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ISOLATED CALMNESS OF EFFICIENT POINT MULTIFUNCTIONS IN PARAMETRIC VECTOR OPTIMIZATION

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Abstract. In the present paper the stability theory of the efficient point multifunction of a parametric vector optimization problem is considered. We establish new sufficient conditions for this efficient point multifunction to be isolatedly calm at a given point in its graph. Furthermore, the elaboration of these conditions in a broad class of conventional vector optimization problems is undertaken as well as examples are also provided for analyzing and illustrating the results obtained.

1. Introduction

This paper deals with the stability theory of parametric vector optimization problems. We first give some notations and definitions.

Let $f : P \times X \to Y$ be a vector function and $C : P \rightrightarrows X$ a multifunction, where P, X and Y are Euclidean spaces equipped with the usual norms. Let $F : P \rightrightarrows Y$ be a multifunction given by

$$F(p) := f(p, C(p)) = \{f(p, x) \,|\, x \in C(p)\}. \tag{1.1}$$

Let $K \subset Y$ be a pointed, closed and convex cone with apex at the origin.

Definition 1.1. We say that $y \in A$ is an *efficient point* of a subset $A \subset Y$ with respect to K if and only if $(y - K) \cap A = \{y\}$. The set of efficient points of A is denoted by $\mathrm{Eff}_K A$. We stipulate that $\mathrm{Eff}_K \emptyset = \emptyset$. When $\mathrm{int}K \neq \emptyset$, an element $y \in A$ is called a *weakly efficient point* of A with respect to K and write $y \in \mathrm{Eff}^w_K A$, if $(y - \mathrm{int}K) \cap A = \emptyset$. We stipulate that $\mathrm{Eff}^w_K \emptyset = \emptyset$. From now on when speaking of weakly efficient points we always assume that K has a nonempty interior.

We consider the following *parametric vector optimization problem*:

$$\mathrm{Eff}_K \left\{f(p, x) \,|\, x \in C(p)\right\}, \tag{1.2}$$

where x is the *unknown* (decision variable) and $p \in P$ is a *parameter*.

The multifunction $\mathcal{F} : P \rightrightarrows Y$ assigns to p the set of efficient points of (1.2), i.e.,

$$\mathcal{F}(p) := \mathrm{Eff}_K \left\{f(p, x) \,|\, x \in C(p)\right\} = \mathrm{Eff}_K F(p), \tag{1.3}$$

is called the *efficient point multifunction* of (1.2).

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Stability analysis in vector optimization problems has been investigated intensively by many researchers. One of the main problems here is to find sufficient conditions for the efficient point multifunction $\mathcal{F}$ to have a certain stability property such as lower (upper) semi-continuous, calmness and continuous properties. For instance, the lower (upper) semi-continuity of the efficient point multifunction have been examined by Penot [28,29]. Other results in this direction for the convex problems can be found in [39]. We can consult the books by Tanino, Sawaragi and Nakayama [37], Luc [24] for more various stability properties of the efficient point multifunction. Using the so-called *domination property* and *containment property*, Bednarczuk [2,3] studied the Hausdorff upper semi-continuity, the K-Hausdorff upper semi-continuity and the lower (upper) semi-continuity of $\mathcal{F}$. Moreover, some of the principal results in [2,3] being valid under weaker assumptions is proved by the authors in a recent paper [11]. More recently, paper [8] gave sufficient conditions in terms of the Fréchet and limiting coderivatives of parametric multifunctions for the efficient solution map to have the calmness.

The contingent derivative has proved to be not only useful in necessary/sufficient conditions of vector optimization (see e.g., [24]) but also successful in sensitivity analysis (see e.g., [19, 20, 35, 36, 38, 39] and the references therein).

In this work the contingent derivative is exploited to study the stability theory of parametric vector optimization problems. Namely, we establish new sufficient conditions for the efficient point multifunction $\mathcal{F}$ of (1.2) to be *isolatedly calm* at a given point in its graph. Furthermore, these conditions are elaborated in a broad class of conventional vector optimization problems.

The rest of the paper is organized as follows. In Section 2, we provide further the basic definitions and notations from set-valued analysis. Then we recall some known auxiliary results which will be useful hereafter. In Section 3 we establish sufficient conditions for the efficient point multifunction $\mathcal{F}$ to have the isolated calmness at a given point in general cases. The further elaboration of these conditions on the concrete/conventional classes in parametric vector optimization problems will be undertaken in the last section. Moreover, examples are also simultaneously provided to analyze and illustrate the obtained results.

2. Preliminaries and auxiliary results

In this section we provide further the basic definitions and notations from set-valued analysis which will be widely used in what follows and also present some auxiliary results which will be useful in the next section. Let $G : P \rightrightarrows Y$ be a multifunction. The *effective domain* and the *graph* of G are defined, respectively, by

$$\text{dom}G = \{p \in P \mid G(p) \neq \emptyset\}, \ \text{gph}G = \{(p, y) \in P \times Y \mid y \in G(p)\}.$$

Let $B_Y := \{y \in Y \mid ||y|| \leq 1\}$. We denote by $\mathcal{N}(y)$ the set of all neighborhoods of $y \in Y$.

Definition 2.1. (i) G is *upper locally Lipschitz* at $\bar{p} \in \text{dom}G$ if there exist $U \in \mathcal{N}(\bar{p})$ and a real number $M > 0$ such that

$$G(p) \subset G(\bar{p}) + M||p - \bar{p}||B_Y \quad \forall p \in U.$$

(ii) We say that G is *isolatedly calm* at $(\bar{p}, \bar{y}) \in \mathrm{gph}\, G$ if there exists $V \in \mathcal{N}(\bar{y})$ such that $V \cap G(\bar{p}) = \{\bar{y}\}$ and the multifunction $p \mapsto V \cap G(p)$ is upper locally Lipschitz at $\bar{p}$.
(iii) G is *pseudo-Lipschitz* at $(\bar{p}, \bar{y}) \in \mathrm{gph} G$ if there exist $U \in \mathcal{N}(\bar{p})$ and $V \in \mathcal{N}(\bar{y})$ and a real number $M > 0$ such that

$$G(p_1) \cap V \subset G(p_2) + M||p_1 - p_2||B_Y \quad \forall p_1, p_2 \in U.$$

Note that the isolated calmness was introduced in [14] under the name *upper Lipschitz continuity at a point*, Levy [22] called it *local upper Lipschitz*. Recently, this property was called *isolated calmness* in [15] and this is the name we use here. The reader is referred to the book [18] for the study in details and the comparison among various concepts of Lipschitz-type stable properties.

Definition 2.2. (i) G is said to be *convex* if

$$\alpha G(p) + (1-\alpha)G(p') \subset G(\alpha p + (1-\alpha)p') \quad \forall p, p' \in P, \forall \alpha \in [0,1].$$

(ii) G is said to be *K-convex* if

$$\alpha G(p) + (1-\alpha)G(p') \subset G(\alpha p + (1-\alpha)p') + K \quad \forall p, p' \in P, \forall \alpha \in [0,1].$$

It is known that (see e.g., [39]) G is convex if and only if $\mathrm{gph} G$ is a convex set in $P \times Y$. Given a subset $\Omega \subset Y$, we denote the interior and the closure of Ω, respectively, by $\mathrm{int}\Omega$ and $\mathrm{cl}\Omega$. Let $\bar{y} \in \mathrm{cl}\Omega$. The *Bouligand tangent cone* to Ω at $\bar{y}$ is defined by

$$T^B(\Omega; \bar{y}) = \{v \in Y \,|\exists \{t_n\} \subset (0, +\infty), t_n \to 0, \exists \{v_n\} \subset Y, v_n \to v \text{ with } \bar{y} + t_n v_n \in \Omega \ \ \forall n \in \mathbb{N}\},$$

where $\mathbb{N} := \{1, 2, \dots\}$. It is well known that this cone is closed.

When Ω is convex, the *normal cone* to $\Omega \subset Y$ at $\bar{y} \in \Omega$ is defined by

$$N(\Omega; \bar{y}) := \{v \in Y \mid \langle v, y - \bar{y}\rangle \le 0 \quad \forall y \in \Omega\},$$

and $N(\Omega; \bar{y}) := \emptyset$ if $\bar{y} \notin \Omega$. Since Ω is convex, it is known that in this case the normal cone to Ω at $\bar{y} \in \Omega$ is precisely the *negative polar cone* of the Bouligand tangent cone to Ω at this point, that is

$$N(\Omega; \bar{y}) := T^B(\Omega; \bar{y})^\circ = \{v \in Y \mid \langle v, y\rangle \le 0 \quad \forall y \in T^B(\Omega; \bar{y})\}.$$

Definition 2.3. Let $(\bar{p}, \bar{y}) \in \mathrm{gph} G$.
(i) (See [1]) The multifunction $DG(\bar{p}, \bar{y}) : P \rightrightarrows Y$ is said to be the *contingent derivative* of G at $(\bar{p}, \bar{y})$ if $\mathrm{gph} DG(\bar{p}, \bar{y}) = T^B(\mathrm{gph} G; (\bar{p}, \bar{y}))$. Equivalently,

$$DG(\bar{p}, \bar{y})(p) = \limsup_{(t,p') \to (0^+, p)} \frac{1}{t}\big(G(\bar{p} + tp') - \bar{y}\big) \quad \forall p \in P.$$

(ii) (See [27]) The multifunction $D_l G(\bar{p}, \bar{y}) : P \rightrightarrows Y$ is said to be the *lower derivative* of G at $(\bar{p}, \bar{y})$ if

$$\begin{aligned} \mathrm{gph} D_l G(\bar{p}, \bar{y}) =& \{(p, y) \in P \times Y \mid \forall \{p_n\} \subset P, p_n \to p, \forall \{t_n\} \subset (0, +\infty), t_n \to 0, \\ & \exists \{y_n\} \subset Y, y_n \to y, \exists n_0 \in \mathbb{N}, \forall n \ge n_0, \bar{y} + t_n y_n \in G(\bar{p} + t_n p_n)\}. \end{aligned}$$

Equivalently,

$$D_lG(\bar{p},\bar{y})(p) = \liminf_{(t,p')\to(0^+,p)} \frac{1}{t}\big(G(\bar{p}+tp')-\bar{y}\big) \quad \forall p \in P.$$

The multifunction G is said to be *semi-differentiable* at $(\bar{p},\bar{y})$ if $DG(\bar{p},\bar{y}) = D_lG(\bar{p},\bar{y})$.

The notion of semi-differentiable multifunction introduced by Penot [27] is *closer* to the classical differentiability of mappings; see [27,31,32] for its various properties and applications therein. Let us recall a result in [31] which will be useful hereafter.

Lemma 2.4 (See [31, Theorem 5.4])**.** *Let X, Y and Z be finite dimensional spaces and let $G : X \rightrightarrows Y$ be a multifunction having the form*

$$G(x) = \{y \in \Omega \mid h(x,y) \in \Theta\},$$

where $h : X \times Y \to Z$ is a continuously differentiable mapping and the sets $\Omega \subset Y$ and $\Theta \subset Z$ are closed and convex. Suppose for $(\bar{x},\bar{y}) \in \operatorname{gph} G$ that the following constraint qualification holds:

The only vector $z \in N\big(\Theta; h(\bar{x},\bar{y})\big)$ satisfying $-z\nabla_y h(\bar{x},\bar{y}) \in N(\Omega;\bar{y})$ is $z = 0$.

Then G is semi-differentiable and pseudo-Lipschitz at $(\bar{x},\bar{y})$ as well as for each $x \in X$,

$$DG(\bar{x},\bar{y})(x) = \{y \in T^B(\Omega;\bar{y}) \mid \nabla_x h(\bar{x},\bar{y})(x) + \nabla_y h(\bar{x},\bar{y})(y) \in T^B\big(\Theta; h(\bar{x},\bar{y})\big)\},$$

where $\nabla h(\bar{x},\bar{y}) := (\nabla_x h(\bar{x},\bar{y}), \nabla_y h(\bar{x},\bar{y}))$ denotes the Fréchet derivative of h at $(\bar{x},\bar{y})$.

Definition 2.5 (See [4])**.** G is called *directionally compact* at $(\bar{p},\bar{y}) \in \operatorname{gph}G$ if for every sequence $\{t_n\} \subset (0,+\infty), t_n \to 0$ and for every sequence $\{h_n\} \subset P, h_n \to h \in P$, any sequence y_n with $\bar{y} + t_n y_n \in G(\bar{p} + t_n h_n)$ for each $n \in \mathbb{N}$ contains a convergent subsequence.

There are several sufficient conditions for the directionally compactness of G at a given point in its graph that can be found in [4]. Here we invoke some other criteria from [10] which will be used in the sequel.

Lemma 2.6 (See [10, Proposition 2.5])**.** *If G is upper locally Lipschitz at $\bar{p} \in \operatorname{dom}G$ with $G(\bar{p}) = \{\bar{y}\}$, then G is directionally compact at $(\bar{p},\bar{y})$.*

Shi [35] introduced the following *derivative* of G at $(\bar{p},\bar{y}) \in \operatorname{gph} G$ in the direction p :

$$\begin{aligned} D^S G(\bar{p},\bar{y})(p) :=& \{y \in Y \mid \exists\{t_n\} \subset (0,+\infty), \exists\{p_n\} \subset P, y_n \in G(p_n) \\ & \text{such that } p_n \to \bar{p},\ t_n(p_n - \bar{p}, y_n - \bar{y}) \to (p,y)\}. \end{aligned}$$

It is easy to check that $\operatorname{gph} DG(\bar{p},\bar{y}) \subset \operatorname{gph} D^S G(\bar{p},\bar{y})$ and $\operatorname{gph} DG(\bar{p},\bar{y}) = \operatorname{gph} D^S G(\bar{p},\bar{y})$ if G is a convex multifunction.

Lemma 2.7 (See [10, Proposition 2.6])**.** *Let $(\bar{p},\bar{y}) \in \operatorname{gph} G$. If the following condition is fulfilled, then G is directionally compact at $(\bar{p},\bar{y})$,*

$$D^S G(\bar{p},\bar{y})(0) = \{0\}. \tag{2.1}$$

Note that the condition (2.1) has served well as a *qualification condition* for having the so-called *proto-differentiability* of G. We refer the reader to [21] for more details and its applications therein.

In what follows we also use the characterization of the isolated calmness of a multifunction in terms of its contingent derivative.

Lemma 2.8 (See [22, Proposition 4.1])**.** *Let* $G : P \rightrightarrows Y$ *be a multifunction and let* $(\bar{p}, \bar{y}) \in \operatorname{gph} G$. *Then* G *is isolatedly calm at* $(\bar{p}, \bar{y})$ *if and only if* $DG(\bar{p}, \bar{y})(0) = \{0\}$.

3. Isolated calmness of the efficient point multifunction in general cases

In this section we provide sufficient conditions for the efficient point multifunction $\mathcal{F}$ in (1.3) to be isolatedly calm at a given point in its graph. To do this, we first need to establish an outer estimate of the contingent derivative of $\mathcal{F}$ at the reference point via the set of weakly efficient points of the contingent derivative of F in (1.1) at the corresponding point.

Proposition 3.1. *Let* $(\bar{p}, \bar{y}) \in \operatorname{gph} \mathcal{F}$. *If one of the following conditions is satisfied:*

(i) F *is semi-differentiable at* $(\bar{p}, \bar{y})$;

(ii) F *is* K*-convex and* $\bar{p} \in \operatorname{int}(\operatorname{dom} F)$,

then

$$D\mathcal{F}(\bar{p}, \bar{y})(p) \subset \operatorname{Eff}^w_K DF(\bar{p}, \bar{y})(p) \quad \forall p \in P. \tag{3.2}$$

Proof. It follows from [17, Theorem 3.6.19]. □

The next example shows that under condition (i) or (ii) the inclusion (3.2) may be not true if the superscription "w" is omitted. In other words, we do not have in general a *sharper* outer estimate $D\mathcal{F}(\bar{p}, \bar{y})(p) \subset \operatorname{Eff}_K DF(\bar{p}, \bar{y})(p)$ for all $p \in P$.

Example 3.2. Let $P = \mathbb{R}, X = Y = \mathbb{R}^2, K = \mathbb{R}^2_+$ and let $f : P \times X \to Y, C : P \rightrightarrows X$ be mappings which are given as follows:

$$f(p, x) = x \quad \forall p \in P,\ \forall x \in X,$$
$$C(p) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq x_1^2\} \quad \forall p \in P.$$

We have

$$F(p) = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq y_1^2\} \quad \forall p \in P,$$
$$\mathcal{F}(p) = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 = y_1^2,\ y_1 \leq 0\} \quad \forall p \in P.$$

Take $\bar{p} = 0$ and $\bar{y} = (0, 0)$, then $\bar{y} \in \mathcal{F}(\bar{p})$. By simple computation, one can find

$$D\mathcal{F}(\bar{p}, \bar{y})(p) = (-\infty, 0] \times \{0\},$$
$$DF(\bar{p}, \bar{y})(p) = \mathbb{R} \times [0, +\infty) \quad \forall p \in P.$$

It is easy to see that F is K-convex and $\bar{p} \in \operatorname{int}(\operatorname{dom} F)$, i.e., the assumption (ii) is fulfilled. Meanwhile, $\operatorname{Eff}_K DF(\bar{p}, \bar{y})(p) = \emptyset$ for all $p \in P$ and therefore the inclusion $D\mathcal{F}(\bar{p}, \bar{y})(p) \subset \operatorname{Eff}_K DF(\bar{p}, \bar{y})(p)$ fails to hold for all $p \in P$.

We are now ready to formulate the main result of this section.

Theorem 3.3. *Let* $(\bar{p}, \bar{y}) \in \mathrm{gph}\mathcal{F}$*. Suppose that* $\mathrm{Eff}^w_K DF(\bar{p}, \bar{y})(0) = \{0\}$*. If one of the following conditions is satisfied:*

(i) *F is semi-differentiable at* $(\bar{p}, \bar{y})$*;*

(ii) *F is K-convex and* $\bar{p} \in \mathrm{int}(\mathrm{dom}\, F)$*,*

then $\mathcal{F}$ is isolatedly calm at $(\bar{p}, \bar{y})$*.*

Proof. Employing Proposition 3.1 and Lemma 2.8, we get the desired result. □

It is worth to noticing that the assumptions in the above theorem are essential. Let us first revisit Example 3.2. Consider $\bar{p} = 0$ and $\bar{y} = (0, 0) \in \mathcal{F}(\bar{p})$. We have $\mathrm{Eff}^w_K DF(\bar{p}, \bar{y})(0) = \mathbb{R} \times \{0\} \neq \{0\}$. Observe that $\mathcal{F}$ is not isolatedly calm at $(\bar{p}, \bar{y})$ although the condition (ii) in Theorem 3.3 holds. In the next example neither (i) nor (ii) in Theorem 3.3 is satisfied, the conclusion of this theorem is invalid.

Example 3.4. Let $P = X = Y = \mathbb{R}, K = \mathbb{R}_+$ and let $f : P \times X \to Y, C : P \rightrightarrows X$ be mappings which are given as follows:

$$f(p, x) = x \quad \forall p \in P,\ \forall x \in X,$$

$$C(p) = \begin{cases} \sqrt{p} & \text{if } p \geq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

We have $\mathcal{F}(p) = F(p) = C(p)$ for all $p \in P$. Take $\bar{p} = 0$ and $\bar{y} = 0$, then $\bar{y} \in \mathcal{F}(\bar{p})$. By computing, we obtain $\mathrm{Eff}^w_K DF(\bar{p}, \bar{y})(0) = \{0\}$. Since $\bar{p} \notin \mathrm{int}(\mathrm{dom}\, F)$, it follows that neither (i) nor (ii) in Theorem 3.3 is satisfied. Actually, $\mathcal{F}$ is not isolatedly calm at $(\bar{p}, \bar{y})$.

4. Isolated calmness of the efficient point multifunction in special/concrete cases

4.1. **Multifunction constraints.** We now consider the problem (1.2) with *constraint mapping* $C : P \rightrightarrows X$. Define $\widetilde{C} : P \times Y \rightrightarrows X$ as follows

$$\widetilde{C}(p, y) = \{x \in C(p) \mid f(p, x) = y\}. \tag{4.1}$$

Our first auxiliary result in this section gives a formula for computing contingent derivatives of F in (1.1) at the reference point via the contingent derivative of the constraint mapping C and the Fréchet derivative of the objective function f at the corresponding points.

Proposition 4.1. *Let* $\bar{p} \in P, \bar{x} \in C(\bar{p})$ *and* $\bar{y} = f(\bar{p}, \bar{x})$*. Suppose that f is Fréchet differentiable at* $(\bar{p}, \bar{x})$ *with the derivative* $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$ *and that* $\widetilde{C}$ *defined in* (4.1) *is directionally compact at* $((\bar{p}, \bar{y}), \bar{x})$*. One has*

$$DF(\bar{p}, \bar{y})(p) = \{\nabla_p f(\bar{p}, \bar{x})(p) + \nabla_x f(\bar{p}, \bar{x})(x) \mid x \in DC(\bar{p}, \bar{x})(p)\} \quad \forall p \in P. \tag{4.2}$$

Furthermore, if C is semi-differentiable at $(\bar{p}, \bar{x})$*, then F is semi-differentiable at* $(\bar{p}, \bar{y})$ *as well.*

Proof. The inclusion "⊃" in (4.2) has been proved in [17, Theorem 2.7.8]. Let us justify the inverse "⊂" in (4.2). For each $p \in P$, take any $y \in DF(\bar{p}, \bar{y})(p)$, i.e.,

$(p, y) \in T^B(\text{gph}F; (\bar{p}, \bar{y}))$. Then there exist sequences $\{t_n\} \subset (0, +\infty), t_n \to 0$ and $\{(p_n, y_n)\} \subset P \times Y, (p_n, y_n) \to (p, y)$ with

$$\bar{y} + t_n y_n \in F(\bar{p} + t_n p_n) \text{ for all } n.$$

Thus it follows from the composite from of F in (1.1) that there exists $\{x_n\} \subset C(\bar{p} + t_n p_n)$ such that

$$\bar{y} + t_n y_n = f(\bar{p} + t_n p_n, x_n) \text{ for all } n, \tag{4.3}$$
$$\text{i.e., } x_n \in \widetilde{C}(\bar{p} + t_n p_n, \bar{y} + t_n y_n) \text{ for all } n.$$

Set $\hat{x}_n = \frac{x_n - \bar{x}}{t_n}$. We get $x_n = \bar{x} + t_n \hat{x}_n \in \widetilde{C}(\bar{p} + t_n p_n, \bar{y} + t_n y_n)$ for all n. Since $\widetilde{C}$ is directionally compact at $((\bar{p}, \bar{y}), \bar{x})$, without loss of generality, we may assume that $\{\hat{x}_n\}$ converges to some $\hat{x} \in X$. Thus $\hat{x} \in DC(\bar{p}, \bar{x})(p)$. By (4.3), we have

$$\begin{aligned} y = \lim_{n\to\infty} y_n &= \lim_{n\to\infty} \frac{f(\bar{p} + t_n p_n, \bar{x} + t_n \hat{x}_n) - f(\bar{p}, \bar{x})}{t_n} \\ &= \nabla_p f(\bar{p}, \bar{x})(p) + \nabla_x f(\bar{p}, \bar{x})(\hat{x}) \end{aligned}$$

and thus (4.2) has been established. We have by definition

$$D_l F(\bar{p}, \bar{y})(p) \subset DF(\bar{p}, \bar{y})(p) \quad \forall p \in P. \tag{4.4}$$

In order to show that F is semi-differentiable at $(\bar{p}, \bar{y})$, it suffices to justify the converse inclusion in (4.4). Let $p \in P$ and take any $y \in DF(\bar{p}, \bar{y})(p)$. By (4.2), there exists $x \in DC(\bar{p}, \bar{x})(p)$ such that $y = \nabla_p f(\bar{p}, \bar{x})(p) + \nabla_x f(\bar{p}, \bar{x})(x)$. From the semi-differentiability of C at $(\bar{p}, \bar{x})$ it follows that for any $\{t_n\} \subset (0, +\infty), t_n \to 0$, and $\{p_n\} \subset P, p_n \to p$, there exist $\{x_n\} \subset X, x_n \to x$ and $n_0 \in \mathbb{N}$ such that

$$\bar{x} + t_n x_n \in C(\bar{p} + t_n p_n) \quad \forall n \geq n_0.$$

Setting

$$y_n := \frac{f(\bar{p} + t_n p_n, \bar{x} + t_n x_n) - f(\bar{p}, \bar{x})}{t_n},$$

we have $\lim_{n\to\infty} y_n = y$ and

$$y_n \in \frac{F(\bar{p} + t_n p_n) - \bar{y}}{t_n} \quad \forall n \geq n_0.$$

Hence $y \in \liminf_{(t,p')\to(0^+,p)} \frac{F(\bar{p}+tp')-\bar{y}}{t} = D_l F(\bar{p}, \bar{y})(p)$ and this completes the proof. □

Remark 4.2. Observe that, in the proof of Proposition 4.1, the multifunction $\widetilde{C}$ is directionally compact at $((\bar{p}, \bar{y}), \bar{x})$ whenever the multifunction C has the corresponding property at $(\bar{p}, \bar{x})$. Thus the conclusion of Proposition 4.1 remains valid if the assumption $\widetilde{C}$ is directionally compact at $((\bar{p}, \bar{y}), \bar{x})$ is replaced by the assumption C is directionally compact at $(\bar{p}, \bar{x})$.

Corollary 4.3. *Let $\bar{p} \in P, \bar{x} \in C(\bar{p})$ and $\bar{y} = f(\bar{p}, \bar{x})$. Suppose that f is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the derivative $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$. If one of the following requirements is fulfilled:*

(i) *C is upper locally Lipschitz at $\bar{p} \in \text{dom}C$ with $C(\bar{p}) = \{\bar{x}\}$;*
(ii) *$D^S C(\bar{p}, \bar{x})(0) = \{0\}$;*

(iii) $\widetilde{C}$ *defined in* (4.1) *is upper locally Lipschitz at* $(\bar{p}, \bar{y}) \in \text{dom}\widetilde{C}$ *with* $\widetilde{C}(\bar{p}, \bar{y}) = \{\bar{x}\}$;

(iv) $D^S\widetilde{C}((\bar{p}, \bar{y}), \bar{x})(0, 0) = \{0\}$,

then (4.2) *holds true.*

Proof. The proof is immediate from Lemmas 2.6, 2.7, Proposition 4.1 and Remark 4.2. □

Note that the formula (4.2) is stated and proved by Tanino [38] under the condition (iii) in Corollary 4.3 and a slightly more exacting assumption that f is a continuously differentiable function; see further [17, Theorem 2.7.8] for a proof which has been undertaken under the condition (i) in Corollary 4.3.

The following proposition gives us another criteria for computing contingent derivatives of F in (1.1) at a given point in its graph.

Proposition 4.4. *Let* $\bar{p} \in P, \bar{x} \in C(\bar{p})$ *and* $\bar{y} = f(\bar{p}, \bar{x})$. *Suppose that* f *is Fréchet differentiable at* $(\bar{p}, \bar{x})$ *with the derivative* $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$. *If one of the following requirements is fulfilled:*

(i) f *is* K*-convex,* C *is a convex multifunction with* $C(\bar{x})$ *being a closed set and* $\bar{x} \in \text{int}(\text{dom}\, C)$, *and* $\widetilde{C}$ *defined in* (4.1) *is isolatedly calm at* $((\bar{p}, \bar{y}), \bar{x})$ *with* $\widetilde{C}(\bar{p}, \bar{y}) = \{\bar{x}\}$;

(ii) $\widetilde{C}$ *defined in* (4.1) *is pseudo-Lipschitz at* $((\bar{p}, \bar{y}), \bar{x})$,

then (4.2) *holds true.*

Proof. The proof of (i) is similar to that given in [39, Proposition 5.2]. We only justify the proposition under condition (ii). In view of [17, Theorem 2.7.8], it suffices to prove that

$$DF(\bar{p}, \bar{y})(p) \subset \{\nabla_p f(\bar{p}, \bar{x})(p) + \nabla_x f(\bar{p}, \bar{x})(x) \mid x \in DC(\bar{p}, \bar{x})(p)\} \quad \forall p \in P. \tag{4.5}$$

For each $p \in P$, take any $y \in DF(\bar{p}, \bar{y})(p)$, i.e., $(p, y) \in T^B(\text{gph}F; (\bar{p}, \bar{y}))$. Then there exist sequences $\{t_n\} \subset (0, +\infty), t_n \to 0$ and $\{(p_n, y_n)\} \subset P \times Y, (p_n, y_n) \to (p, y)$ with

$$\bar{y} + t_n y_n \in F(\bar{p} + t_n p_n) \text{ for all } n \in \mathbb{N}.$$

Since $\widetilde{C}$ is pseudo-Lipschitz at $((\bar{p}, \bar{y}), \bar{x})$, there exist $U_1 \in \mathcal{N}(\bar{p}), U_2 \in \mathcal{N}(\bar{y}), V \in \mathcal{N}(\bar{x})$ and $M > 0$ such that

$$\widetilde{C}(p_1, y_1) \cap V \subset \widetilde{C}(p_2, y_2) + M(||p_1 - p_2||^2 + ||y_1 - y_2||^2)^{\frac{1}{2}} B_X \quad \forall p_1, p_2 \in U_1, \forall y_1, y_2 \in U_2. \tag{4.6}$$

Choose $\delta > 0$ such that $\bar{p} + \delta B_P \subset U_1, \bar{y} + \delta B_Y \subset U_2$. It follows from (4.6) that

$$\bar{x} \in \widetilde{C}(\bar{p}, \bar{y}) \cap V \subset \widetilde{C}(\bar{p} + tp', \bar{y} + ty') + Mt(||p'||^2 + ||y'||^2)^{\frac{1}{2}} B_X \quad \forall t \in (0, \delta), \forall p' \in P, ||tp'|| \leq \delta, \forall y' \in Y, ||ty'|| \leq \delta. \tag{4.7}$$

Since $t_n \to 0$ and $(p_n, y_n) \to (p, y)$, without loss of generality, we may assume that there exists $M_1 > 0$ such that $(||p_n||^2 + ||y_n||^2)^{\frac{1}{2}} \leq M_1$ and $t_n \in (0, \delta), ||t_n p_n|| \leq$

$\delta, ||t_n y_n|| \le \delta$ for all n. So, by (4.7) there exists $\{x_n\} \subset \widetilde{C}(\bar{p} + t_n p_n, \bar{y} + t_n y_n)$ such that $||\bar{x} - x_n|| \le MM_1 t_n$ for all n. Set $\hat{x}_n = \frac{x_n - \bar{x}}{t_n}$. Then $||\hat{x}_n|| \le MM_1$ for all n and

$$x_n = \bar{x} + t_n \hat{x}_n \in \widetilde{C}(\bar{p} + t_n p_n, \bar{y} + t_n y_n) \quad \forall n. \tag{4.8}$$

Since X is finite dimensional, without loss of generality, we may assume that $\{\hat{x}_n\}$ converges to some $\hat{x} \in X$. This together with the fact that

$$\bar{x} + t_n \hat{x}_n \in \widetilde{C}(\bar{p} + t_n p_n, \bar{y} + t_n y_n) \subset C(\bar{p} + t_n p_n) \text{ for all } n$$

yields $\hat{x} \in DC(\bar{p}, \bar{x})(p)$. It follows from (4.8) that $\bar{y} + t_n y_n = f(\bar{p} + t_n p_n, x_n)$ for all n. We have

$$\begin{aligned} y = \lim_{n\to\infty} y_n &= \lim_{n\to\infty} \frac{f(\bar{p} + t_n p_n, \bar{x} + t_n \hat{x}_n) - f(\bar{p}, \bar{x})}{t_n} \\ &= \nabla_p f(\bar{p}, \bar{x})(p) + \nabla_x f(\bar{p}, \bar{x})(\hat{x}) \end{aligned}$$

and thus (4.5) has been justified. □

The criteria for the mapping $\widetilde{C}$ to be pseudo-Lipschitz or upper locally Lipschitz can be found in [38, 39] (also see [30] for the more general mappings).

We now provide sufficient conditions in the presence of the objective function f and the constraint mapping C for the efficient point multifunction $\mathcal{F}$ in (1.3) to be isolatedly calm at the reference point.

Theorem 4.5. *Let $\bar{p} \in P$ and let $\bar{x} \in C(\bar{p})$ be such that $\bar{y} = f(\bar{p}, \bar{x}) \in \mathcal{F}(\bar{p})$. Suppose that f is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the derivative $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$. Assume further that* (4.2) *holds and that*

$$\mathrm{Eff}^w_K\{\nabla_x f(\bar{p}, \bar{x})(x) \mid x \in DC(\bar{p}, \bar{x})(0)\} = \{0\}.$$

If C is semi-differentiable at $(\bar{p}, \bar{x})$, then $\mathcal{F}$ is isolatedly calm at $(\bar{p}, \bar{y})$.

Proof. Combining Theorem 3.3 and Proposition 4.1, we get the desired result. □

4.2. **Operator constraints.** We now consider the problem (1.2) with the constraint mapping $C : P \rightrightarrows X$ given in the form

$$C(p) := \{x \in \Omega \mid h(p, x) \in \Theta\}, \tag{4.9}$$

where $h : P \times X \to Z$ is a single-valued mapping between finite dimensional spaces. Here $\Omega \subset X, \Theta \subset Z$ are nonempty, closed and convex. Constraints of type (4.9) are known as *operator constraints.* They include geometric, functional, and other types of constraints under appropriate specifications of h and Θ, see [25, 26, 32] for more discussions and examples.

The following theorem gives a sufficient condition ensuring the efficient point multifunction $\mathcal{F}$ in (1.3) with constraints given by (4.9) to be isolatedly calm at the point under consideration.

Theorem 4.6. *Let $\mathcal{F}$ be the efficient point multifunction of* (1.2) *with the constraint mapping C given by* (4.9) *and let $\bar{p} \in P$, $\bar{x} \in C(\bar{p})$ be such that $\bar{y} = f(\bar{p}, \bar{x}) \in \mathcal{F}(\bar{p})$. Suppose that f is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the derivative $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$ and h in* (4.9) *is continuously differentiable at $(\bar{p}, \bar{x})$ with the* *derivative* $\nabla h(\bar{p},\bar{x}) := (\nabla_p h(\bar{p},\bar{x}), \nabla_x h(\bar{p},\bar{x}))$. *Assume further that* (4.2) *holds and that the following constraint qualification is fulfilled:*

(4.10)

The only vector $z \in N\big(\Theta; h(\bar{p},\bar{x})\big)$ *satisfying* $-z\nabla_x h(\bar{p},\bar{x}) \in N(\Omega;\bar{x})$ *is* $z = 0$.

If the following is valid,

$$\mathrm{Eff}^w_K\{\nabla_x f(\bar{p},\bar{x})(x) \mid x \in T^B(\Omega;\bar{x}), \nabla_x h(\bar{p},\bar{x})(x) \in T^B\big(\Theta; h(\bar{p},\bar{x})\big)\} = \{0\}, \tag{4.11}$$

then $\mathcal{F}$ *is isolatedly calm at* $(\bar{p},\bar{y})$.

Proof. According to Lemma 2.4, we have C is semi-differentiable at $(\bar{p},\bar{x})$ and

$$DC(\bar{p},\bar{x})(p) = \{x \in T^B(\Omega;\bar{x}) \mid \nabla_p h(\bar{p},\bar{x})(p) + \nabla_x h(\bar{p},\bar{x})(x) \in T^B\big(\Theta; h(\bar{p},\bar{x})\big)\}$$

for all $p \in P$. To finish the proof it remains to apply Theorem 4.5. □

The next theorem shows that the assumption (4.11) always holds in the following case.

Theorem 4.7. *Let* $\mathcal{F}$ *be the efficient point multifunction of* (1.2) *with the constraint mapping* C *given by* (4.9) *and let* $\bar{p} \in P$, $\bar{x} \in C(\bar{p})$ *be such that* $\bar{y} = f(\bar{p},\bar{x}) \in \mathcal{F}(\bar{p})$. *Suppose that* f *is Fréchet differentiable at* $(\bar{p},\bar{x})$ *with the derivative* $\nabla f(\bar{p},\bar{x}) := (\nabla_p f(\bar{p},\bar{x}), \nabla_x f(\bar{p},\bar{x}))$ *and* h *in* (4.9) *is continuously differentiable at* $(\bar{p},\bar{x})$ *with the derivative* $\nabla h(\bar{p},\bar{x}) := (\nabla_p h(\bar{p},\bar{x}), \nabla_x h(\bar{p},\bar{x}))$. *Assume further that* (4.2) *holds and that the following constraint qualification is fulfilled:*

The only vector $z \in N\big(\Theta; h(\bar{p},\bar{x})\big)$ *satisfying* $-z\nabla_x h(\bar{p},\bar{x}) \in N(\Omega;\bar{x})$ *is* $z = 0$.

If $C(\bar{p},\bar{x}) = \{\bar{x}\}$, *then* $\mathcal{F}$ *is isolatedly calm at* $(\bar{p},\bar{y})$.

Proof. It follows from Lemma 2.4 that C is pseudo-Lipschitz at $(\bar{p},\bar{x})$. Moreover, $C(\bar{p},\bar{x}) = \{\bar{x}\}$, we therefore have that C is isolatedly calm at $(\bar{p},\bar{x})$. By Lemma 2.8, $DC(\bar{p},\bar{x})(0) = \{0\}$. Using again Lemma 2.4 we have that C is semi-differentiable at $(\bar{p},\bar{x})$. Applying now Theorem 4.5, we get the desired result. □

4.3. **Constraints described by finitely many equalities and inequalities.** Next we consider the problem (1.2) with the *functional constraints* described by finitely many equalities and inequalities given as follows

$$\begin{aligned} C(p) := \big\{x \in X \mid\ & g_i(p,x) \le 0,\ i = 1,\dots,m, \\ & g_i(p,x) = 0,\ \ i = m+1,\dots,m+r\big\}, \end{aligned} \tag{4.12}$$

where $g_i,\ i = 1,\dots,m+r$, are real-valued functions on the space $P \times X$. Constraints of this type can be treated as a particular case of the operator constraints (4.9) with $h : P \times X \to \mathbb{R}^{m+r}$ defined by

$$h(p,x) := (g_1(p,x),\dots,g_{m+r}(p,x)), \tag{4.13}$$

$\Omega = X$ and $\Theta \subset \mathbb{R}^{m+r}$ defined by

$$\begin{aligned} \Theta := \big\{(\alpha_1,\dots,\alpha_{m+r}) \in \mathbb{R}^{m+r} \mid\ & \alpha_i \le 0,\ i = 1,\dots,m, \\ & \alpha_i = 0,\ \ i = m+1,\dots,m+r\big\}. \end{aligned} \tag{4.14}$$

However, constraints of type (4.12) is a conventional and remarkable class in parametric nonlinear programs and parametric vector optimization. The following corollaries provide sufficient conditions for the efficient point multifunction $\mathcal{F}$ in (1.3) with constraints given by (4.12) to be isolatedly calm at the reference point via the *Mangasarian-Fromovitz constraint qualification* which is as follows:

(4.15)

$$\begin{aligned}&\text{the gradients } \nabla g_{m+1}(\bar p,\bar x),\ldots,\nabla g_{m+r}(\bar p,\bar x) \text{ are linearly independent, and}\\ &\text{there is } u\in P\times X \text{ such that } \langle\nabla g_i(\bar p,\bar x),u\rangle=0 \text{ for } i=m+1,\ldots,m+r\\ &\text{and that } \langle\nabla g_i(\bar p,\bar x),u\rangle<0 \text{ whenever } i=1,\ldots,m \text{ with } g_i(\bar p,\bar x)=0.\end{aligned}$$

Corollary 4.8. *Let $\mathcal{F}$ be the efficient point multifunction of* (1.2) *with the constraint mapping C given by* (4.12) *and let $\bar p\in P$, $\bar x\in C(\bar p)$ be such that $\bar y = f(\bar p,\bar x)\in\mathcal{F}(\bar p)$. Suppose that f is Fréchet differentiable at $(\bar p,\bar x)$ with the derivative $\nabla f(\bar p,\bar x):=(\nabla_p f(\bar p,\bar x),\nabla_x f(\bar p,\bar x))$ and that $g_i, i=1,\ldots,m+r$, in* (4.12) *are continuously differentiable at $(\bar p,\bar x)$. Assume further that* (4.2) *holds and that the Mangasarian-Fromovitz constraint qualification* (4.15) *is fulfilled. If the following is valid, then $\mathcal{F}$ is isolatedly calm at $(\bar p,\bar y)$,*

$$\begin{aligned}\mathrm{Eff}^w_K\big\{\nabla_x f(\bar p,\bar x)(x)\mid x\in X,&\nabla_x g_i(\bar p,\bar x)(x)\le 0 \textit{ for all } i\in I(\bar p,\bar x)\\ &\nabla_x g_i(\bar p,\bar x)(x)=0 \textit{ for } i=m+1,\ldots,m+r\big\}=\{0\},\end{aligned}$$

where $I(\bar p,\bar x):=\big\{i\in\{1,\ldots,m\}\mid g_i(\bar p,\bar x)=0\big\}$ denotes the index set of active inequality constraints in (4.12) *at $(\bar p,\bar x)$.*

Proof. Observe that the condition (4.10) which is fulfilled is precisely the Mangasarian-Fromovitz constraint qualification (4.15) and the fact that

$$\begin{aligned}T^B\big(\Theta;h(\bar p,\bar x)\big)=\{(\lambda_1,\ldots,\lambda_{m+r})\in\mathbb{R}^{m+r}|&\lambda_i\le 0 \text{ for all } i\in I(\bar p,\bar x)\\ &\lambda_i=0 \text{ for } i=m+1,\ldots,m+r\},\end{aligned}$$

where h and Θ were defined in (4.13) and (4.14) respectively. Thus the proof is immediate from Theorem 4.6. □

4.4. **Constraints described by an arbitrary (possibly infinite) number of inequalities.** In this subsection we consider the problem (1.2) with the constraint mapping $C\colon P\rightrightarrows X$ defined by

$$C(p):=\{x\in X\;:\;g_t(p,x)\le 0,\, t\in T\},\tag{4.16}$$

where T is an *arbitrary* (possibly *infinite*) index set and for each $t\in T, g_t: P\times X\to\overline{\mathbb{R}}$ is proper, lower semicontinuous (l.s.c.) and convex.

Constraints of type (4.16) are known as *semi-infinite/infinite* inequality constraints. It is well known that models of semi-infinite optimization cover, e.g., pollution control models, engineering design, control of robots, mechanical stress of materials, optimal experimental design in regression and the popular semi-definite programming. Semi-infinite optimization programming and its wide applications have attracted much attention from many researchers. We refer readers to the books [16,33] for more details and discussions and some recent papers [5–7,9,12,13] for references.

Denote by $\mathbb{R}^{(T)}$ (respectively, $\mathbb{R}^{(T)}_+$) the collection of all the functions $\lambda : T \to \mathbb{R}$ taking nonzero (respectively, nonnegative) values only at finitely many points of T, and $\text{supp}\,\lambda := \{t \in T \,|\, \lambda_t \neq 0\}$. Given $u \in \mathbb{R}^{(T)}$ and $\lambda \in \mathbb{R}^{(T)}_+$, we put $\langle \lambda, u\rangle = \sum_{t\in \text{supp}\,\lambda} \lambda_t u_t$. In connection with (4.16), we use the set of *active constraint multipliers* defined by

$$A(\bar{p},\bar{x}) := \{\lambda \in \mathbb{R}^{(T)}_+ \,|\, \lambda_t g_t(\bar{p},\bar{x}) = 0 \ \text{ for all } t \in \text{supp}\,\lambda\}. \tag{4.17}$$

Definition 4.9. Let C be defined in (4.16) and let $(\bar{p},\bar{x}) \in \text{gph}C$. We say that C satisfies the *regular constraint qualification* (RCQ) at $(\bar{p},\bar{x})$ if

$$N\big(\text{gph}\,C; (\bar{p},\bar{x})\big) = \bigcup_{\lambda\in A(\bar{p},\bar{x})} \Big[\sum_{t\in\text{supp}\,\lambda} \lambda_t \partial g_t(\bar{p},\bar{x})\Big], \tag{4.18}$$

where the symbol ∂ stands for the subdifferential in the sense of convex analysis.

Various criteria for the validity of this qualification condition can be found in [7,13,23].

The following proposition gives a criterion for computing the contingent derivative of the constraint mapping C in (4.16) at a given point.

Proposition 4.10. *Let $(\bar{p},\bar{x}) \in \text{gph}C$. Suppose that C in* (4.16) *satisfies* (RCQ) *in* (4.18). *Then*

$$DC(\bar{p},\bar{x})(p) = \Big\{x \in X \Big| \sum_{t\in\text{supp}\,\lambda} \lambda_t \partial g_t(\bar{p},\bar{x})(p,x) \leq 0, \ \forall \lambda \in A(\bar{p},\bar{x})\Big\} \ \forall p \in P. \tag{4.19}$$

Proof. Note that $\text{gph}\,C = \big\{(p,x) \in P \times X \mid g_t(p,x) \leq 0 \text{ for all } t \in T\big\}$ is convex. Since C satisfies (RCQ) in (4.18), it holds

$$N\big(\text{gph}\,C; (\bar{p},\bar{x})\big) = \bigcup_{\lambda\in A(\bar{p},\bar{x})} \Big[\sum_{t\in\text{supp}\,\lambda} \lambda_t \partial g_t(\bar{p},\bar{x})\Big]. \tag{4.20}$$

Since $T^B\big(\text{gph}\,C; (\bar{p},\bar{x})\big)$ is closed, we have

$$T^B\big(\text{gph}\,C; (\bar{p},\bar{x})\big) = \big(T^B\big(\text{gph}\,C; (\bar{p},\bar{x})\big)^\circ\big)^\circ = N\big(\text{gph}\,C; (\bar{p},\bar{x})\big)^\circ. \tag{4.21}$$

Combining (4.20) with (4.21) and using the definition of the contingent derivative, we get (4.19). The proof is complete. □

Our first result in this subsection provides a sufficient condition for verifying the isolated calmness of the efficient point multifunction $\mathcal{F}$ in (1.3) with the constraints of nondifferentiable functions at a given point.

Theorem 4.11. *Let $\mathcal{F}$ be the efficient point multifunction of* (1.2) *with the constraint mapping C given by* (4.16) *and let $\bar{p} \in P, \bar{x} \in C(\bar{p})$ be such that $\bar{y} = f(\bar{p},\bar{x}) \in \mathcal{F}(\bar{p})$. Suppose that f is Fréchet differentiable at $(\bar{p},\bar{x})$ with the derivative $\nabla f(\bar{p},\bar{x}) := (\nabla_p f(\bar{p},\bar{x}), \nabla_x f(\bar{p},\bar{x}))$. Assume further that* (4.2) *holds and that C satisfies* (RCQ) *in* (4.18) *as well as*

$$\text{Eff}^w_K\Big\{\nabla_x f(\bar{p},\bar{x})(x) \mid x \in X, \sum_{t\in\text{supp}\,\lambda} \lambda_t \partial g_t(\bar{p},\bar{x})(0,x) \leq 0, \ \forall \lambda \in A(\bar{p},\bar{x})\Big\} = \{0\}.$$

If C is semi-differentiable at $(\bar{p},\bar{x})$, then $\mathcal{F}$ is isolatedly calm at $(\bar{p},\bar{y})$.

Proof. The proof follows from Proposition 4.10 and Theorem 4.5. □

Remark 4.12. It is worth to mention here that since C is a convex multifunction, one thus gets the semi-differentiability of C at $(\bar{p}, \bar{x})$ as long as $\bar{p} \in \operatorname{int}(\operatorname{dom} C)$ (see [17, Theorem 2.7.6]).

As an immediate consequence of Theorem 4.11, we have the following result which gives a criterion for verifying the isolated calmness of the efficient point multifunction $\mathcal{F}$ in (1.3) with the constraints of differentiable functions at the point under consideration.

Corollary 4.13. *Let $\mathcal{F}$ be the efficient point multifunction of* (1.2) *with the constraint mapping C given by* (4.16) *and let $\bar{p} \in P, \bar{x} \in C(\bar{p})$ be such that $\bar{y} = f(\bar{p}, \bar{x}) \in \mathcal{F}(\bar{p})$. Suppose that f is Fréchet differentiable at $(\bar{p}, \bar{x})$ with the derivative $\nabla f(\bar{p}, \bar{x}) := (\nabla_p f(\bar{p}, \bar{x}), \nabla_x f(\bar{p}, \bar{x}))$ and that all $g_t, t \in T$, in* (4.16) *are Fréchet differentiable at this point. Assume further that* (4.2) *holds and that C satisfies* (RCQ) *in* (4.18) *as well as*

(4.22)
$$\operatorname{Eff}_K^w\Big\{\nabla_x f(\bar{p}, \bar{x})(x) \mid x \in X, \sum_{t \in \operatorname{supp} \lambda} \lambda_t \nabla_x g_t(\bar{p}, \bar{x})(x) \leq 0, \ \forall \lambda \in A(\bar{p}, \bar{x})\Big\} = \{0\}.$$

If C is semi-differentiable at $(\bar{p}, \bar{x})$, then $\mathcal{F}$ is isolatedly calm at $(\bar{p}, \bar{y})$.

Finally we present an example which aims for illustrating the results obtained.

Example 4.14. Let $T = [0, 1] \cup \{-1\} \cup \{2\}, P = \mathbb{R}, X = Y = \mathbb{R}^2, K = \mathbb{R}^2_+$ and let $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, g_t : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}, t \in T$ be mappings which are given as follows:

$$f(p, x) = (p + x_1, x_2) \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \ \forall p \in \mathbb{R},$$

$$g_t(p, x) = tp - tx_1 - (1 - t)x_2 \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \ \forall p \in \mathbb{R}.$$

We consider the problem (1.2) with C defined in (4.16). By simple computation, one can find

$$C(p) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid -p + x_1 - 2x_2 \leq 0, 2p - 2x_1 + x_2 \leq 0\},$$
$$F(p) = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid -2p + y_1 - 2y_2 \leq 0, 4p - 2y_1 + y_2 \leq 0\} \quad \forall p \in P.$$

In particular, for $\bar{p} = 0$,

$$C(\bar{p}) = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 - 2x_2 \leq 0, -2x_1 + x_2 \leq 0\},$$
$$F(\bar{p}) = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_1 - 2y_2 \leq 0, -2y_1 + y_2 \leq 0\},$$

and thus $\bar{x} = (0, 0) \in C(\bar{p})$ as well as $\bar{y} = f(\bar{p}, \bar{x}) = (0, 0) \in \mathcal{F}(\bar{p})$. Observe that C is a convex multifunction and $\bar{p} \in \operatorname{int}(\operatorname{dom} C)$ and therefore it is semi-differentiable at $(\bar{p}, \bar{x})$ (see [17, Theorem 2.7.6]).

For each $t \in T$, we have

$$g_t^*(p, x) = \begin{cases} 0 & \text{if } (p, x) = (t, -t, t - 1) \\ +\infty & \text{if } (p, x) \neq (t, -t, t - 1), \end{cases} \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \ \forall p \in \mathbb{R},$$

$$\operatorname{epi} g_t^* = \{t\} \times \{-t\} \times \{t - 1\} \times \mathbb{R}_+,$$

where g_t^* denotes the *conjugate function* of g_t. Thus $cone\Big(\bigcup_{t\in T}\operatorname{epi}g_t^*\Big) = \mathbb{R}_+ \times \mathbb{R}^2_- \times \mathbb{R}_+$ which is closed in $\mathbb{R}^4$, where $cone(\Omega)$ stands for the *convex conical hull* of Ω. So it follows from [7, Theorem 3.7] that C satisfies (RCQ) in (4.18). Using now Proposition 4.10, we obtain

$$DC(\bar{p},\bar{x})(p) = \big\{x = (x_1,x_2) \in \mathbb{R}^2 \mid tp - tx_1 - (1-t)x_2 \le 0\ \forall t \in T\big\} = C(p) \quad \forall p \in P.$$

Thus for each $p \in P$,

$$\{\nabla_p f(\bar{p},\bar{x})(p) + \nabla_x f(\bar{p},\bar{x})(x) |\ x \in DC(\bar{p},\bar{x})(p)\} = F(p).$$

Besides, $DF(\bar{p},\bar{y})(p) = \{y \in \mathbb{R}^2 \mid -2p + y_1 - 2y_2 \le 0, 4p - 2y_1 + y_2 \le 0\}$ for all $p \in P$. So (4.2) is valid. Similarly, it is easy to check that (4.22) is satisfied. We now assert by Corollary 4.13 that $\mathcal{F}$ is isolatedly calm at $(\bar{p},\bar{y})$.

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