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POSITIVE STOCHASTIC MATRICES AS CONTRACTION MAPS

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Abstract. It is shown that if $A = (a_{ij})$ is an $n \times n$ positive stochastic matrix, then A is a contraction from the unit $(n-1)$-simplex into itself with respect to the ℓ_1 norm, with the contraction constant $\frac{1}{2} \max\limits_{j \neq j'} \sum\limits_{i=1}^{n} |a_{ij} - a_{ij'}|$.

Let Δ^{n-1} be the unit $(n-1)$-simplex in $\mathbb{R}^n$, i.e.,

$$\Delta^{n-1} = \{x \in \mathbb{R}^n \,;\, x_1 + x_2 + \cdots + x_n = 1 \text{ with all } x_i \geq 0\}.$$

Let $A = (a_{ij})$ be an $n \times n$ real matrix. The matrix A is called *stochastic* if $a_{ij} \geq 0$ for all $i, j = 1, \ldots, n$ and $\sum_{i=1}^{n} a_{ij} = 1$ for all $j = 1, \ldots, n$. The matrix A is said to be *positive* if $a_{ij} > 0$ for all $i, j = 1, \ldots, n$.

We shall establish the following:

Theorem. *If $A = (a_{ij})$ is an $n \times n$ positive stochastic matrix, then A is a contraction from the unit $(n-1)$-simplex into itself with respect to the ℓ_1 norm, with the contraction constant $\frac{1}{2} \max\limits_{j \neq j'} \sum\limits_{i=1}^{n} |a_{ij} - a_{ij'}|$.*

Proof. We consider the vector subspace of $\mathbb{R}^n$ given by

$$M = \{x \in \mathbb{R}^n \,;\, x_1 + x_2 + \cdots + x_n = 0\}.$$

For each $x \in M$, let $(Ax)_i$ denote the ith coordinate of Ax. Since A is stochastic, we have

$$\sum_{i=1}^{n} (Ax)_i = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} x_j = 0$$

for each $x \in M$, so that $AM \subset M$. Since for any $x \in M$ with $x \neq 0$ some component of x must be negative, it follows from the positivity and stochasticity of A that if $x \in M$ with $x \neq 0$ then

$$\begin{aligned} \|Ax\|_1 &= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \\ &< \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |x_j| \\ &= \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} |x_j| \end{aligned}$$

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$$= \|x\|_1.$$

Thus

$$\|Ax\|_1 \leq \alpha \|x\|_1 \text{ for all } x \in M,$$

where $\alpha = \max\{\|Ax\|_1 \,;\, x \in M, \|x\|_1 = 1\} < 1$. The constant α can be computed explicitly as follows. Let $S = \{x \in M \,;\, \|x\|_1 = 1\}$ be the unit sphere of M, and denote by S_e the set of extreme points of S. Then S_e is composed of all vectors with each vector having exactly two nonzero components $\frac{1}{2}$ and $-\frac{1}{2}$. A computation shows that

$$\alpha = \max_{x \in S} \|Ax\|_1 = \max_{x \in S_e} \|Ax\|_1 = \frac{1}{2} \max_{j \neq j'} \sum_{i=1}^{n} |a_{ij} - a_{ij'}|.$$

For all $x, y \in \Delta^{n-1}$, we have $x - y \in M$, so that

$$\|Ax - Ay\|_1 \leq \frac{1}{2} \max_{j \neq j'} \sum_{i=1}^{n} |a_{ij} - a_{ij'}| \; \|x - y\|_1,$$

and the proof is complete. □

Observe that if A is a positive stochastic matrix, then $A\Delta^{n-1}$ is contained in the interior of Δ^{n-1}. This observation with the theorem above and the Banach contraction mapping principle implies that *if A is a positive stochastic matrix then A has a unique fixed point $\hat{x}$ in Δ^{n-1} with all $\hat{x}_i > 0$ and $\lim_{k \to \infty} A^k x = \hat{x}$ for any x in Δ^{n-1}. Moreover, for any $x \in \Delta^{n-1}$ and any $k = 1, 2, \ldots,$*

$$\|A^k x - \hat{x}\|_1 \leq \frac{\left(\frac{1}{2} \max_{j \neq j'} \sum_{i=1}^{n} |a_{ij} - a_{ij'}| \right)^k}{1 - \frac{1}{2} \max_{j \neq j'} \sum_{i=1}^{n} |a_{ij} - a_{ij'}|} \|Ax - x\|_1.$$

The first part of this remarkable result was given in Ortega [1, p.216]. Ortega's proof is based on Perron's theorem about the spectrum of a positive matrix. The "moreover" part is the "error estimate" aspect of the Banach contraction mapping principle and provides a computational method for approximation of fixed points. It may be mentioned that our proof of the theorem shows that a positive stochastic matrix provides an example that a nonexpansive map on the whole space may become a contraction in a certain smaller set.

References

[1] J. M. Ortega, *Matrix Theory, A Second Course*, Plenum Press, New, York,1987.

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