 6.3. *The following assertions are equivalent:*

(i) *The family $\{\delta_C; G\}$ satisfies the $C(f, g; A)$.*

(ii) *The stable strong Lagrange duality holds between $(P(S))$ and $(D(S))$.*

(iii) *The family $\{f, g, \delta_C; \lambda G : \lambda \in S^{\oplus}\}$ satisfies the stable Farkas rule.*

Furthermore, if g is lsc, then each of assertions (i)-(iii) *is equivalent to the following one:*

(iv) *The family $\{\delta_C; G\}$ satisfies the $C(f - g, A)$.*

Proof. Consider the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ with T and f_t, g_t, $t \in T$, defined by (6.4). As we mentioned earlier, the dual problem (6.5) for conic problem (6.1) and the Farkas rule (1.8) for the family $\{\delta_C; G\}$ are respectively equivalent to the corresponding ones for problem (1.1) and the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$. Furthermore, by (6.6) and definition, the $C(f, g; A)$ for the family $\{\delta_C; G\}$ coincides with the conical $(WEHP)$ for the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$. Then, (i)⇔(ii) follows from Theorem 5.5 and (i)⇔(iii) follows from Theorem 4.4; while the equivalence (i)⇔(iv) holds by Remark 6.2. The proof is complete. □

Example 6.4. Let $X = Y = C := \mathbb{R}$ and $S := [0, +\infty)$. Let $h, G : \mathbb{R} \to \overline{\mathbb{R}}$ be defined respectively by $h := f - g, G(x) := -x$ for each $x \in \mathbb{R}$, where $f := \delta_{(-\infty,0]}$ and

$$g(x) := \begin{cases} 0, & x < 0, \\ 1, & x = 0, \\ +\infty, & x > 0 \end{cases} \quad \text{for each } x \in \mathbb{R}.$$

Then f, g is proper convex, G is $\mathbb{R}$-convex. Consider the system (6.2). Then one has that

$$A = \{x \in \mathbb{R} : G(x) \in -S\} = [0, +\infty).$$

Note that for each $\lambda \in S^{\oplus} = [0, +\infty)$, one has that for each $x \in \mathbb{R}$,

$$(h + \delta_A)(x) = \begin{cases} -1, & x = 0, \\ +\infty, & x \neq 0 \end{cases} \quad \text{and} \quad (h + \delta_C + \lambda G)(x) = \begin{cases} -\lambda x, & x < 0, \\ -\lambda x - 1, & x = 0, \\ +\infty, & x > 0. \end{cases}$$

Hence, for each $x^* \in \mathbb{R}$, $(h + \delta_A)^*(x^*) = 1$ and

$$(h + \delta_C + \lambda G)^*(x^*) = \begin{cases} 1, & x^* \geq -\lambda, \\ +\infty, & x^* < -\lambda. \end{cases}$$

Thus,

$$\operatorname{epi}(h + \delta_A)^* = \bigcup_{\lambda \in [0,+\infty)} \operatorname{epi}(h + \delta_C + \lambda G)^* = \mathbb{R} \times [1, +\infty).$$

Therefore, (6.8) holds and the family $\{\delta_C; G\}$ satisfies the $C(h, A)$. However, it is easy to see that $\operatorname{dom} g^* = [0, +\infty)$ and for each $\lambda \in [0, +\infty)$,

$$(f + \delta_C + \lambda G)^*(x^*) = \begin{cases} 0, & x^* \geq -\lambda, \\ +\infty, & x^* < -\lambda \end{cases} \quad \text{for each } x^* \in \mathbb{R}.$$

This implies that

$$K = \bigcup_{\lambda \in [0,+\infty)} \bigcap_{u^* \in [0,+\infty)} \{\operatorname{epi}(f + \delta_C + \lambda G)^* - (u^*, 0)\} = \mathbb{R} \times [0, +\infty).$$

Hence, $\operatorname{epi}(h + \delta_A)^* \neq K$ and the $C(f, g; A)$ does not holds. Therefore, the $C(h, A)$ does not coincide with the $C(f, g; A)$.

We end this paper with the following remark which shows that some results for the case of convex conical programming (i.e., the case when $g = 0$) can not be extended to the case of DC conical programming.

Remark 6.5. Suppose that $g = 0$. Then the following assertions hold by [14, Proposition 6.4 and Theorem 6.8]:

(a) If f is lsc, C is closed and G is S-epi-closed, then

$$\operatorname{epi}(f - g + \delta_A)^* = \operatorname{cl}\Big(\bigcup_{\lambda \in S^{\oplus}} \operatorname{epi}(f - g + \delta_C + \lambda G)^*\Big), \tag{6.9}$$

and so

$$C(f - g; A) \Longleftrightarrow \bigcup_{\lambda \in S^{\oplus}} \operatorname{epi}(f - g + \delta_C + \lambda G)^* \text{ is weak}^*\text{-closed.} \tag{6.10}$$

(b) If $f - g$ is continuous at some point in A, then

$$C(0, A) \Longrightarrow C(f - g, A). \tag{6.11}$$

The following example shows that, if $g \neq 0$, then each of (6.9), (6.10) and (6.11) is no longer true (even in the case when f, g and G are continuous).

Example 6.6. Let $X = Y = C := \mathbb{R}$ and $S := [0, +\infty)$. Let $h, G : \mathbb{R} \to \overline{\mathbb{R}}$ be defined respectively by $h := f - g, G(x) := x - 1$ for each $x \in \mathbb{R}$, where $f := \delta_{[0,+\infty)}$ and

$$g(x) := \begin{cases} x^2, & x \geq 0, \\ +\infty, & x < 0 \end{cases} \quad \text{for each } x \in \mathbb{R}.$$

Then f, g is proper convex, G is $\mathbb{R}$-convex. Consider the system (6.2). Then one has that

$$A = \{x \in \mathbb{R} : G(x) \in -S\} = (-\infty, 1].$$

Hence, h is continuous at some point in A and $\operatorname{epi}\delta_A^* = \{(x, y) : x \geq 0, y \geq x\}$. It is easy to see that for each $\lambda \in S^{\oplus} = [0, +\infty)$ and for each $x^* \in \mathbb{R}$,

$$(\delta_C + \lambda G)^*(x^*) = \begin{cases} \lambda, & x^* = \lambda, \\ +\infty, & x^* \neq \lambda. \end{cases}$$

Then

$$\bigcup_{\lambda \geq 0} \operatorname{epi}(\delta_C + \lambda G)^* = \{(x, y) : x \geq 0, y \geq x\} = \operatorname{epi}\delta_A^*.$$

This implies that the family $\{\delta_C; g\}$ satisfies the condition $C(0, A)$. Note that for each $x \in \mathbb{R}$,

$$h(x) = \begin{cases} -x^2, & x \geq 0, \\ +\infty, & x < 0, \end{cases} \quad \text{and} \quad (h + \delta_A)(x) = \begin{cases} -x^2, & 0 \leq x \leq 1, \\ +\infty, & x > 1 \text{ or } x < 0. \end{cases}$$

It follows that for each $x^* \in \mathbb{R}$,

$$(h + \delta_A)^*(x^*) = \begin{cases} x^* + 1, & x^* \geq 0, \\ 0, & x^* < 0. \end{cases}$$

Therefore,

$$\operatorname{epi}(h + \delta_A)^* = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y > x + 1\} \cup \{(x, y) \in \mathbb{R}^2 : x < 0, y \geq 0\}.$$

However, for each $\lambda \in [0, +\infty)$ and for each $x \in \mathbb{R}$,

$$(h + \delta_C + \lambda G)(x) = \begin{cases} -x^2 + \lambda x - \lambda, & x \geq 0, \\ +\infty, & x < 0. \end{cases}$$

Thus, $(h + \delta_C + \lambda G)^*(x^*) = +\infty$ for each $x^* \in \mathbb{R}$ and hence $\operatorname{epi}(h + \delta_C + \lambda G)^* = \emptyset$. This implies that

$$\bigcup_{\lambda \in [0, +\infty)} \operatorname{epi}(h + \delta_C + \lambda G)^* = \emptyset \neq \operatorname{epi}(h + \delta_A)^*.$$

Hence, the condition $C(h, A)$ does not hold. Therefore, (6.11) does not hold. Moreover, note that

$$\bigcup_{\lambda \in [0, +\infty)} \operatorname{epi}(h + \delta_C + \lambda G)^* \text{ is w}^*\text{-closed},$$

it follows that (6.9) and (6.10) do not hold.

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