Journal of Nonlinear and Convex Analysis
Volume 14, Number 4, 2013, 687–717

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

TWO TURNPIKE RESULTS FOR DYNAMIC DISCRETE TIME ZERO-SUM GAMES

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Abstract. In this paper we study turnpike properties of approximate solutions for a class of dynamic discrete-time two-player zero-sum games without using convexity-concavity assumptions. These properties describe the structure of approximate solutions which is independent of the length of the interval, for all sufficiently large intervals.

1. Introduction

The study of the existence and the structure of (approximate) solutions of optimal control problems and dynamic zero-sum games defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research [3, 7-10, 12, 15, 17, 19, 20, 24-27, 36, 37, 39]. These problems arise in engineering [2, 43], in models of economic growth [11, 13, 22, 23, 29, 33, 34, 39, 40], in stochastic and differential dynamic games [1, 6, 14, 16, 28], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [5, 35] and in the theory of thermodynamical equilibrium for materials [18, 21].

In this paper we study the structure of approximate solutions for a class of dynamic discrete-time two-player zero-sum games without using standard convexity-concavity assumptions and establish two turnpike results. These results describe the structure of approximate solutions which is independent of the length of the interval, for all sufficiently large intervals. We show, roughly speaking, that approximate solutions are determined mainly by the objective function, and are essentially independent of the choice of interval and endpoint conditions. Turnpike results are well known in mathematical economics and optimal control (see [22, 33, 36, 37, 39, 40, 43] and the references mentioned there).

Let (X, ρ_X) and (Y, ρ_Y) be compact metric spaces equipped with the metrics ρ_X and ρ_Y respectively. We consider the set $X \times X \times Y \times Y$ equipped with the product topology induced by the metric

$$\rho((x_1, x_2, y_1, y_2), (x_1', x_2', y_1', y_2')) = \rho_X(x_1, x_1') + \rho_X(x_2, x_2') + \rho_Y(y_1, y_1') + \rho_Y(y_2, y_2'),$$
$$x_1, x_2, x_1', x_2' \in X,\ y_1, y_2, y_1', y_2' \in Y.$$

Denote by $C(X \times X \times Y \times Y)$ the set of all continuous functions $f : X \times X \times Y \times Y \to R^1$. For each $f \in C(X \times X \times Y \times Y)$ set

$$\|f\| = \sup\{|f(x_1, x_2, y_1, y_2)| :\ x_1, x_2 \in X,\ y_1, y_2 \in Y\}.$$

Clearly, $(C(X \times X \times Y \times Y), \|\cdot\|)$ is a Banach space.

2010 *Mathematics Subject Classification.* 49J99, 91A05, 91A25.

Key words and phrases. Approximate solution, dynamic game, turnpike property, zero-sum game.

Let $f \in C(X \times X \times Y \times Y)$. We associate with f a dynamic discrete-time two-player zero-sum game and study its optimal solutions. Namely, given an integer $n \geq 1$ we consider a discrete-time two-player zero-sum game over the interval $[0, n]$. For this game $\{\{x_i\}_{i=0}^n : x_i \in X,\ i = 0, \dots n\}$ is the set of strategies for the first player, $\{\{y_i\}_{i=0}^n : y_i \in Y,\ i = 0, \dots n\}$ is the set of strategies for the second player, and the cost for the first player associated with the strategies $\{x_i\}_{i=0}^n$, $\{y_i\}_{i=0}^n$ is given by $\sum_{i=0}^{n-1} f_i(x_i, x_{i+1}, y_i, y_{i+1})$.

Denote by $\mathcal{A}_0$ the set of all $f \in C(X \times X \times Y \times Y)$ for which there exist $x_f \in X$ and $y_f \in Y$ such that

(1.1)

$$f(x_f, x_f, y, y) \leq f(x_f, x_f, y_f, y_f) \leq f(x, x, y_f, y_f) \text{ for all } x \in X \text{ and all } y \in Y.$$

The following result is proved in Section 2.

Proposition 1.1. *$\mathcal{A}_0$ is a closed subset of the Banach space $(C(X \times X \times Y \times Y), \|\cdot\|)$.*

In this paper we consider a dynamic discrete-time two-player zero-sum game associated with $f \in \mathcal{A}_0$. If $f \in \mathcal{A}_0$ and $x_f \in X$ and $y_f \in Y$ satisfy (1.1), then the pair (x_f, y_f) is a saddle point for the function $\bar{f}(x, y) := f(x, x, y, y)$, $x \in X$, $y \in Y$.

It should be mentioned that in [38, 42] this class of games was considered in the case when X and Y are convex subsets of finite-dimensional Euclidean spaces and an objective function $f \in C(X \times X \times Y \times Y)$ satisfies standard convexity-concavity assumptions which, of course, imply the existence of a pair $(x_f, y_f) \in X \times Y$ satisfying (1.1) [4, 38, 42]. Clearly, there exist objective functions f which do not satisfy convexity-concavity assumptions and for which (1.1) holds with a pair $(x_f, y_f) \in X \times Y$. The main goal of our paper is to extend the turnpike results of [38, 42] obtained under convexity-concavity assumptions, to the class of games considered here.

Let us now define approximate solutions (saddle points) of our dynamic games.

Let $f \in C(X \times X \times Y \times Y)$, integers $n_2 > n_1$ and $M \geq 0$. A pair of sequences $\{\bar{x}_i\}_{i=n_1}^{n_2} \subset X$, $\{\bar{y}_i\}_{i=n_1}^{n_2} \subset Y$ is called (f, M)-good [17, 38, 39, 42] if the following properties hold:

(i) for each sequence $\{x_i\}_{i=n_1}^{n_2} \subset X$ satisfying $x_{n_1} = \bar{x}_{n_1}$, $x_{n_2} = \bar{x}_{n_2}$,

$$M + \sum_{i=n_1}^{n_2-1} f(x_i, x_{i+1}, \bar{y}_i, \bar{y}_{i+1}) \geq \sum_{i=n_1}^{n_2-1} f(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}); \tag{1.2}$$

(ii) for each sequence $\{y_i\}_{i=n_1}^{n_2} \subset Y$ satisfying $y_{n_1} = \bar{y}_{n_1}$, $y_{n_2} = \bar{y}_{n_2}$,

$$M + \sum_{i=n_1}^{n_2-1} f(\bar{x}_i, \bar{x}_{i+1}, \bar{y}_i, \bar{y}_{i+1}) \geq \sum_{i=n_1}^{n_2-1} f(\bar{x}_i, \bar{x}_{i+1}, y_i, y_{i+1}). \tag{1.3}$$

If a pair of sequences $\{x_i\}_{i=n_1}^{n_2} \subset X$, $\{y_i\}_{i=n_1}^{n_2} \subset Y$ is $(f, 0)$-good then it is called (f)-optimal.

A pair of sequences $\{\tilde{x}_i\}_{i=0}^{\infty} \subset X$, $\{\tilde{y}_i\}_{i=0}^{\infty} \subset Y$ is called (f, M)-good if for each natural number n the pair of sequences $\{\tilde{x}_i\}_{i=0}^n$, $\{\tilde{y}_i\}_{i=0}^n$ is (f, M)-good.

A pair of sequences $\{\tilde{x}_i\}_{i=0}^{\infty} \subset X$, $\{\tilde{y}_i\}_{i=0}^{\infty} \subset Y$ is called (f)-good if it is (f, M)-good with some $M > 0$.

A pair of sequences $\{\tilde{x}_i\}_{i=0}^{\infty} \subset X$, $\{\tilde{y}_i\}_{i=0}^{\infty} \subset Y$ is called (f)-optimal if for each natural number n the pair of sequences $\{\tilde{x}_i\}_{i=0}^{n}$, $\{\tilde{y}_i\}_{i=0}^{n}$ is (f)-optimal.

The following result is proved in Section 3.

Proposition 1.2. *Let $f \in \mathcal{A}_0$, $x_f \in X$ and $y_f \in Y$ satisfy (1.1) and let $\bar{x}_i^f = x_f$, $\bar{y}_i^f = y_f$ for all integers $i \geq 0$. Then the pair $\{\bar{x}_i^f\}_{i=0}^{\infty}$, $\{\bar{y}_i^f\}_{i=0}^{\infty}$ is (f)-optimal if and only if it is (f)-good.*

The following result is proved in Section 4.

Proposition 1.3. *Let $f \in \mathcal{A}_0$, $x_f \in X$ and $y_f \in Y$ satisfy (1.1) and let $\bar{x}_i^f = x_f$, $\bar{y}_i^f = y_f$ for all integers $i \geq 0$. Then the following properties are equivalent:*

(i) *the pair $\{\bar{x}_i^f\}_{i=0}^{\infty}$, $\{\bar{y}_i^f\}_{i=0}^{\infty}$ is (f)-optimal;*

(ii) *for each natural number n and each pair of sequences $\{x_i\}_{i=0}^{n} \subset X$, $\{y_i\}_{i=0}^{n} \subset Y$,*

$$-4\|f\| + \sum_{i=0}^{n-1} f(x_f, x_f, y_i, y_{i+1}) \leq nf(x_f, x_f, y_f, y_f) \leq \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) + 4\|f\|;$$

(iii) *there is $c > 0$ such that for each natural number n and each pair of sequences $\{x_i\}_{i=0}^{n} \subset X$, $\{y_i\}_{i=0}^{n} \subset Y$,*

$$-c + \sum_{i=0}^{n-1} f(x_f, x_f, y_i, y_{i+1}) \leq nf(x_f, x_f, y_f, y_f) \leq \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) + c.$$

Note that analogs of properties (i)-(iii) are used in the infinite horizon optimal control and they are usually posed, when one obtains a turnpike result where the turnpike is a singleton. See, for example, [40]. It was shown in [38] that these properties hold if X and Y are convex sets in finite-dimensional Euclidean spaces and f satisfies the convexity-concavity assumptions.

Proposition 1.3 implies the following result.

Proposition 1.4. *Let $f \in \mathcal{A}_0$, $x_f \in X$ and $y_f \in Y$ satisfy (1.1) and let property (iii) of Proposition 1.3 hold. Then the following assertions hold.*

1. *For each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ either the sequence*

$$\Big\{ \Big| \sum_{i=0}^{n-1} [f(x_i, x_{i+1}, y_f, y_f) - f(x_f, x_f, y_f, y_f)] \Big| \Big\}_{n=1}^{\infty}$$

is bounded or

$$\lim_{n \to \infty} \sum_{i=0}^{n-1} [f(x_i, x_{i+1}, y_f, y_f) - f(x_f, x_f, y_f, y_f)] = \infty.$$

2. *For each sequence $\{y_i\}_{i=0}^{\infty} \subset Y$ either the sequence*

$$\Big\{ \Big| \sum_{i=0}^{n-1} [f(x_f, x_f, y_i, y_{i+1}) - f(x_f, x_f, y_f, y_f)] \Big| \Big\}_{n=1}^{\infty}$$

is bounded or

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}[f(x_f,x_f,y_i,y_{i+1})-f(x_f,x_f,y_f,y_f)]=-\infty.$$

Denote by $\mathcal{A}$ the set of all $f\in\mathcal{A}_0$ for which there exist $x_f\in X$ and $y_f\in Y$ satisfying (1.1) and $c_f>0$ such that for each natural number n and each pair of sequences $\{x_i\}_{i=0}^n\subset X,\ \{y_i\}_{i=0}^n\subset Y$,

$$-c_f+\sum_{i=0}^{n-1}f(x_f,x_f,y_i,y_{i+1})\le nf(x_f,x_f,y_f,y_f)\le\sum_{i=0}^{n-1}f(x_i,x_{i+1},y_f,y_f)+c_f. \tag{1.4}$$

Remark 1.5. In view of Proposition 1.3 we may assume without loss of generality that $c_f=4\|f\|$. In the sequel we associate with each $f\in\mathcal{A}$ points $x_f\in X$ and $y_f\in Y$ satisfying (1.1) and such that (1.4) holds with $c_f=4\|f\|$ for each natural number n and each pair of sequences $\{x_i\}_{i=0}^n\subset X,\ \{y_i\}_{i=0}^n\subset Y$.

The following result is proved in Section 6.

Proposition 1.6. *$\mathcal{A}$ is a closed subset of the Banach space $(C(X\times X\times Y\times Y),\|\cdot\|)$.*

The next result is proved in Section 7.

Proposition 1.7. *Let $f\in\mathcal{A}$, $z_1,z_2\in X$, $\xi_1,\xi_2\in Y$, n be a natural number and let*

$$x_0=z_1,\ x_n=z_2,\ x_i=x_f \text{ for all integers } i \text{ satisfying } 0<i<n, \tag{1.5}$$

$$y_0=\xi_1,\ y_n=\xi_2,\ y_i=y_f \text{ for all integers } i \text{ satisfying } 0<i<n. \tag{1.6}$$

Then the pair of sequences $\{x_i\}_{i=0}^n,\{y_i\}_{i=0}^n$ is $(f,12\|f\|)$-good.

In this paper we establish a turnpike property of (f)-good pairs of sequences which means that they spend most of the time in a small neighborhood of the pair (x_f,y_f). It is known in the optimal control theory that turnpike properties of approximately optimal solutions are deduced from an asymptotic turnpike property of solutions of corresponding infinite horizon optimal control problems [37, 39, 40].

Let $f\in\mathcal{A}$. We say that f possesses the asymptotic turnpike property (or, briefly, (ATP)) if for each pair $\tilde{x}\in X$, $\tilde{y}\in Y$ satisfying

$$f(\tilde{x},\tilde{x},y,y)\le f(\tilde{x},\tilde{x},\tilde{y},\tilde{y})\le f(x,x,\tilde{y},\tilde{y}) \text{ for all } x\in X \text{ and all } y\in Y$$

we have $\tilde{x}=x_f$ and $\tilde{y}=y_f$ and if for each pair of sequences $\{x_i\}_{i=0}^\infty\subset X$, $\{y_i\}_{i=0}^\infty\subset Y$ satisfying

$$\sup\Big\{\sum_{i=0}^{n-1}f(x_i,x_{i+1},y_f,y_f)-nf(x_f,x_f,y_f,y_f):\ n \text{ is a natural number}\Big\}<\infty$$

and

$$\inf\Big\{\sum_{i=0}^{n-1}f(x_f,x_f,y_i,y_{i+1})-nf(x_f,x_f,y_f,y_f):\ n \text{ is a natural number}\Big\}>-\infty$$

we have

$$\lim_{i\to\infty}\rho_X(x_i,x_f)=0,\ \lim_{i\to\infty}\rho_Y(y_i,y_f)=0.$$

Let $f \in \mathcal{A}$. Denote by $S(f)$ the set of all pairs $(x,y) \in X \times Y$ such that

$$f(x,x,\xi,\xi) \le f(x,x,y,y) \le f(z,z,y,y) \text{ for all } z \in X \text{ and all } \xi \in Y. \tag{1.7}$$

Clearly, $S(f) \neq \emptyset$ and for all $(x_1,y_1),\ (x_2,y_2) \in S(f)$,

$$f(x_1,x_1,y_1,y_1) = f(x_2,x_2,y_2,y_2). \tag{1.8}$$

We consider the topological subspace $\mathcal{A} \subset C(X \times X \times Y \times Y)$ with the relative topology induced by the metric $d(f,g) = \|f-g\|$, $f,g \in \mathcal{A}$.

The following two theorems are the main results of the paper.

Theorem 1.8. *Let f possess (ATP) and $M, \epsilon > 0$. Then there exist natural numbers l and Q and a positive number δ such that for each $g \in \mathcal{A}$ satisfying $\|f-g\| \le \delta$, each integer $T > Ql$ and each (g,M)-good pair of of sequences $\{x_i\}_{i=0}^T \subset X$, $\{y_i\}_{i=0}^T \subset Y$ there exist a natural number $q \le Q$ and sequences of integers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0,T]$ such that*

$$0 \le b_i - a_i \le l,\ i = 1,\dots,q,$$

$$\rho_X(x_i,x_f) \le \epsilon,\ \rho_Y(y_i,y_f) \le \epsilon$$

for all integers $i \in [0,T] \setminus \cup_{j=1}^q [a_j,b_j]$.

Note that Theorem 1.8 shows that the turnpike phenomenon is stable under small perturbations of the objective function f.

Theorem 1.9. *There exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{A}$ such that each $f \in \mathcal{F}$ possesses (ATP).*

Theorem 1.9 shows that a generic (typical) function $f \in \mathcal{A}$ possesses (ATP). Results of this kind for classes of single-player control systems have been established in [36, 37, 39]. Note that the generic approach of [36, 37, 39] is not limited to the turnpike property, but is also applicable to other problems in optimization and nonlinear analysis [4, 30-32, 41].

Theorem 1.8 is proved in Section 9 while the proof of Theorem 1.9 is given in Section 10. Section 8 contains auxiliary results.

2. Proof of Proposition 1.1

Let $\{f_n\}_{n=1}^\infty \subset \mathcal{A}_0$, $f \in C(X \times X \times Y \times Y)$ and

$$\lim_{n\to\infty} \|f_n - f\| = 0. \tag{2.1}$$

For each integer $n \ge 1$ there exist $x_n \in X$ and $y_n \in Y$ such that for each $x \in X$ and each $y \in Y$,

$$f_n(x_n,x_n,y,y) \le f_n(x_n,x_n,y_n,y_n) \le f_n(x,x,y_n,y_n). \tag{2.2}$$

Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that there exist

$$x_* = \lim_{n\to\infty} x_n,\ y_* = \lim_{n\to\infty} y_n. \tag{2.3}$$

By (2.1), (2.2) and (2.3), for each $x \in X$ and each $y \in Y$,

$$f(x_*,x_*,y,y) = \lim_{n\to\infty} f(x_n,x_n,y,y) = \lim_{n\to\infty} f_n(x_n,x_n,y,y)$$

$$\leq \lim_{n\to\infty} f_n(x_n, x_n, y_n, y_n) = \lim_{n\to\infty} f(x_n, x_n, y_n, y_n) = f(x_*, x_*, y_*, y_*),$$

$$\begin{aligned} f(x_*, x_*, y_*, y_*) &= \lim_{n\to\infty} f_n(x_n, x_n, y_n, y_n) \leq \lim_{n\to\infty} f_n(x, x, y_n, y_n) \\ &= \lim_{n\to\infty} f(x, x, y_n, y_n) = f(x, x, y_*, y_*) \end{aligned}$$

and

$$f(x_*, x_*, y, y) \leq f(x_*, x_*, y_*, y_*) \leq f(x, x, y_*, y_*).$$

Thus $f \in \mathcal{A}_0$ and Proposition 1.1 is proved.

3. Proof of Proposition 1.2

Clearly, if the pair $\{\bar{x}_i^f\}_{i=0}^\infty$, $\{\bar{y}_i^f\}_{i=0}^\infty$ is (f)-optimal it is also (f)-good.

Assume that the pair $\{\bar{x}_i^f\}_{i=0}^\infty$, $\{\bar{y}_i^f\}_{i=0}^\infty$ is (f)-good and show that it is (f)-optimal. There is $M > 0$ such that the pair $\{\bar{x}_i^f\}_{i=0}^\infty$, $\{\bar{y}_i^f\}_{i=0}^\infty$ is (f, M)-good.

Let n be a natural number. In order to complete the proof of the proposition it is sufficient to show that the pair $\{\bar{x}_i^f\}_{i=0}^n$, $\{\bar{y}_i^f\}_{i=0}^n$ is (f)-optimal.

Assume that

$$\{z_i\}_{i=0}^n \subset X,\ \{\xi_i\}_{i=0}^n \subset Y,\ z_0 = z_n = x_f,\ \xi_0 = \xi_n = y_f. \tag{3.1}$$

There exist $\{z_i\}_{i=n+1}^\infty \subset X$, $\{\xi_i\}_{i=n+1}^\infty \subset Y$ such that

$$z_{i+n} = z_i,\ \xi_{i+n} = \xi_i \text{ for all integers } i \geq 0. \tag{3.2}$$

Since the pair of sequences $\{\bar{x}_i^f\}_{i=0}^\infty$, $\{\bar{y}_i^f\}_{i=0}^\infty$ is (f, M)-good it follows from (1.2), (1.3), (3.1) and (3.2) that for any natural number k,

$$\begin{aligned} M &\geq \sum_{i=0}^{kn-1} [f(x_f, x_f, y_f, y_f) - f(z_i, z_i, y_f, y_f)] \\ &= k\sum_{i=0}^{n-1}[f(x_f, x_f, y_f, y_f) - f(z_i, z_i, y_f, y_f)], \end{aligned}$$

$$\sum_{i=0}^{n-1} f(z_i, z_{i+1}, y_f, y_f) \geq nf(x_f, x_f, y_f, y_f)$$

and

$$\begin{aligned} M &\geq \sum_{i=0}^{kn-1} [f(x_f, x_f, \xi_i, \xi_{i+1}) - f(x_f, x_f, y_f, y_f)] \\ &= k\sum_{i=0}^{n-1}[f(x_f, x_f, \xi_i, \xi_{i+1}) - f(x_f, x_f, y_f, y_f)], \end{aligned}$$

$$\sum_{i=0}^{n-1} f(x_f, x_f, \xi_i, \xi_{i+1}) \leq nf(x_f, x_f, y_f, y_f).$$

Thus the pair of sequences $\{\bar{x}_i^f\}_{i=0}^n$, $\{\bar{y}_i^f\}_{i=0}^n$ is (f)-optimal. Proposition 1.2 is proved.

4. Proof of Proposition 1.3

We show that (i) implies (ii). Assume that property (i) holds, n be a natural number and $\{x_i\}_{i=0}^n \subset X,\ \{y_i\}_{i=0}^n \subset Y$. Set

$$x'_0 = x_f,\ x'_i = x_{i-1},\ i = 1, \dots, n+1,\ x'_{n+2} = x_f, \tag{4.1}$$

$$y'_0 = y_f,\ y'_i = y_{i-1},\ i = 1, \dots, n+1,\ y'_{n+2} = y_f.$$

By property (i) and (4.1),

$$\begin{aligned}(n+2)f(x_f, x_f, y_f, y_f) &= \sum_{i=0}^{n+1} f(\bar{x}_i^f, \bar{x}_{i+1}^f, \bar{y}_i^f, \bar{y}_{i+1}^f)\\ &\le \sum_{i=0}^{n+1} f(x'_i, x'_{i+1}, y_f, y_f)\\ &\le \sum_{i=0}^{n+1} f(x_i, x_{i+1}, y_f, y_f) + 2\|f\|,\end{aligned}$$

$$\sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) \ge nf(x_f, x_f, y_f, y_f) - 4\|f\|,$$

$$\begin{aligned}(n+2)f(x_f, x_f, y_f, y_f) &= \sum_{i=0}^{n+1} f(\bar{x}_i^f, \bar{x}_{i+1}^f, \bar{y}_i^f, \bar{y}_{i+1}^f)\\ &\ge \sum_{i=0}^{n+1} f(x_f, x_f, y'_i, y'_{i+1})\\ &\ge \sum_{i=0}^{n-1} f(x_f, x_f, y_i, y_{i+1}) - 2\|f\|,\end{aligned}$$

$$\sum_{i=0}^{n-1} f(x_f, x_f, y_i, y_{i+1}) \le nf(x_f, x_f, y_f, y_f) + 4\|f\|.$$

Thus property (ii) holds.

Clearly, property (ii) implies property (iii).

Assume that property (iii) holds. In order to complete the proof of the proposition it is sufficient to show that property (i) holds. Let n be a natural number. We show that the pair $\{\bar{x}_i^f\}_{i=0}^n,\ \{\bar{y}_i^f\}_{i=0}^n$ is (f)-optimal.

Assume that

$$\{x_i\}_{i=0}^n \subset X,\ \{y_i\}_{i=0}^n \subset Y,\ x_0 = x_n = x_f,\ y_0 = y_n = y_f. \tag{4.2}$$

There exist $\{x_i\}_{i=n+1}^\infty \subset X,\ \{y_i\}_{i=n+1}^\infty \subset Y$ such that for each integer $i \ge 0$,

$$x_{i+n} = x_i,\ y_{i+n} = y_i. \tag{4.3}$$

It follows from property (iii), (4.2) and (4.3) that for any natural number k,

$$-c + k\sum_{i=0}^{n-1} f(x_f, x_f, y_i, y_{i+1}) = -c + \sum_{i=0}^{nk-1} f(x_f, x_f, y_i, y_{i+1}) \le nkf(x_f, x_f, y_f, y_f)$$

$$\leq \sum_{i=0}^{nk-1} f(x_i, x_{i+1}, y_f, y_f) + c = k \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) + c,$$

$$-c/k + \sum_{i=0}^{n-1} f(x_f, x_f, y_i, y_{i+1}) \leq nf(x_f, x_f, y_f, y_f) \leq \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f) + c/k.$$

Since k is any natural number we conclude that

$$\sum_{i=0}^{n-1} f(x_f, x_f, y_i, y_{i+1}) \leq nf(x_f, x_f, y_f, y_f) \leq \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_f, y_f)$$

and that $\{\bar{x}_i^f\}_{i=0}^n$, $\{\bar{y}_i^f\}_{i=0}^n$ is an (f)-optimal pair of sequences. Proposition 1.3 is proved.

5. Good pairs of sequences

Proposition 5.1. *Let $f \in \mathcal{A}_0$, $x_f \in X$ and $y_f \in Y$ satisfy*

(5.1)

$$f(x_f, x_f, y, y) \leq f(x_f, x_f, y_f, y_f) \leq f(x, x, y_f, y_f) \text{ for all } x \in X \text{ and all } y \in Y$$

and let $c > 0$ be such that for each natural number k and each pair of sequences $\{x_i\}_{i=0}^k \subset X$, $\{y_i\}_{i=0}^k \subset Y$,

$$(5.2) \quad -c + \sum_{i=0}^{k-1} f(x_f, x_f, y_i, y_{i+1}) \leq kf(x_f, x_f, y_f, y_f) \leq \sum_{i=0}^{k-1} f(x_i, x_{i+1}, y_f, y_f) + c.$$

Let $M \geq 0$, n be a natural number and $\{u_i\}_{i=0}^n \subset X$, $\{v_i\}_{i=0}^n \subset Y$ be an (f, M)-good pair of sequences. Then

$$\Big| \sum_{i=0}^{n-1} f(u_i, u_{i+1}, y_f, y_f) - nf(x_f, x_f, y_f, y_f) \Big| \leq c + 2M + 8\|f\|,$$

$$\Big| \sum_{i=0}^{n-1} f(x_f, x_f, v_i, v_{i+1}) - nf(x_f, x_f, y_f, y_f) \Big| \leq c + 2M + 8\|f\|.$$

Proof. By (5.2),

$$(5.3) \quad -c + \sum_{i=0}^{n-1} f(x_f, x_f, v_i, v_{i+1}) \leq nf(x_f, x_f, y_f, y_f) \leq \sum_{i=0}^{n-1} f(u_i, u_{i+1}, y_f, y_f) + c.$$

Set

$$(5.4) \quad u'_0 = u_0,\ u'_n = u_n,\ u'_i = x_f \text{ for all integers } i \text{ satisfying } 0 < i < n,$$

$$v'_0 = v_0,\ v'_n = v_n,\ v'_i = y_f \text{ for all integers } i \text{ satisfying } 0 < i < n.$$

Since $\{u_i\}_{i=0}^n$, $\{v_i\}_{i=0}^n$ is an (f, M)-good pair of sequences it follows from (5.3) and (5.4) that

$$nf(x_f, x_f, y_f, y_f) - c - M - 4\|f\| \quad \leq \quad \sum_{i=0}^{n-1} f(u_i, u_{i+1}, y_f, y_f) - M - 4\|f\|$$

$$\begin{aligned}
&\le -M + \sum_{i=0}^{n-1} f(u_i, u_{i+1}, v'_i, v'_{i+1}) \\
&\le \sum_{i=0}^{n-1} f(u_i, u_{i+1}, v_i, v_{i+1}) \\
&\le \sum_{i=0}^{n-1} f(u'_i, u'_{i+1}, v_i, v_{i+1}) + M \\
&\le \sum_{i=0}^{n-1} f(x_f, x_f, v_i, v_{i+1}) + M + 4\|f\| \\
&\le nf(x_f, x_f, y_f, y_f) + c + M + 4\|f\|.
\end{aligned}$$

This implies that

$$\Big|\sum_{i=0}^{n-1} f(u_i, u_{i+1}, y_f, y_f) - nf(x_f, x_f, y_f, y_f)\Big| \le c + 2M + 8\|f\|,$$

$$\Big|\sum_{i=0}^{n-1} f(x_f, x_f, v_i, v_{i+1}) - nf(x_f, x_f, y_f, y_f)\Big| \le c + 2M + 8\|f\|.$$

Proposition 5.1 is proved. □

6. Proof of Proposition 1.5

Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{A}$, $f \in C(X \times X \times Y \times Y)$ and

$$\lim_{n\to\infty} \|f_n - f\| = 0. \tag{6.1}$$

By Proposition 1.1, $f \in \mathcal{A}_0$. By definition of $\mathcal{A}$ (see (1.4)), Proposition 1.3 and Remark 1.5, for any natural number n there exist $x_{f_n} \in X$ and $y_{f_n} \in Y$ such that

$$\begin{aligned}
f_n(x_{f_n}, x_{f_n}, y, y) &\le f_n(x_{f_n}, x_{f_n}, y_{f_n}, y_{f_n}) \\
&\le f_n(x, x, y_{f_n}, y_{f_n}) \text{ for all } x \in X \text{ and all } y \in Y
\end{aligned} \tag{6.2}$$

and that for each natural number k and each pair of sequences $\{x_i\}_{i=0}^{k} \subset X$, $\{y_i\}_{i=0}^{k} \subset Y$,

$$\begin{aligned}
-4\|f_n\| + \sum_{i=0}^{k-1} f_n(x_{f_n}, x_{f_n}, y_i, y_{i+1}) &\le kf_n(x_{f_n}, x_{f_n}, y_{f_n}, y_{f_n}) \\
&\le \sum_{i=0}^{k-1} f_n(x_i, x_{i+1}, y_{f_n}, y_{f_n}) + 4\|f_n\|.
\end{aligned} \tag{6.3}$$

Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that there exist

$$x^* = \lim_{n\to\infty} x_{f_n},\ y^* = \lim_{n\to\infty} y_{f_n}. \tag{6.4}$$

By (6.1), (6.2) and (6.4), for each $x \in X$ and each $y \in Y$,

$$f(x^*, x^*, y, y) = \lim_{n\to\infty} f(x_{f_n}, x_{f_n}, y, y) = \lim_{n\to\infty} f_n(x_{f_n}, x_{f_n}, y, y)$$

$$
\begin{aligned}
&\le \lim_{n\to\infty} f_n(x_{f_n}, x_{f_n}, y_{f_n}, y_{f_n}) = \lim_{n\to\infty} f(x_{f_n}, x_{f_n}, y_{f_n}, y_{f_n})\\
&= f(x^*, x^*, y^*, y^*) = \lim_{n\to\infty} f_n(x_{f_n}, x_{f_n}, y_{f_n}, y_{f_n})\\
&\le \lim_{n\to\infty} f_n(x, x, y_{f_n}, y_{f_n}) = \lim_{n\to\infty} f(x, x, y_{f_n}, y_{f_n})\\
&= f(x, x, y^*, y^*)
\end{aligned}
$$

and

$$f(x^*, x^*, y, y) \le f(x^*, x^*, y^*, y^*) \le f(x, x, y^*, y^*) \text{ for all } x \in X \text{ and all } y \in Y.$$

Let k be a natural number and $\{x_i\}_{i=0}^k \subset X$, $\{y_i\}_{i=0}^k \subset Y$. By (6.1), (6.3) and (6.4),

$$
\begin{aligned}
-4\|f\| + \sum_{i=0}^{k-1} f(x^*, x^*, y_i, y_{i+1}) &= \lim_{n\to\infty} [-4\|f_n\| + \sum_{i=0}^{k-1} f(x_{f_n}, x_{f_n}, y_i, y_{i+1})]\\
&= \lim_{n\to\infty} [-4\|f_n\| + \sum_{i=0}^{k-1} f_n(x_{f_n}, x_{f_n}, y_i, y_{i+1})]\\
&\le \lim_{n\to\infty} k f_n(x_{f_n}, x_{f_n}, y_{f_n}, y_{f_n})\\
&= \lim_{n\to\infty} k f(x_{f_n}, x_{f_n}, y_{f_n}, y_{f_n}) = k f(x^*, x^*, y^*, y^*)\\
&= \lim_{n\to\infty} k f_n(x_{f_n}, x_{f_n}, y_{f_n}, y_{f_n})\\
&\le \lim_{n\to\infty} [4\|f_n\| + \sum_{i=0}^{k-1} f_n(x_i, x_{i+1}, y_{f_n}, y_{f_n})]\\
&= 4\|f\| + \lim_{n\to\infty} \sum_{i=0}^{k-1} f(x_i, x_{i+1}, y_{f_n}, y_{f_n})\\
&= 4\|f\| + \sum_{i=0}^{k-1} f(x_i, x_{i+1}, y^*, y^*).
\end{aligned}
$$

Thus $f \in \mathcal{A}$ with $x_f = x^*$ and $y_f = y^*$ and Proposition 1.6 is proved.

7. Proof of Proposition 1.7

By (1.4), (1.5) and (1.6), each pair of sequences $\{u_i\}_{i=0}^n \subset X$, $\{v_i\}_{i=0}^n \subset Y$,

$$
\begin{aligned}
&\sum_{i=0}^{n-1} f(u_i, u_{i+1}, y_i, y_{i+1}) - \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_i, y_{i+1})\\
&\quad \ge \sum_{i=0}^{n-1} f(u_i, u_{i+1}, y_f, y_f) - 4\|f\| - n f(x_f, x_f, y_f, y_f) - 4\|f\|\\
&\quad \ge -12\|f\|,
\end{aligned}
$$

$$\sum_{i=0}^{n-1} f(x_i, x_{i+1}, v_i, v_{i+1}) - \sum_{i=0}^{n-1} f(x_i, x_{i+1}, y_i, y_{i+1})$$
$$\leq \sum_{i=0}^{n-1} f(x_f, x_f, v_i, v_{i+1}) + 4\|f\| - nf(x_f, x_f, y_f, y_f) + 4\|f\|$$
$$\leq 12\|f\|.$$

Proposition 1.7 is proved.

8. Auxiliary results

Assume that $f \in \mathcal{A}$ possesses (ATP). We suppose that the sum over empty set is zero.

Lemma 8.1. *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $g \in \mathcal{A}_0$ satisfying $\|g - f\| \leq \delta$, each $x \in X$ and each $y \in Y$ satisfying*

$$g(x, x, \xi, \xi) \leq g(x, x, y, y) \leq g(z, z, y, y) \text{ for all } z \in X \text{ and all } \xi \in Y$$

the inequalities

$$\rho_X(x, x_f) \leq \epsilon, \ \rho_Y(y, y_f) \leq \epsilon$$

hold.

Proof. Assume the contrary. Then there exist sequences $\{g_n\}_{n=1}^{\infty} \subset \mathcal{A}_0$, $\{x_n\}_{n=1}^{\infty} \subset X$ and $\{y_n\}_{n=1}^{\infty} \subset Y$ such that

$$\lim_{n\to\infty} \|f - g_n\| = 0 \tag{8.1}$$

and for all integers $n \geq 1$,

$$g_n(x_n, x_n, \xi, \xi) \leq g_n(x_n, x_n, y_n, y_n) \leq g_n(z, z, y_n, y_n) \text{ for all } z \in X \text{ and all } \xi \in Y, \tag{8.2}$$

$$\rho_X(x_n, x_f) + \rho_Y(y_n, y_f) \geq \epsilon. \tag{8.3}$$

Extracting a subsequence and re-indexing, if necessary, we may assume without loss of generality that there exist $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$. Arguing as in the proof of Proposition 1.6 and using (8.1) and (8.2) we obtain that for all $z \in X$ and all $\xi \in Y$,

$$\begin{aligned} f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_n, \xi, \xi) &\leq f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n, \lim_{n\to\infty} y_n) \\ &\leq f(z, z, \lim_{n\to\infty} y_n, \lim_{n\to\infty} y_n). \end{aligned}$$

Since f possesses (ATP) we conclude that $\lim_{n\to\infty} x_n = x_f$ and $\lim_{n\to\infty} y_n = y_f$. This contradicts (8.3). The contradiction we have reached proves Lemma 8.1. □

Lemma 8.2. *Let $M, \epsilon > 0$ and τ_0 be a natural number. Then there exists an integer $\tau > \tau_0$ such that for each integer $T \geq \tau$, each sequence $\{x_i\}_{i=0}^{T} \subset X$ satisfying*

$$\sum_{i=0}^{T-1} f(x_i, x_{i+1}, y_f, y_f) \leq Tf(x_f, x_f, y_f, y_f) + M$$

and each integer $s \in [0, T-\tau]$ *there is an integer* s_0 *such that* $[s_0, s_0+\tau_0] \subset [s, s+\tau]$ *and*

$$\rho_X(x_i, x_f) \le \epsilon,\ i = s_0, \dots, s_0 + \tau_0.$$

Proof. Let us assume the contrary. Then for each integer $k > \tau_0$ there exist and integer $T_k \ge k$, a sequence $\{x_i^{(k)}\}_{i=0}^{T_k} \subset X$ satisfying

$$\sum_{i=0}^{T_k-1} f(x_i^{(k)}, x_{i+1}^{(k)}, y_f, y_f) \le T_k f(x_f, x_f, y_f, y_f) + M, \tag{8.4}$$

and an integer $s_k \in [0, T_k - k]$ such that the following property holds:

(P1) for each integer p satisfying $[p, p+\tau_0] \subset [s_k, s_k + k]$ we have

$$\max\{\rho_X(x_i^{(k)}, x_f) :\ i = p, \dots, p + \tau_0\} > \epsilon. \tag{8.5}$$

For each integer $k > \tau_0$ set

$$u_i^{(k)} = x_{i+s_k}^{(k)},\ i = 0, \dots, k. \tag{8.6}$$

By (8.6) and (P1), for each integer $k > \tau_0$ the following property holds:

(P2) for each integer p satisfying $[p, p+\tau_0] \subset [0, k]$ we have

$$\max\{\rho_X(u_i^{(k)}, x_f) :\ i = p, \dots, p + \tau_0\} > \epsilon. \tag{8.7}$$

Let $k > \tau_0$ be an integer. By (8.4), (8.6) and the definition of $\mathcal{A}$ (see (1.4)),

$$\begin{aligned}\sum_{i=0}^{k-1} f(u_i^{(k)}, u_{i+1}^{(k)}, y_f, y_f) &= \sum_{i=s_k}^{s_k+k-1} f(x_i^{(k)}, x_{i+1}^{(k)}, y_f, y_f) = \sum_{i=0}^{T_k-1} f(x_i^{(k)}, x_{i+1}^{(k)}, y_f, y_f)\\ &\quad - \sum\{f(x_i^{(k)}, x_{i+1}^{(k)}, y_f, y_f) :\ i \text{ is an integer such that } 0 \le i < s_k\}\\ &\quad - \sum\{f(x_i^{(k)}, x_{i+1}^{(k)}, y_f, y_f) :\ i \text{ is an integer such that } s_k + k \le i < T\}\\ &\le T_k f(x_f, x_f, y_f, y_f) + M - s_k f(x_f, x_f, y_f, y_f) + c_f\\ &\quad -(T_k - s_k - k) f(x_f, x_f, y_f, y_f) + c_f = k f(x_f, x_f, y_f, y_f) + M + 2c_f.\end{aligned} \tag{8.8}$$

Let $q > \tau_0$ be an integer. By (8.8) and the definition of $\mathcal{A}$ (see (1.4)), for each integer $k > q$,

$$\begin{aligned}\sum_{i=0}^{q-1} f(u_i^{(k)}, u_{i+1}^{(k)}, y_f, y_f) &= \sum_{i=0}^{k-1} f(u_i^{(k)}, u_{i+1}^{(k)}, y_f, y_f) - \sum_{i=q}^{k-1} f(u_i^{(k)}, u_{i+1}^{(k)}, y_f, y_f)\\ &\le k f(x_f, x_f, y_f, y_f) + M + 2c_f - (k-q) f(x_f, x_f, y_f, y_f)\\ &\quad + c_f = q f(x_f, x_f, y_f, y_f) + M + 3c_f.\end{aligned} \tag{8.9}$$

Clearly, there exists a strictly increasing sequence of natural numbers $\{k_j\}_{j=1}^{\infty}$ with $k_1 > \tau_0$ such that for each integer $i \ge 0$, the sequence $\{u_i^{(k_j)}\}_{j=1}^{\infty}$ converges. For each integer $i \ge 0$ set

$$u_i = \lim_{j \to \infty} u_i^{(k_j)}. \tag{8.10}$$

By (8.9) and (8.10) for each integer $q > \tau_0$,

$$\sum_{i=0}^{q-1} f(u_i, u_{i+1}, y_f, y_f) \leq q f(x_f, x_f, y_f, y_f) + M + 3c_f$$

and combined with (ATP) this implies that $\lim_{i\to\infty} u_i = x_f$. Clearly, there is an integer $p > 1$ such that

$$\rho_X(u_i, x_f) \leq \epsilon/4 \text{ for all integers } i \geq p. \tag{8.11}$$

By (8.10) there is an integer $j \geq 1$ such that

$$k_j > p + \tau_0 + 4,$$

$$\rho_X(u_i, u_i^{(k_j)}) \leq \epsilon/4,\ i = 0, \dots, p + \tau_0 + 4. \tag{8.12}$$

By (8.11) and (8.12) for all $i = p, \dots, p + \tau_0$,

$$\rho_X(u_i^{(k_j)}, x_f) \leq \rho_X(u_i^{(k_j)}, u_i) + \rho_X(u_i, x_f) \leq \epsilon/2.$$

This contradicts (P2). The contradiction we have reached proves Lemma 8.2. □

Analogously to Lemma 8.2 we can prove the following auxiliary result.

Lemma 8.3. *Let $M, \epsilon > 0$ and τ_0 be a natural number. Then there exists an integer $\tau > \tau_0$ such that for each integer $T \geq \tau$, each sequence $\{y_i\}_{i=0}^T \subset Y$ satisfying*

$$\sum_{i=0}^{T-1} f(x_f, x_f, y_i, y_{i+1}) \geq T f(x_f, x_f, y_f, y_f) - M$$

and each integer $s \in [0, T - \tau]$ there is an integer s_0 such that $[s_0, s_0 + \tau_0] \subset [s, s + \tau]$ and

$$\rho_Y(y_i, y_f) \leq \epsilon,\ i = s_0, \dots, s_0 + \tau_0.$$

Lemma 8.4. *Let $M, \epsilon > 0$. Then there exists and integer $\tau_1 > 1$ such that for each integer $T \geq \tau_1$, each pair of sequences $\{x_i\}_{i=0}^T \subset X$ and $\{y_i\}_{i=0}^T \subset Y$ satisfying*

$$\sum_{i=0}^{T-1} f(x_i, x_{i+1}, y_f, y_f) \leq T f(x_f, x_f, y_f, y_f) + M, \tag{8.13}$$

$$\sum_{i=0}^{T-1} f(x_f, x_f, y_i, y_{i+1}) \geq T f(x_f, x_f, y_f, y_f) - M \tag{8.14}$$

and each integer $s \in [0, T - \tau_1]$ there is an integer s_0 such that

$$\{s_0, s_0 + 1\} \subset [s, s + \tau_1],$$

$$\rho_X(x_i, x_f) \leq \epsilon \text{ and } \rho_Y(y_i, y_f) \leq \epsilon,\ i = s_0, s_0 + 1.$$

Proof. By Lemma 8.3 (with $\tau_0 = 1$) there exists an integer $\tau_0 > 1$ such that the following property holds:

(P3) for each integer $T \geq \tau_0$, each sequence $\{y_i\}_{i=0}^T \subset Y$ satisfying

$$\sum_{i=0}^{T-1} f(x_f, x_f, y_i, y_{i+1}) \geq Tf(x_f, x_f, y_f, y_f) - M - 2c_f$$

and each integer $s \in [0, T-\tau_0]$ there is an integer s_0 such that $\{s_0, s_0+1\} \subset [s, s+\tau_0]$ and

$$\rho_Y(y_i, y_f) \leq \epsilon, \ i = s_0, s_0 + 1.$$

By Lemma 8.2 there is an integer $\tau_1 > \tau_0$ such that the following property holds:

(P4) for each integer $T \geq \tau_1$, each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying

$$\sum_{i=0}^{T-1} f(x_i, x_{i+1}, y_f, y_f) \leq Tf(x_f, x_f, y_f, y_f) + M$$

and each integer $s \in [0, T-\tau_1]$ there is an integer s_0 such that $[s_0, s_0+\tau_0] \subset [s, s+\tau_1]$ and

$$\rho_X(x_i, x_f) \leq \epsilon, \ i = s_0, \dots, s_0 + \tau_0.$$

Assume that an integer $T \geq \tau_1$, $\{x_i\}_{i=0}^T \subset X$, $\{y_i\}_{i=0}^T \subset Y$, (8.13) and (8.14) hold and that an integer $s \in [0, T - \tau_1]$. By (8.13) and (P4), there is an integer q_0 such that

$$[q_0, q_0 + \tau_0] \subset [s, s + \tau_1], \tag{8.15}$$

$$\rho_X(x_i, x_f) \leq \epsilon, \ i = q_0, \dots, q_0 + \tau_0.$$

By (8.14) and the definition of $\mathcal{A}$ (see (1.4)),

$$\begin{aligned}
\sum_{i=q_0}^{q_0+\tau_0-1} f(x_f, x_f, y_i, y_{i+1}) &= \sum_{i=0}^{T-1} f(x_f, x_f, y_i, y_{i+1}) \\
&\quad - \sum \{f(x_f, x_f, y_i, y_{i+1}) : \ i \text{ is an integer such that } 0 \leq i < q_0\} \\
&\quad - \sum \{f(x_f, x_f, y_i, y_{i+1}) : \ i \text{ is an integer such that } q_0 + \tau_0 \leq i < T\} \\
&\geq Tf(x_f, x_f, y_f, y_f) - M - q_0 f(x_f, x_f, y_f, y_f) - c_f \\
&\quad - (T - q_0 - \tau_0) f(x_f, x_f, y_f, y_f) - c_f = \tau_0 f(x_f, x_f, y_f, y_f) - M - 2c_f.
\end{aligned} \tag{8.16}$$

By (8.16) and (P3) (with $T = \tau_0$) there is an integer s_0 such that $\{s_0, s_0 + 1\} \subset [q_0, q_0 + \tau_0]$ and

$$\rho_Y(y_i, y_f) \leq \epsilon, \ i = s_0, s_0 + 1.$$

This completes the proof of Lemma 8.4. □

Lemma 8.4 and Proposition 5.1 imply the following result.

Lemma 8.5. *Let $M, \epsilon > 0$. Then there exists and integer $\tau_1 > 1$ such that for each integer $T \geq \tau_1$, each (f, M)-good pair of sequences $\{x_i\}_{i=0}^T \subset X$ and $\{y_i\}_{i=0}^T \subset Y$ and each integer $s \in [0, T - \tau_1]$ there is an integer s_0 such that*

$$\{s_0, s_0 + 1\} \subset [s, s + \tau_1],$$

$$\rho_X(x_i, x_f) \leq \epsilon \text{ and } \rho_Y(y_i, y_f) \leq \epsilon,\ i = s_0, s_0 + 1.$$

Lemma 8.6. *Let $M, \epsilon > 0$. Then there exist and integer $\tau > 1$ and $\delta > 0$ such that for each $g \in C(X \times X \times Y \times Y)$ satisfying $\|g - f\| \leq \delta$ and each (g, M)-good pair of sequences $\{x_i\}_{i=0}^\tau \subset X$ and $\{y_i\}_{i=0}^\tau \subset Y$ there is an integer s_0 such that*

$$\{s_0, s_0 + 1\} \subset [0, \tau], \tag{8.17}$$

$$\rho_X(x_i, x_f) \leq \epsilon \text{ and } \rho_Y(y_i, y_f) \leq \epsilon,\ i = s_0, s_0 + 1.$$

Proof. By Lemma 8.5 there is an integer $\tau > 1$ such that the following property holds:

(P5) for each $(f, M + 1)$-good pair of sequences $\{x_i\}_{i=0}^\tau \subset X$ and $\{y_i\}_{i=0}^\tau \subset Y$ there is an integer s_0 such that (8.17) holds.

Set

$$\delta = (8\tau)^{-1}. \tag{8.18}$$

Let $g \in C(X \times X \times Y \times Y)$ satisfy

$$\|g - f\| \leq \delta \tag{8.19}$$

and let $\{x_i\}_{i=0}^\tau \subset X$ and $\{y_i\}_{i=0}^\tau \subset Y$ be an (g, M)-good pair of sequences. By (8.18) and (8.19), $\{x_i\}_{i=0}^\tau$ and $\{y_i\}_{i=0}^\tau$ is an $(f, M + 1)$-good pair of sequences. Together with (P5) this implies that there exists an integer s_0 such that (8.17) holds. Lemma 8.6 is proved. □

Lemma 8.6 implies the following result.

Lemma 8.7. *Let $M, \epsilon > 0$. Then there exist an integer $\tau > 1$ and $\delta > 0$ such that for each $g \in C(X \times X \times Y \times Y)$ satisfying $\|g - f\| \leq \delta$, each integer $T \geq \tau$, each (g, M)-good pair of sequences $\{x_i\}_{i=0}^T \subset X$ and $\{y_i\}_{i=0}^T \subset Y$ and each integer $s \in [0, T - \tau]$ there is an integer s_0 such that*

$$\{s_0, s_0 + 1\} \subset [s, s + \tau],$$

$$\rho_X(x_i, x_f) \leq \epsilon \text{ and } \rho_Y(y_i, y_f) \leq \epsilon,\ i = s_0, s_0 + 1.$$

Lemma 8.8. *Let $M, \epsilon > 0$. Then there exist an integer $\tau > 1$ and $\delta > 0$ such that for each $g \in \mathcal{A}$ satisfying $\|g - f\| \leq \delta$, each $(z, \xi) \in S(g)$ and each sequence $\{x_i\}_{i=0}^\tau \subset X$ satisfying*

$$\sum_{i=0}^{\tau-1} g(x_i, x_{i+1}, \xi, \xi) \leq \tau g(z, z, \xi, \xi) + M \tag{8.20}$$

there is an integer $s_0 \in [0, \tau - 1]$ such that

$$\rho_X(x_i, x_f) \leq \epsilon,\ i = s_0, s_0 + 1. \tag{8.21}$$

Proof. By Lemma 8.2 (with $\tau_0 = 1$) there exists an integer $\tau > 1$ such that the following property holds:

(P6) for each sequence $\{x_i\}_{i=0}^{\tau} \subset X$ satisfying

$$\sum_{i=0}^{\tau-1} f(x_i, x_{i+1}, y_f, y_f) \le \tau f(x_f, x_f, y_f, y_f) + M + 4$$

there is an integer $s_0 \in [0, \tau - 1]$ such that (8.21) holds.

Since the function f is uniformly continuous on the space $X \times X \times Y \times Y$ there is $\delta_1 > 0$ such that

$$|f(u_1, u_2, v_1, v_2) - f(u_1', u_2', v_1', v_2')| \le (8\tau)^{-1} \tag{8.22}$$

for all $u_1, u_2, u_1', u_2' \in X$ and all $v_1, v_2, v_1', v_2' \in Y$ satisfying $\rho_X(u_i, u_i') \le \delta_1$, $\rho_Y(v_i, v_i') \le \delta_1$, $i = 1, 2$.

By Lemma 8.1 there exists

$$\delta \in (0, (8\tau)^{-1}) \tag{8.23}$$

such that for each $g \in \mathcal{A}$ satisfying $\|g - f\| \le \delta$ and each $(z, \xi) \in S(g)$

$$\rho_X(z, x_f) \le \delta_1, \ \rho_Y(\xi, y_f) \le \delta_1. \tag{8.24}$$

Assume that $g \in \mathcal{A}$ satisfies

$$\|g - f\| \le \delta, \tag{8.25}$$

$$(z, \xi) \in S(g) \tag{8.26}$$

and that $\{x_i\}_{i=0}^{\tau} \subset X$ satisfies (8.20). By (8.20), (8.23) and (8.25),

$$\sum_{i=0}^{\tau-1} f(x_i, x_{i+1}, \xi, \xi) \le \tau f(z, z, \xi, \xi) + M + 2\delta\tau \le \tau f(z, z, \xi, \xi) + M + 1. \tag{8.27}$$

By (8.24), (8.25), (8.26) and the choice of δ (see (8.23)),

$$\rho_X(z, x_f) \le \delta_1, \ \rho_Y(\xi, y_f) \le \delta_1. \tag{8.28}$$

By (8.27), (8.28) and the choice of δ_1 (see (8.22)),

$$\begin{aligned} \sum_{i=0}^{\tau-1} f(x_i, x_{i+1}, y_f, y_f) &\le \sum_{i=0}^{\tau-1} f(x_i, x_{i+1}, \xi, \xi) + 8^{-1} \\ &\le \tau f(z, z, \xi, \xi) + M + 8^{-1} + 1 \\ &\le \tau f(x_f, x_f, y_f, y_f) + M + 2. \end{aligned} \tag{8.29}$$

By (8.29) and property (P6) there is an integer $s_0 \in [0, \tau - 1]$ such that (8.21) holds. Lemma 8.8 is proved. □

Analogously to Lemma 8.8 we can prove the following result.

Lemma 8.9. *Let $M, \epsilon > 0$. Then there exist an integer $\tau > 1$ and $\delta > 0$ such that for each $g \in \mathcal{A}$ satisfying $\|g - f\| \le \delta$, each $(z, \xi) \in S(g)$ and each sequence $\{y_i\}_{i=0}^{\tau} \subset Y$ satisfying*

$$\sum_{i=0}^{\tau-1} g(z, z, y_i, y_{i+1}) \ge \tau g(z, z, \xi, \xi) - M$$

there is an integer $s_0 \in [0, \tau - 1]$ such that

$$\rho_Y(y_i, y_f) \le \epsilon, \ i = s_0, s_0 + 1.$$

Let $g \in \mathcal{A}$, integers $T_2 > T_1 \ge 0$, $z_1, z_2 \in X$ and $\xi_1, \xi_2 \in Y$. Set

$$\sigma_X(g, T_1, T_2, z_1, z_2, \xi_1, \xi_2)$$
$$= \inf\Big\{ \sum_{i=T_1}^{T_2-1} g(x_i, x_{i+1}, \xi, \xi) : \{x_i\}_{i=T_1}^{T_2} \subset X, \ x_{T_1} = z_1, \ x_{T_2} = z_2 \Big\}, \tag{8.30}$$

$$\sigma_Y(g, T_1, T_2, z_1, z_2, \xi_1, \xi_2)$$
$$= \sup\Big\{ \sum_{i=T_1}^{T_2-1} g(z_1, z_2, y_i, y_{i+1}) : \{y_i\}_{i=T_1}^{T_2} \subset Y, \ y_{T_1} = \xi_1, \ y_{T_2} = \xi_2 \Big\}. \tag{8.31}$$

It is easy to see that the following result holds.

Lemma 8.10. *Let $g \in \mathcal{A}$, integers $T_2 > T_1 \ge 0$, $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in X$, $\xi_1, \xi_2, \tilde{\xi}_1, \tilde{\xi}_2 \in Y$. Then*

$$|\sigma_X(g, T_1, T_2, z_1, z_2, \xi_1, \xi_2) - \sigma_X(g, T_1, T_2, \tilde{z}_1, \tilde{z}_2, \xi_1, \xi_2)| \le 4\|g\|,$$

$$|\sigma_Y(g, T_1, T_2, z_1, z_2, \xi_1, \xi_2) - \sigma_Y(g, T_1, T_2, z_1, z_2, \tilde{\xi}_1, \tilde{\xi}_2)| \le 4\|g\|,$$

$$|\sigma_X(g, T_1, T_2, z_1, z_2, y_g, y_g) - (T_2 - T_1) g(x_g, x_g, y_g, y_g)| \le 4\|g\|,$$

$$|\sigma_Y(g, T_1, T_2, x_g, x_g, \xi_1, \xi_2) - (T_2 - T_1) g(x_g, x_g, y_g, y_g)| \le 4\|g\|.$$

Lemmas 8.8 and 8.10 imply the following result.

Lemma 8.11. *Let $M, \epsilon > 0$. Then there exist $\delta \in (0, 1)$ and an integer $\tau > 1$ such that for each $g \in \mathcal{A}$ satisfying $\|g - f\| \le \delta$, each $(x_g, y_g) \in S(g)$, each integer $T \ge \tau$, each sequence $\{x_i\}_{i=0}^{T} \subset X$ satisfying*

$$\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) \le \sigma_X(g, 0, T, x_0, x_T, y_g, y_g) + M$$

and each integer $p \in [0, T - \tau]$ there is an integer j such that $\{j, j+1\} \subset [p, p + \tau]$ and

$$\rho_X(x_i, x_f) \le \epsilon, \ i = j, j + 1.$$

Lemmas 8.9 and 8.11 imply the following result.

Lemma 8.12. *Let $M, \epsilon > 0$. Then there exist $\delta \in (0,1)$ and an integer $\tau > 1$ such that for each $g \in \mathcal{A}$ satisfying $\|g - f\| \leq \delta$, each $(x_g, y_g) \in S(g)$, each integer $T \geq \tau$, each sequence $\{y_i\}_{i=0}^T \subset Y$ satisfying*

$$\sum_{i=0}^{T-1} g(x_g, x_g, y_i, y_{i+1}) \geq \sigma_Y(g, 0, T, x_g, x_g, y_0, y_T) - M$$

and each integer $p \in [0, T - \tau]$ there is an integer j such that $\{j, j+1\} \subset [p, p+\tau]$ and

$$\rho_Y(y_i, y_f) \leq \epsilon, \ i = j, j+1.$$

It is easy to see that the following lemma holds.

Lemma 8.13. *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each pair of integers $T_2 > T_1 \geq 0$, each $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in X$ and each $\xi_1, \xi_2, \tilde{\xi}_1, \tilde{\xi}_2 \in Y$ satisfying*

$$\rho_X(z_i, \tilde{z}_i) \leq \delta, \ i = 1, 2, \ \rho_Y(\xi_i, \tilde{\xi}_i) \leq \delta, \ i = 1, 2$$

the following inequalities hold:

$$|\sigma_X(f, T_1, T_2, z_1, z_2, \xi_1, \xi_2) - \sigma_X(f, T_1, T_2, \tilde{z}_1, \tilde{z}_2, \xi_1, \xi_2)| \leq \epsilon,$$

$$|\sigma_Y(f, T_1, T_2, z_1, z_2, \xi_1, \xi_2) - \sigma_Y(f, T_1, T_2, z_1, z_2, \tilde{\xi}_1, \tilde{\xi}_2)| \leq \epsilon.$$

Lemma 8.14. *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each integer $T > 0$ and each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying*

$$x_0 = x_f, \ x_T = x_f,$$

$$\sum_{i=0}^{T-1} f(x_i, x_{i+1}, y_f, y_f) \leq \sigma_X(f, 0, T, x_f, x_f, y_f, y_f) + \delta$$

the following inequality holds:

$$\rho_X(x_i, x_f) \leq \epsilon, \ i = 0, \dots, T.$$

Proof. Assume the contrary. Then for each natural number k there exists a sequence $\{x_i^{(k)}\}_{i=0}^{T_k} \subset X$ where T_k is a natural number such that

$$x_0^{(k)} = x_f, \ x_{T_k}^{(k)} = x_f, \tag{8.32}$$

$$\sum_{i=0}^{T_k-1} f(x_i^{(k)}, x_{i+1}^{(k)}, y_f, y_f) \leq \sigma_X(f, 0, T_k, x_f, x_f, y_f, y_f) + 2^{-k}, \tag{8.33}$$

$$\max\{\rho_X(x_i^{(k)}, x_f) : \ i = 0, \dots, T_k\} > \epsilon. \tag{8.34}$$

There is a sequence $\{x_i\}_{i=0}^\infty \subset X$ such that

$$x_i = x_i^{(1)}, \ i = 0, \dots, T_1 \tag{8.35}$$

and for each integer $k \geq 1$,

$$x_{\sum_{i=1}^k T_i + j} = x_j^{(k+1)}, \ j = 1, \dots, T_{k+1}. \tag{8.36}$$

By (1.4) and Proposition 1.3, for any integer $k \geq 1$,

$$\sigma_X(f, 0, T_k, x_f, x_f, y_f, y_f) = T_k f(x_f, x_f, y_f, y_f). \tag{8.37}$$

It follows from (8.33), (8.35), (8.36) and (8.37) that

$$\begin{aligned} \sum \{ f(x_i, x_{i+1}, y_f, y_f) : \ i = 0, \dots, \sum_{j=1}^{k} T_j - 1 \} \ &\leq \ \sum_{j=1}^{k} (T_j f(x_f, x_f, y_f, y_f) + 2^{-j}) \\ &\leq \ \Big(\sum_{j=1}^{k} T_j \Big) f(x_f, x_f, y_f, y_f) + 1. \end{aligned} \tag{8.38}$$

In view (8.38) and (ATP), $\lim_{i \to \infty} \rho_X(x_i, x_f) = 0$. Then there is a natural number i_0 such that $\rho_X(x_i, x_f) \leq \epsilon/2$ for all integers $i \geq i_0$. Together with (8.35) and (8.36) this implies that for all natural number $k \geq i_0$,

$$\rho_X(x_i^{(k)}, x_f) \leq \epsilon/2, \ i = 0, \dots, T_k.$$

This contradicts (8.34). The contradiction we have reached proves Lemma 8.14. □

Analogously to Lemma 8.14 we can prove the following result.

Lemma 8.15. *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each integer $T > 0$ and each sequence $\{y_i\}_{i=0}^{T} \subset Y$ satisfying*

$$y_0 = y_f, \ y_T = y_f,$$

$$\sum_{i=0}^{T-1} f(x_f, x_f, y_i, y_{i+1}) \geq \sigma_Y(f, 0, T, x_f, x_f, y_f, y_f) - \delta$$

the following inequality holds:

$$\rho_Y(y_i, y_f) \leq \epsilon, \ i = 0, \dots, T.$$

It is easy to see that the following lemma holds.

Lemma 8.16. *Let $\epsilon > 0$ and τ be a natural number. Then there exists $\delta > 0$ such that for each $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in X$ and each $\xi_1, \xi_2, \tilde{\xi}_1, \tilde{\xi}_2 \in Y$ satisfying*

$$\rho_X(z_i, \tilde{z}_i) \leq \delta, \ i = 1, 2, \ \rho_Y(\xi_i, \tilde{\xi}_i) \leq \delta, \ i = 1, 2$$

the following inequalities hold:

$$|\sigma_X(f, 0, \tau, z_1, z_2, \xi_1, \xi_2) - \sigma_X(f, 0, \tau, \tilde{z}_1, \tilde{z}_2, \tilde{\xi}_1, \tilde{\xi}_2)| \leq \epsilon,$$

$$|\sigma_Y(f, 0, \tau, z_1, z_2, \xi_1, \xi_2) - \sigma_Y(f, 0, \tau, \tilde{z}_1, \tilde{z}_2, \tilde{\xi}_1, \tilde{\xi}_2)| \leq \epsilon.$$

Lemmas 8.13 and 8.14 imply the following result.

Lemma 8.17. *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each integer $T > 0$ and each sequence $\{x_i\}_{i=0}^{T} \subset X$ satisfying*

$$\rho_X(x_0, x_f) \leq \delta, \ \rho_X(x_T, x_f) \leq \delta,$$

$$\sum_{i=0}^{T-1} f(x_i, x_{i+1}, y_f, y_f) \leq \sigma_X(f, 0, T, x_0, x_T, y_f, y_f) + \delta$$

the following inequality holds:

$$\rho_X(x_i, x_f) \le \epsilon,\ i = 0, \dots, T.$$

Lemmas 8.13 and 8.15 imply the following result.

Lemma 8.18. *Let $\epsilon > 0$. Then there exists $\delta > 0$ such that for each integer $T > 0$ and each sequence $\{y_i\}_{i=0}^T \subset Y$ satisfying*

$$\rho_Y(y_0, y_f) \le \delta,\ \rho_Y(y_T, y_f) \le \delta,$$

$$\sum_{i=0}^{T-1} f(x_f, x_f, y_i, y_{i+1}) \ge \sigma_Y(f, 0, T, x_f, x_f, y_0, y_T) - \delta$$

the following inequality holds:

$$\rho_Y(y_i, y_f) \le \epsilon,\ i = 0, \dots, T.$$

Lemma 8.19. *Let $\epsilon > 0$. Then there exists $\delta \in (0, 1)$ such that for each $g \in \mathcal{A}$ satisfying $\|f - g\| \le \delta$, each $(x_g, y_g) \in S(g)$, each integer $T > 0$ and each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying*

$$\rho_X(x_0, x_f) \le \delta,\ \rho_X(x_T, x_f) \le \delta, \tag{8.39}$$

$$\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) \le \sigma_X(g, 0, T, x_0, x_T, y_g, y_g) + \delta \tag{8.40}$$

the following inequality holds:

$$\rho_X(x_i, x_f) \le \epsilon,\ i = 0, \dots, T.$$

Proof. By Lemma 8.17 there exists $\delta_1 \in (0, 1)$ such that the following property holds:

(P7) for each integer $T > 0$ and each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying

$$\rho_X(x_0, x_f) \le \delta_1,\ \rho_X(x_T, x_f) \le \delta_1,$$

$$\sum_{i=0}^{T-1} f(x_i, x_{i+1}, y_f, y_f) \le \sigma_X(f, 0, T, x_0, x_T, y_f, y_f) + \delta_1$$

we have

$$\rho_X(x_i, x_f) \le \epsilon,\ i = 0, \dots, T.$$

By Lemma 8.11 there exist

$$\delta_2 \in (0, \delta_1) \tag{8.41}$$

and an integer $\tau_0 > 1$ such that the following property holds:

(P8) for each $g \in \mathcal{A}$ satisfying $\|g - f\| \le \delta_2$, each $(x_g, y_g) \in S(g)$, each integer $T \ge \tau_0$, each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying

$$\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) \le \sigma_X(g, 0, T, x_0, x_T, y_g, y_g) + 4$$

and each integer $p \in [0, T - \tau_0]$ there is an integer j such that $\{j, j+1\} \subset [p, p + \tau_0]$ and

$$\rho_X(x_i, x_f) \le \delta_1,\ i = j, j+1.$$

Since the function f is continuous there is

$$\delta_3 \in (0, (2\tau_0 + 1)^{-1} 8^{-1} \delta_2) \tag{8.42}$$

such that the following property holds:

(P9) for each $z_1, z_2, \tilde{z}_1, \tilde{z}_2 \in X$ and each $\xi_1, \xi_2, \tilde{\xi}_1, \tilde{\xi}_2 \in Y$ satisfying

$$\rho_X(z_i, \tilde{z}_i) \le \delta_3,\ i = 1, 2,\ \rho_Y(\xi_i, \tilde{\xi}_i) \le \delta_3\ i = 1, 2$$

we have

$$|f(z_1, z_2, \xi_1, \xi_2) - f(\tilde{z}_1, \tilde{z}_2, \tilde{\xi}_1, \tilde{\xi}_2)| \le 8^{-1}(2\tau_0 + 1)^{-1}\delta_2.$$

By Lemma 8.1 there is

$$\delta \in (0, \delta_3) \tag{8.43}$$

such that the following property holds:

(P10) for each $g \in \mathcal{A}$ satisfying $\|g - f\| \le \delta$ and each $(x, y) \in S(g)$ we have

$$\rho_X(x, x_f) \le \delta_3,\ \rho_Y(y, y_f) \le \delta_3.$$

Assume that $g \in \mathcal{A}$,

$$\|f - g\| \le \delta,\ (x_g, y_g) \in S(g), \tag{8.44}$$

an integer $T > 0$ and a sequence $\{x_i\}_{i=0}^T \subset X$ satisfies (8.39) and (8.40). By (8.40), (8.42), (8.43), (8.44) and property (P8) there exist a strictly increasing sequence of nonnegative integers $\{t_i\}_{i=0}^q$ where q is a natural number such that

$$t_0 = 0,\ t_q = T,\ t_{i+1} - t_i \le \tau_0 \text{ for all } i = 0, \dots, q-1, \tag{8.45}$$

$$\rho_X(x_{t_i}, x_f) \le \delta_1,\ i = 0, \dots, q. \tag{8.46}$$

Let $j \in \{0, \dots, T\}$. We show that $\rho_X(x_j, x_f) \le \epsilon$. By (8.45) there is an integer $k \in [0, q-1]$ such that

$$j \in [t_k, t_{k+1}]. \tag{8.47}$$

By (8.40) and (8.46),

$$\rho_X(x_{t_k}, x_f) \le \delta_1,\ \rho_X(x_{t_k+1}, x_f) \le \delta_1, \tag{8.48}$$

$$\sum_{i=t_k}^{t_{k+1}-1} g(x_i, x_{i+1}, y_g, y_g) \le \sigma_X(g, 0, t_{k+1} - t_k, x_{t_k}, x_{t_{k+1}}, y_g, y_g) + \delta. \tag{8.49}$$

By (8.44), (8.45) and (8.49),

$$\begin{aligned}\sum_{i=t_k}^{t_{k+1}-1} f(x_i, x_{i+1}, y_g, y_g) &\le \sum_{i=t_k}^{t_{k+1}-1} g(x_i, x_{i+1}, y_g, y_g) + \delta\tau_0 \\ &\le \sigma_X(g, 0, t_{k+1} - t_k, x_{t_k}, x_{t_{k+1}}, y_g, y_g) + \delta(\tau_0 + 1) \\ &\le \sigma_X(f, 0, t_{k+1} - t_k, x_{t_k}, x_{t_{k+1}}, y_g, y_g) + \delta(2\tau_0 + 1).\end{aligned} \tag{8.50}$$

By (8.44) and (P10),

$$\rho_X(x_g, x_f) \le \delta_3,\ \rho_Y(y_g, y_f) \le \delta_3. \tag{8.51}$$

In view of (8.45), (8.51) and (P9),

$$\Big|\sum_{i=t_k}^{t_{k+1}-1} f(x_i, x_{i+1}, y_g, y_g) - \sum_{i=t_k}^{t_{k+1}-1} f(x_i, x_{i+1}, y_f, y_f)\Big| \le \tau_0(2\tau_0+1)^{-1}8^{-1}\delta_2, \tag{8.52}$$

$$\begin{aligned}&|\sigma_X(f, 0, t_{k+1}-t_k, x_{t_k}, x_{t_{k+1}}, y_g, y_g) - \sigma_X(f, 0, t_{k+1}-t_k, x_{t_k}, x_{t_{k+1}}, y_f, y_f)\\ &\quad \le \tau_0(2\tau_0+1)^{-1}8^{-1}\delta_2.\end{aligned} \tag{8.53}$$

By (8.52), (8.50), (8.53), (8.42), (8.43) and (8.41),

$$\begin{aligned}\sum_{i=t_k}^{t_{k+1}-1} f(x_i, x_{i+1}, y_f, y_f) &\le \sum_{i=t_k}^{t_{k+1}-1} f(x_i, x_{i+1}, y_g, y_g) + 16^{-1}\delta_2\\ &\le \sigma_X(f, 0, t_{k+1}-t_k, x_{t_k}, x_{t_{k+1}}, y_g, y_g) + \delta(2\tau_0+1) + 16^{-1}\delta_2\\ &\le \sigma_X(f, 0, t_{k+1}-t_k, x_{t_k}, x_{t_{k+1}}, y_f, y_f) + 16^{-1}\delta_2 + \delta(2\tau_0+1) + 16^{-1}\delta_2\\ &\le \sigma_X(f, 0, t_{k+1}-t_k, x_{t_k}, x_{t_k+1}, y_f, y_f) + \delta_1.\end{aligned} \tag{8.54}$$

By (8.47), (8.48), (8.54) and (P7), $\rho_X(x_g, x_f) \le \epsilon$. Lemma 8.19 is proved. □

Analogously to Lemma 8.19 we can prove the following result.

Lemma 8.20. *Let $\epsilon > 0$. Then there exists $\delta \in (0, 1)$ such that for each $g \in \mathcal{A}$ satisfying $\|f - g\| \le \delta$, each $(x_g, y_g) \in S(g)$, each integer $T > 0$ and each sequence $\{y_i\}_{i=0}^T \subset Y$ satisfying*

$$\rho_Y(y_0, y_f) \le \delta,\ \rho_Y(y_T, y_f) \le \delta,$$

$$\sum_{i=0}^{T-1} g(x_g, x_g, y_i, y_{i+1}) \ge \sigma_Y(g, 0, T, x_g, x_g, y_0, y_T) - \delta$$

the following inequality holds:

$$\rho_Y(y_i, y_f) \le \epsilon,\ i = 0, \dots, T.$$

Lemma 8.21. *Let $M, \epsilon > 0$. Then there exist natural numbers l and Q and a positive number δ such that for each $g \in \mathcal{A}$ satisfying $\|f - g\| \le \delta$, each $(x_g, y_g) \in S(g)$, each integer $T > Ql$ and each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying*

$$\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) \le Tg(x_g, x_g, y_g, y_g) + M \tag{8.55}$$

there exist a natural number $q \le Q$ and sequences of integers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0, T]$ such that

$$0 \le b_i - a_i \le l,\ i = 1, \dots, q,$$

$$\rho_X(x_i, x_f) \le \epsilon$$

for all integers $i \in [0, T] \setminus \cup_{j=1}^q [a_j, b_j]$.

Proof. By Lemma 8.19 there exists $\delta_0 \in (0,1)$ such that the following property holds:

(P11) for each $g \in \mathcal{A}$ satisfying $\|f-g\| \le \delta_0$, each $(x_g, y_g) \in S(g)$, each integer $T > 0$ and each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying

$$\rho_X(x_0, x_f) \le \delta_0, \ \rho_X(x_T, x_f) \le \delta_0,$$

$$\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) \le \sigma_X(g, 0, T, x_0, x_T, y_g, y_g) + \delta_0$$

we have

$$\rho_X(x_i, x_f) \le \epsilon, \ i = 0, \dots, T.$$

By Lemma 8.11 there exist a positive number

$$\delta < \min\{\delta_0, M/8\}$$

and an integer $\tau_0 > 1$ such that the following property holds:

(P12) for each $g \in \mathcal{A}$ satisfying $\|g-f\| \le \delta$, each $(x_g, y_g) \in S(g)$, each integer $T \ge \tau_0$ and each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying

$$\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) \le \sigma_X(g, 0, T, x_0, x_T, y_g, y_g) + M$$

and each integer $p \in [0, T-\tau_0]$ there is an integer j such that $\{j, j+1\} \subset [p, p+\tau_0]$ and

$$\rho_X(x_i, x_f) \le \delta_0, \ i = j, j+1.$$

Choose natural numbers

$$Q > 6 + 3\delta_0^{-1}(4(\|f\|+1)+M), \ l > 4(\tau_0+1). \tag{8.56}$$

Assume that $g \in \mathcal{A}$,

$$\|f-g\| \le \delta, \ (x_g, y_g) \in S(g), \tag{8.57}$$

an integer $T > Ql$ and that a sequence $\{x_i\}_{i=0}^T \subset X$ satisfies (8.55).

Set $t_0 = 0$. Assume that an integer $k \ge 0$ and we have defined a strictly increasing sequence of integers t_p, $p = 0, \dots, k$ such that $t_k < T$ and that for each integer p satisfying $p < k$,

$$t_{p+1} - t_p \ge 2, \tag{8.58}$$

$$\sum_{j=t_p}^{t_{p+1}-1} g(x_j, x_{j+1}, y_g, y_g) > \sigma_X(g, 0, t_{p+1}-t_p, x_{t_p}, x_{t_{p+1}}, y_g, y_g) + \delta_0, \tag{8.59}$$

$$\sum_{j=t_p}^{t_{p+1}-2} g(x_j, x_{j+1}, y_g, y_g) \le \sigma_X(g, 0, t_{p+1}-t_p-1, x_{t_p}, x_{t_{p+1}-1}, y_g, y_g) + \delta_0. \tag{8.60}$$

(Clearly, for $k = 0$ this assumption holds.) If

$$\sum_{j=t_k}^{T-1} g(x_j, x_{j+1}, y_g, y_g) \le \sigma_X(g, 0, T-t_k, x_{t_k}, x_T, y_g, y_g) + \delta_0,$$

then set $q = k + 1$, $t_q = T$ and the construction is completed.

Assume that

$$\sum_{j=t_k}^{T-1} g(x_j, x_{j+1}, y_g, y_g) > \sigma_X(g, 0, T - t_k, x_{t_k}, x_T, y_g, y_g) + \delta_0.$$

Then there exists an integer t_{k+1} such that $t_k + 1 < t_{k+1} \le T$ (it is possible that $t_{k+1} = T$) such that

$$\sum_{j=t_k}^{t_{k+1}-1} g(x_j, x_{j+1}, y_g, y_g) > \sigma_X(g, 0, t_{k+1} - t_k, x_{t_k}, x_{t_{k+1}}, y_g, y_g) + \delta_0,$$

$$\sum_{j=t_k}^{t_{k+1}-2} g(x_j, x_{j+1}, y_g, y_g) \le \sigma_X(g, 0, t_{k+1} - t_k - 1, x_{t_k}, x_{t_{k+1}-1}, y_g, y_g) + \delta_0.$$

Clearly, the assumption made for k also holds for $k + 1$ (if $k + 1 < q$) and the construction is completed after a finite number of steps and the last element of the sequence satisfies $t_q = T$.

It follows from the construction of the sequence $\{t_i\}_{i=0}^q$ that for each integer p satisfying $0 \le p < q - 1$, (8.58)-(8.60) hold and if $t_q - t_{q-1} > 1$, then (8.60) holds with $p = q - 1$.

By (8.55), (8.30), (8.31), Proposition 1.3, (1.4), Lemma 8.10 and (8.57),

$$\begin{aligned}\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) &\le M + Tg(x_g, x_g, y_g, y_g) \\ &= M + \sigma_X(g, 0, T, x_g, x_g, y_g, y_g) \\ &\le M + \sigma_X(g, 0, T, x_0, x_T, y_g, y_g) + 4\|g\| \\ &\le M + 4(\|f\| + 1) + \sigma_X(g, 0, T, x_0, x_T, y_g, y_g). \qquad (8.61)\end{aligned}$$

By (8.59) and (8.61),

$$\begin{aligned}4(\|f\| + 1) + M &\ge \sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) - \sigma_X(g, 0, T, x_0, x_T, y_g, y_g) \\ &\ge \sum \Big\{ \sum_{i=t_p}^{t_{p+1}-1} g(x_i, x_{i+1}, y_g, y_g) - \sigma_X(g, t_p, t_{p+1}, x_{t_p}, x_{t_{p+1}}, y_g, y_g) : \\ &\qquad p \in \{0, \dots, q-1\} \text{ and } p < q - 1 \Big\} \ge \delta_0(q - 2),\end{aligned}$$

$$q \le 2 + \delta_0^{-1}(4(\|f\| + 1) + M). \qquad (8.62)$$

Set

$$E = \{p \in \{0, \dots, q - 1\} : t_{p+1} - t_p > 4(\tau_0 + 1)\}. \qquad (8.63)$$

Let $p \in E$. Then (8.60) holds. By (8.57), (8.60), (8.63) and (P12), there exist integers

$$j_{1,p} \in [t_p, t_p + \tau_0],\ j_{2,p} \in [t_{p+1} - 1 - \tau_0,\ t_{p+1} - 1] \qquad (8.64)$$

such that

$$\rho_X(x_{j_{s,p}}, x_f) \le \delta_0, \ s = 1, 2. \tag{8.65}$$

By (8.64), the inclusion $p \in E$, (8.63), (8.65), (8.60), (8.57) and (P11),

$$\rho_X(x_i, x_f) \le \epsilon, \ i \in \{j_{1,p}, \dots, j_{2,p}\},$$

$$\rho_X(x_i, x_f) \le \epsilon, \ i \in \{t_p + \tau_0, \dots, t_{p+1} - 1 - \tau_0\} \tag{8.66}$$

for all $p \in E$. By (8.66),

$$\{i \in \{0, \dots, T\} : \rho_X(x_i, x_f) > \epsilon\} \subset \cup\{\{t_i, \dots, t_{i+1}\} : \ i \in \{0, \dots, q-1\} \setminus E\}$$

$$\cup(\cup\{(\{t_p, \dots, t_p + \tau_0 - 1\} \cup \{t_{p+1} - \tau_0, \dots, t_{p+1}\}) : \ p \in E\}). \tag{8.67}$$

Consider the collection of closed intervals

$$[t_i, t_{i+1}], \ i \in \{0, \dots, q-1\} \setminus E, \ [t_p, t_p + \tau_0 - 1], \ p \in E, \ [t_{p+1} - \tau_0, t_{p+1}], \ p \in E.$$

Clearly, the number of intervals in this collection does not exceed

$$3q \le 6 + 3\delta_0^{-1}(4(\|f\| + 1) + M) < Q$$

(see (8.56) and (8.62)) and by (8.56) and (8.63) the lengths of all these intervals does not exceed $4(\tau_0 + 1) \le l$.

Lemma 8.21 is proved. □

Analogously to Lemma 8.21 we can prove the following result.

Lemma 8.22. *Let $M, \epsilon > 0$. Then there exist natural numbers l and Q and a positive number δ such that for each $g \in \mathcal{A}$ satisfying $\|f - g\| \le \delta$, each $(x_g, y_g) \in S(g)$, each integer $T > Ql$ and each sequence $\{y_i\}_{i=0}^T \subset Y$ satisfying*

$$\sum_{i=0}^{T-1} g(x_g, x_g, y_i, y_{i+1}) \ge Tg(x_g, x_g, y_g, y_g) - M$$

there exist a natural number $q \le Q$ and sequences of integers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0, T]$ such that

$$0 \le b_i - a_i \le l, \ i = 1, \dots, q,$$

$$\rho_Y(y_i, y_f) \le \epsilon$$

for all integers $i \in [0, T] \setminus \cup_{j=1}^q [a_j, b_j]$.

9. Proof of Theorem 1.8

By Lemma 8.21 there exist natural numbers l_1 and Q_1 and a positive number $\delta_1 < 1$ such that the following property holds:

(P13) for each $g \in \mathcal{A}$ satisfying $\|f - g\| \le \delta_1$, each $(x_g, y_g) \in S(g)$, each integer $T > Q_1 l_1$ and each sequence $\{x_i\}_{i=0}^T \subset X$ satisfying

$$\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) \le Tg(x_g, x_g, y_g, y_g) + 2M + 12(\|f\| + 1)$$

there exist a natural number $q \le Q_1$ and sequences of integers $\{a_i\}_{i=1}^q$, $\{b_i\}_{i=1}^q \subset [0, T]$ such that

$$0 \le b_i - a_i \le l_1, \ i = 1, \dots, q,$$

$$\rho_X(x_i, x_f) \le \epsilon$$

for all integers $i \in [0,T] \setminus \cup_{j=1}^{q}[a_j, b_j]$.

By Lemma 8.22 there exist natural numbers l_2 and Q_2 and a positive number $\delta_2 < 1$ such that the following property holds:

(P14) for each $g \in \mathcal{A}$ satisfying $\|f-g\| \le \delta_2$, each $(x_g, y_g) \in S(g)$, each integer $T > Q_2 l_2$ and each sequence $\{y_i\}_{i=0}^{T} \subset Y$ satisfying

$$\sum_{i=0}^{T-1} g(x_g, x_g, y_i, y_{i+1}) \ge Tg(x_g, x_g, y_g, y_g) - 2M - 12(\|f\|+1)$$

there exist a natural number $q \le Q_2$ and sequences of integers $\{a_i\}_{i=1}^{q}$, $\{b_i\}_{i=1}^{q} \subset [0,T]$ such that

$$0 \le b_i - a_i \le l_2,\ i = 1, \dots, q,$$
$$\rho_Y(y_i, y_f) \le \epsilon$$

for all integers $i \in [0,T] \setminus \cup_{j=1}^{q}[a_j, b_j]$.

Put

$$\delta = \min\{\delta_1, \delta_2\},\ l = \max\{l_1, l_2\},\ Q = Q_1 + Q_2. \tag{9.1}$$

Assume that

$$g \in \mathcal{A},\ \|f-g\| \le \delta, \tag{9.2}$$

an integer $T > Ql$ and that $\{x_i\}_{i=0}^{T} \subset X$, $\{y_i\}_{i=0}^{T} \subset Y$ is a (g, M)-good pair of sequences. Let

$$(x_g, y_g) \in S(g) \tag{9.3}$$

be an element of $S(g)$ associated with g (see the definition of $\mathcal{A}$, (1.4) and Remark 1.5). Then

$$g(x_g, x_g, \xi, \xi) \le g(x_g, x_g, y_g, y_g) \le g(z, z, y_g, y_g) \text{ for all } z \in X \text{ and all } \xi \in Y \tag{9.4}$$

and for each natural number n and each pair of sequences $\{z_i\}_{i=0}^{n} \subset X$, $\{\xi_i\}_{i=0}^{n} \subset Y$,

$$-4\|g\| + \sum_{i=0}^{n-1} g(x_g, x_g, \xi_i, \xi_{i+1}) \le ng(x_g, x_g, y_g, y_g) \le \sum_{i=0}^{n-1} g(z_i, z_{i+1}, y_g, y_g) + 4\|g\|. \tag{9.5}$$

By (9.2), (9.3), (9.4), (9.5) and Proposition 5.1,

$$\Big|\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) - Tg(x_g, x_g, y_g, y_g)\Big| \le 2M + 12\|g\| \le 2M + 12(\|f\|+1), \tag{9.6}$$

$$\Big|\sum_{i=0}^{T-1} g(x_g, x_g, y_i, y_{i+1}) - Tg(x_g, x_g, y_g, y_g)\Big| \le 2M + 12\|g\| \le 2M + 12(\|f\|+1). \tag{9.7}$$

By properties (P13) and (P14), (9.1), (9.2), (9.3), (9.6) and (9.7) there exist natural numbers $q_1 \le Q_1$, $q_2 \le Q_2$ and sequences of integers

$$\{a_{1,i}\}_{i=1}^{q_1},\ \{b_{1,i}\}_{i=1}^{q_1}, \{a_{2,i}\}_{i=1}^{q_2},\ \{b_{2,i}\}_{i=1}^{q_2} \subset [0,T]$$

such that

$$0 \le b_{1,i} - a_{1,i} \le l_1,\ i = 1, \dots, q_1,$$

$$0 \le b_{2,i} - a_{2,i} \le l_2,\ i = 1, \dots, q_2,$$
$$\rho_X(x_i, x_f) \le \epsilon,\ \rho_Y(y_i, y_f) \le \epsilon$$

for all integers

$$i \in [0, T] \setminus [\cup(\cup_{j=1}^{q_1}[a_{1,j}, b_{1,j}]) \cup (\cup_{j=1}^{q_2}[a_{2,j}, b_{2,j}])].$$

Theorem 1.8 is proved.

10. Proof of Theorem 1.9

Let $\mathcal{E}$ be the set of all $f \in \mathcal{A}$ which possess (ATP).

Lemma 10.1. *$\mathcal{E}$ is an everywhere dense subset of $\mathcal{A}$.*

Proof. Let $f \in \mathcal{A}$ and $r \in (0, 1]$ and let $(x_f, y_f) \in S(f)$ be an element of $S(f)$ associated with f such that (1.1) holds and (1.4) holds with $c_f = 4\|f\|$. Set

$$\begin{aligned} f_r(x_1, x_2, y_1, y_2) &= f(x_1, x_2, y_1, y_2) + r\rho_X(x_1, x_f) \\ &\quad + r\rho_X(x_2, x_f) - r\rho_Y(y_1, y_f) - r\rho_Y(y_2, y_f). \end{aligned} \tag{10.1}$$

Clearly,

$$f_r \in \mathcal{A}_0,\ (x_f, y_f) \in S(f_r) \text{ and } f_r \in \mathcal{A}. \tag{10.2}$$

Let

$$(x, y) \in S(f_r). \tag{10.3}$$

Then by (10.2), (10.3) and the definition of $S(\cdot)$ (see (1.7)),

$$\begin{aligned} f_r(x, x, y_f, y_f) &\le f_r(x, x, y, y) \le f_r(x_f, x_f, y, y) \\ &\le f_r(x_f, x_f, y_f, y_f) \le f_r(x, x, y_f, y_f). \end{aligned} \tag{10.4}$$

By (10.1), (10.4) and the inclusion $(x_f, y_f) \in S(f)$,

$$f(x, x, y_f, y_f) + 2r\rho_X(x, x_f) = f(x_f, x_f, y_f, y_f) \le f(x, x, y_f, y_f),$$
$$x = x_f$$

and

$$f(x_f, x_f, y, y) \le f(x_f, x_f, y_f, y_f) = f(x_f, x_f, y, y) - 2r\rho_Y(y, y_f),$$
$$y = y_f.$$

Thus $S(f_r) = \{(x_f, y_f)\}$. We show that f_r possesses (ATP).

Let $\{x_i\}_{i=0}^{\infty} \subset X$ satisfy

$$\sup\Big\{\sum_{i=0}^{n-1} f_r(x_i, x_{i+1}, y_f, y_f) - nf_r(x_f, x_f, y_f, y_f) :\ n = 1, 2, \dots\Big\} < \infty.$$

By the inequality above, (10.1) and Proposition 1.4,

$$\sum_{i=0}^{\infty}(\rho_X(x_i, x_f) + \rho_X(x_{i+1}, x_f)) < \infty.$$

This implies that $\rho_X(x_i, x_f) \to 0$ as $i \to \infty$.

Analogously we can show that for each sequence $\{y_i\}_{i=0}^{\infty} \subset Y$ satisfying

$$\inf\Big\{\sum_{i=0}^{n-1} f_r(x_f, x_f, y_i, y_{i+1}) - n f_r(x_f, x_f, y_f, y_f) :\ n = 1, 2, \dots\Big\} > -\infty$$

we have $\rho_Y(y_i, y_f) \to 0$ as $i \to \infty$. Thus f_r possesses (ATP). Since $\|f - f_r\| \to 0$ as $r \to 0^+$ we conclude that Lemma 10.1 is proved. □

Let $f \in \mathcal{E}$, $(x_f, y_f) \in S(f)$ and n be a natural number. By Lemmas 8.1, 8.21 and 8.22 there exists an open neighborhood $V(f, n)$ of f in $\mathcal{A}$ such that the following properties hold:

(i) for each $g \in V(f, n)$ and each $(z, \xi) \in S(g)$,

$$\rho_X(z, x_f) \le (4n)^{-1},\ \rho_Y(\xi, y_f) \le (4n)^{-1};$$

(ii) for each $g \in V(f, n)$, each $(x_g, y_g) \in S(g)$ and each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfying

$$\sup\Big\{\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) - T g(x_g, x_g, y_g, y_g) :\ T = 1, 2, \dots\Big\} \le n,$$

for all sufficiently large natural numbers i, $\rho_X(x_i, x_f) \le (4n)^{-1}$;

(iii) for each $g \in V(f, n)$, each $(x_g, y_g) \in S(g)$ and each sequence $\{y_i\}_{i=0}^{\infty} \subset Y$ satisfying

$$\inf\Big\{\sum_{i=0}^{T-1} g(x_g, x_g, y_i, y_{i+1}) - T g(x_g, x_g, y_g, y_g) :\ T = 1, 2, \dots\Big\} \ge -n,$$

for all sufficiently large natural numbers i, $\rho_Y(y_i, y_f) \le (4n)^{-1}$.

Set

$$\mathcal{F} = \cap_{n=1}^{\infty} \cup \{V(f, n) :\ n = 1, 2, \dots\}. \tag{10.5}$$

Clearly, $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $\mathcal{A}$.

Let $g \in \mathcal{F}$. By (10.5) for any natural number n there is $f_n \in \mathcal{E}$ such that

$$g \in V(f_n, n). \tag{10.6}$$

Let

$$(z_1, \xi_1),\ (z_2, \xi_2) \in S(g) \tag{10.7}$$

and let n be a natural number. By (10.6), (10.7) and property (i),

$$\rho_X(z_i, x_{f_n}),\ \rho_Y(\xi_i, y_{f_n}) \le (4n)^{-1},\ i = 1, 2,$$

$$\rho_X(z_1, z_2),\ \rho_Y(\xi_1, \xi_2) \le (2n)^{-1}$$

and since n is an arbitrary natural number we conclude that $z_1 = z_2$, $\xi_1 = \xi_2$ and $S(g)$ is a singleton: $S(g) = \{(x_g, y_g)\}$ and

$$\rho_X(x_g, x_{f_n}),\ \rho_Y(y_g, y_{f_n}) \le (4n)^{-1} \text{ for all integers } n \ge 1. \tag{10.8}$$

Assume that $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies

$$\sup\Big\{\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) - T g(x_g, x_g, y_g, y_g) :\ T = 1, 2, \dots\Big\} < \infty.$$

Choose a natural number n such that

$$2n^{-1} < \epsilon, \tag{10.9}$$

$$\sup\left\{\sum_{i=0}^{T-1} g(x_i, x_{i+1}, y_g, y_g) - Tg(x_g, x_g, y_g, y_g) : T = 1, 2, \dots\right\} \leq n.$$

By (10.6), (10.8), (10.9) and property (ii), for all sufficiently large natural numbers i,

$$\rho_X(x_i, x_{f_n}) \leq (4n)^{-1},$$

$$\rho_X(x_i, x_g) \leq \rho_X(x_i, x_{f_n}) + \rho_X(x_{f_n}, x_g) \leq (2n)^{-1} < \epsilon.$$

Since ϵ is an arbitrary positive number we conclude that $\lim_{i\to\infty} \rho_X(x_i, x_g) = 0$.

Analogously we can show that if $\{y_i\}_{i=0}^{\infty} \subset Y$ satisfies

$$\inf\left\{\sum_{i=0}^{T-1} g(x_g, x_g, y_i, y_{i+1}) - Tg(x_g, x_g, y_g, y_g) : T = 1, 2, \dots\right\} > -\infty,$$

then $\lim_{i\to\infty} \rho_Y(y_i, y_g) = 0$. Thus g possesses (ATP). Theorem 1.9 is proved.

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Manuscript received July 12, 2012
revised August 14, 2012

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