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HOMOTOPY PRINCIPLES FOR d-ESSENTIAL ACYCLIC MAPS

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Abstract. This paper discusses acyclic maps and we present a definition of a d–essential map in this setting. Continuation theorems are presented for this type of map.

1. Introduction

The notion of an essential map was introduced by Granas [3] and he showed for single valued maps that if a map F is essential and $F \cong G$ then G is essential. This notion was extended to d–essential maps by Precup [6]. In this paper we discuss d–essential maps for a very large class of maps, namely the class of acyclic maps.

Let X and Z be subsets of Hausdorff topological spaces. We will consider maps $F : X \to K(Z)$; here $K(Z)$ denotes the family of nonempty compact subsets of Z. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F : X \to K(Z)$ is acyclic if F is upper semicontinuous with acyclic values.

2. d–essential maps

Let E be a normed linear space and U an open subset of E.

Definition 2.1. We say $F \in AC(\overline{U}, E)$ if $F : \overline{U} \to K(E)$ is an acyclic compact map; here $\overline{U}$ denotes the closure of U in E.

Definition 2.2. We say $F \in AC_{\partial U}(\overline{U}, E)$ if $F \in AC(\overline{U}, E)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in E.

For any map $F \in AC(\overline{U}, E)$ let $F^\star = I \times F : \overline{U} \to K(\overline{U} \times E)$, with $I : \overline{U} \to \overline{U}$ given by $I(x) = x$, and let

$$d : \left\{ (F^\star)^{-1} \ (B) \right\} \cup \{\emptyset\} \ \to \ \Omega \tag{2.1}$$

be any map with values in the nonempty set Ω; here $B = \left\{ (x, x) : \ x \in \overline{U} \right\}$.

Definition 2.3. Let $F \in AC_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to K(\overline{U} \times E)$ is d–essential if for every map $J \in AC_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ we have that $d\left((F^\star)^{-1} \ (B) \right) = d\left((J^\star)^{-1} \ (B) \right) \neq d(\emptyset)$.

Remark 2.4. If $F^\star$ is d–essential then

$$\emptyset \neq (F^\star)^{-1} \ (B) = \{ x \in \overline{U} : \ (x, F(x)) \cap (x, x) \neq \emptyset \},$$

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and this together with $x \notin F(x)$ for $x \in \partial U$ implies that there exists $x \in U$ with $(x,x) \in F^\star(x)$ (i.e. $x \in F(x)$).

Theorem 2.5. *Let E be a normed linear space, U an open subset of E, $B = \{(x,x): x \in \overline{U}\}$ and d be the map defined in (2.1). Suppose $G \in AC_{\partial U}(\overline{U}, E)$, $H : \overline{U} \times [0,1] \to K(E)$ is an upper semicontinuous compact map, $H_t : \overline{U} \to K(E)$ has acyclic values for each $t \in [0,1]$ (here $H_t(x) = H(x,t)$) and assume the following hold:*

$$H(x,0) = G(x) \quad \text{for } x \in \overline{U} \tag{2.2}$$

$$G^\star = I \times G : \overline{U} \to K(\overline{U} \times E) \quad \text{is } d\text{-essential} \tag{2.3}$$

and

$$x \notin H(x,t) \quad \text{for } x \in \partial U \quad \text{and} \quad t \in (0,1]. \tag{2.4}$$

Let $F(x) = H(x,1)$ for $x \in \overline{U}$ and $F^\star = I \times F$. Then

$$d\left((F^\star)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) \neq d(\emptyset).$$

Proof. Let $H^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by

$$H^\star(x,\lambda) = (x\,,\,H(x,\lambda)).$$

Consider

$$D = \left\{x \in \overline{U}: (x,x) \in H^\star(x,t) \ \text{ for some } \ t \in [0,1]\right\}.$$

Notice $D \neq \emptyset$ since for $t = 0$, $H^\star(x,0) = G^\star(x)$ and $G^\star$ is d–essential (i.e. in particular there exists $x \in U$ with $(x,x) \in (x, G(x)) = H^\star(x,0)$). Also D is closed in E. To see this let $\{x_n\}_{n=1}^\infty \subseteq D$ with $x_n \to x \in \overline{U}$. Now there exists $t_n \in (0,1]$ with

$$x_n \in H(x_n, t_n) \ \text{ for each } \ n \in \{1,2,...\}.$$

Without loss of generality assume $t_n \to t \in [0,1]$ so $(x_n, t_n) \to (x,t)$. Now since $H : \overline{U} \times [0,1] \to K(E)$ is a upper semicontinuous map we have $x \in H(x,t)$. As a result $(x,x) \in H^\star(x,t)$, so D is closed. Next notice (2.4), with $G \in AC_{\partial U}(\overline{U}, E)$, guarantees that $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to K(E)$ by $R_\mu(x) = H(x, \mu(x)) = H_{\mu(x)}(x) = H \circ \tau(x)$ and let $R_\mu^\star = I \times R_\mu$; here $\tau : \overline{U} \to \overline{U} \times [0,1]$ is given by $\tau(x) = (x, \mu(x))$. Notice $R_\mu \in AC(\overline{U}, E)$ (note $H : \overline{U} \times [0,1] \to E$ is an upper semicontinuous compact map and $H_t : \overline{U} \to K(E)$ has acyclic values for each $t \in [0,1]$) and notice $R_\mu|_{\partial U} = G|_{\partial U}$ since $\mu(\partial U) = 0$. Thus $R_\mu \in AC_{\partial U}(\overline{U}, E)$ with $R_\mu|_{\partial U} = G|_{\partial U}$ and since $G^\star$ is d–essential we have

$$d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right) \neq d(\emptyset). \tag{2.5}$$

Also notice since $\mu(D) = 1$ that

$$\begin{aligned}
\left(R_\mu^\star\right)^{-1}(B) &= \left\{x \in \overline{U}: (x,x) \cap (x, H(x,\mu(x)) \neq \emptyset\right\} \\
&= \left\{x \in \overline{U}: (x,x) \cap (x, H(x,1) \neq \emptyset\right\} \\
&= (F^\star)^{-1}(B).
\end{aligned}$$

This together with (2.5) yields

$$d\left((F^\star)^{-1}\,(B)\right)=d\left((G^\star)^{-1}\,(B)\right)\neq d(\emptyset).$$

□

Remark 2.6. From the proof we see in Theorem 2.5 that we can replace E being a normed linear space with E being a normal topological space. Also the map H being compact could be replaced by H condensing or indeed H satisfying other compactness type conditions (see [1]).

Remark 2.7. We remark here that acyclic maps are a special case of maps presented in [4, 5] so the proof presented in Theorem 2.5 is the same as that in [4]. For completeness we include the proof here. Our goal is this paper is to establish a result so that the map $F^\star$ (in Theorem 2.5) is d–essential. Unfortunately the ideas in [4] do not work for acyclic maps so new ideas are needed here (see Theorem 2.9).

Remark 2.8. If we discuss the existence of fixed points the function d is

$$d(Q)=\begin{cases}1 & \text{if } \emptyset\neq Q\subseteq\overline{U}\\ 0 & \text{if } Q=\emptyset\end{cases}$$

whereas if we discuss degree theory the values of d are usually integers which can be obtained by means of degree. Recall [2] a map $F\in AC_{\partial U}(\overline{U},E)$ is essential in $AC_{\partial U}(\overline{U},E)$ if for any map $J\in AC_{\partial U}(\overline{U},E)$ with $J|_{\partial U}=F|_{\partial U}$ we have that there exists a $x\in U$ with $x\in F\,(x)$. Notice a map $F\in AC_{\partial U}(\overline{U},E)$ is essential in $AC_{\partial U}(\overline{U},E)$ implies that $F^\star=I\times F$ is d_1–essential where

$$d_1(Q)=\begin{cases}1 & \text{if } \emptyset\neq Q\subseteq\overline{U}\\ 0 & \text{if } Q=\emptyset.\end{cases}$$

To see this suppose $F\in AC_{\partial U}(\overline{U},E)$ is essential in $AC_{\partial U}(\overline{U},E)$. Then for any $J\in AC_{\partial U}(\overline{U},E)$ with $J|_{\partial U}=F|_{\partial U}$ there exists $x\in U$ with $x\in J(x)$. Thus $(x,x)\in(x,J(x))\equiv J^\star(x)$ and so $(J^\star)^{-1}\,(B)\neq\emptyset$ (in particular $(F^\star)^{-1}\,(B)\neq\emptyset$). Hence $d_1\left((J^\star)^{-1}\,(B)\right)=1$ and $d_1\left((F^\star)^{-1}\,(B)\right)=1$ so $d_1\left((J^\star)^{-1}\,(B)\right)=d_1\left((F^\star)^{-1}\,(B)\right)\neq d_1(\emptyset)$.

We now present a result which guarantees that $F^\star$ in Theorem 2.5 is d–essential.

Theorem 2.9. *Let E be an infinite dimensional normed linear space, U an open convex subset of E with $0\in U$, $B=\{(x,x):\ x\in\overline{U}\}$ and d be the map defined in (2.1). Suppose $F\in AC_{\partial U}(\overline{U},E)$, $G\in AC_{\partial U}(\overline{U},E)$ with $G|_{\partial U}=F|_{\partial U}$. In addition assume*

$$G^\star=I\times G:\overline{U}\to K(\overline{U}\times E)\ \textit{ is } d\textit{-essential.}\tag{2.6}$$

Let $F^\star=I\times F$. Then $d\left((F^\star)^{-1}\,(B)\right)=d\left((G^\star)^{-1}\,(B)\right)\neq d(\emptyset)$ and $F^\star:\overline{U}\to K(\overline{U}\times E)$ is d–essential.

Proof. We know [2] there exists a continuous retraction $r:\overline{U}\to\partial U$. Let

$$H(x,\lambda)=\begin{cases}G(2\,\lambda\,r(x)+(1-2\,\lambda)\,x)=G\circ j\,(x,\lambda) & \text{for } (x,\lambda)\in\overline{U}\times\left[0,\frac{1}{2}\right]\\ F((2-2\,\lambda)\,r(x)+(2\,\lambda-1)\,x)=F\circ k\,(x,\lambda) & \text{for } (x,\lambda)\in\overline{U}\times\left[\frac{1}{2},1\right]\end{cases}$$

where $j : \overline{U} \times \left[0, \frac{1}{2}\right] \to \overline{U}$ is given by $j(x,\lambda) = 2\,\lambda\, r(x) + (1-2\,\lambda)\, x$ and $k : \overline{U} \times \left[\frac{1}{2}, 1\right] \to \overline{U}$ is given by $k(x,\lambda) = (2-2\,\lambda)\, r(x) + (2\,\lambda - 1)\, x$. Note for $x \in \overline{U}$ that $G \circ j\left(x, \frac{1}{2}\right) = G(r(x))$ and $F \circ k\left(x, \frac{1}{2}\right) = F(r(x)) = G(r(x)) = G \circ j\left(x, \frac{1}{2}\right)$ since $G|_{\partial U} = F|_{\partial U}$. Note $H : \overline{U} \times [0,1] \to K(E)$ is an upper semicontinuous compact map and $H_t : \overline{U} \to K(E)$ has acyclic values for each $t \in [0,1]$ (here $H_t(x) = H(x,t)$). Also notice $x \notin H(x,t)$ for $x \in \partial U$ and $t \in (0,1]$ since if there exists a $x \in \partial U$ and without loss of generality a $\lambda \in \left(0, \frac{1}{2}\right]$ with $x \in H(x,\lambda)$ then since $r(x) = x$ we have $x \in G(2\,\lambda\, x + (1-2\,\lambda)\, x) = G(x)$, a contradiction. Now Theorem 2.5 guarantees that

$$d\left(\left(F^\star\right)^{-1}(B)\right) = d\left(\left(G^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{2.7}$$

Next we show $F^\star : \overline{U} \to K(\overline{U} \times E)$ is d–essential. Let $J \in AC_{\partial U}(\overline{U}, E)$ be any map with $J|_{\partial U} = F|_{\partial U}$. We must show if $J^\star = I \times J$ then

$$d\left(\left(F^\star\right)^{-1}(B)\right) = d\left(\left(J^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{2.8}$$

Let

$$Q(x,\lambda) = \begin{cases} G(2\,\lambda\, r(x) + (1-2\,\lambda)\, x) = G \circ j\,(x,\lambda) & \text{for } (x,\lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right] \\ J((2-2\,\lambda)\, r(x) + (2\,\lambda - 1)\, x) = J \circ k\,(x,\lambda) & \text{for } (x,\lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]. \end{cases}$$

Note for $x \in \overline{U}$ that $G \circ j\left(x, \frac{1}{2}\right) = G(r(x))$ and $J \circ k\left(x, \frac{1}{2}\right) = J(r(x)) = F(r(x)) = G(r(x)) = G \circ j\left(x, \frac{1}{2}\right)$ since $J|_{\partial U} = F|_{\partial U} = G|_{\partial U}$.

Let $Q^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by

$$Q^\star(x,\lambda) = (x\,,\, Q(x,\lambda))$$

and consider

$$D = \left\{x \in \overline{U} :\ (x,x) \in Q^\star(x,t) \ \text{ for some } \ t \in [0,1]\right\}.$$

The same reasoning as in Theorem 2.5 guarantees that $D \neq \emptyset$ is closed. Suppose there exists $x \in D$ with $x \in \partial U$. Then $x \in Q(x,\lambda)$ for some $\lambda \in [0,1]$. Suppose $\lambda \in \left[0, \frac{1}{2}\right]$. Then since $x \in \partial U$ we have $r(x) = x$ so $x \in G(2\,\lambda\, x + (1-2\,\lambda)\, x) = G(x)$, a contradiction. Next suppose $\lambda \in \left[\frac{1}{2}, 1\right]$. Then $x \in J((2-2\,\lambda)\, r(x) + (2\,\lambda - 1)\, x) = J(x) = F(x)$ since $J|_{\partial U} = F|_{\partial U}$, a contradiction. Thus $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $\Phi_\mu : \overline{U} \to K(E)$ by $\Phi_\mu(x) = Q(x, \mu(x)) = Q_{\mu(x)}(x) = Q \circ \tau(x)$ and let $\Phi_\mu^\star = I \times \Phi_\mu$; here $\tau : \overline{U} \to \overline{U} \times [0,1]$ is given by $\tau(x) = (x, \mu(x))$. Notice $\Phi_\mu \in AC(\overline{U}, E)$ with $\Phi_\mu|_{\partial U} = G|_{\partial U}$ since $\mu(\partial U) = 0$. Since $G^\star$ is d–essential we have

$$d\left(\left(\Phi_\mu^\star\right)^{-1}(B)\right) = d\left(\left(G^\star\right)^{-1}(B)\right) \neq d(\emptyset). \tag{2.9}$$

However notice since $\mu(D) = 1$ that

$$\begin{aligned} \left(\Phi_\mu^\star\right)^{-1}(B) &= \left\{x \in \overline{U} :\ (x,x) \cap (x, Q(x,\mu(x)) \neq \emptyset\right\} \\ &= \left\{x \in \overline{U} :\ (x,x) \cap (x, Q(x,1) \neq \emptyset\right\} \\ &= \left(J^\star\right)^{-1}(B). \end{aligned}$$

and so with (2.9) we have

$$d\left((J^\star)^{-1}\,(B)\right) = d\left((G^\star)^{-1}\,(B)\right) \neq d(\emptyset). \tag{2.10}$$

Combine (2.7) and (2.10) and we have

$$d\left((F^\star)^{-1}\,(B)\right) = d\left((J^\star)^{-1}\,(B)\right) \neq d(\emptyset).$$

□

For completeness we present a more general formulation for new d–essential maps (a bigger class than in Definition 2.3). Let E be a normed linear space and U an open subset of E.

Definition 2.10. Let $F, G \in AC_{\partial U}(\overline{U}, E)$. We say $F \cong G$ in $AC_{\partial U}(\overline{U}, E)$ if there exists a upper semicontinuous compact map $\Psi : \overline{U} \times [0,1] \to K(E)$ with $\Psi_t \in AC_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$, $\Psi_0 = F$ and $\Psi_1 = G$ (here $\Psi_t(x) = \Psi(x,t)$).

Remark 2.11. It is easy to see that $\cong$ is an equivalence relation in $AC_{\partial U}(\overline{U}, E)$.

Definition 2.12. Let $F \in AC_{\partial U}(\overline{U}, E)$ with $F^\star = I \times F$. We say $F^\star : \overline{U} \to K(\overline{U} \times E)$ is d–essential if for every map $J \in AC_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $AC_{\partial U}(\overline{U}, E)$ we have that $d\left((F^\star)^{-1}\,(B)\right) = d\left((J^\star)^{-1}\,(B)\right) \neq d(\emptyset)$. Otherwise $F^\star$ is d–inessential. It is easy to check that this means either $d\left((F^\star)^{-1}\,(B)\right) = d(\emptyset)$ or there exists a map $J \in AC_{\partial U}(\overline{U}, E)$ with $J^\star = I \times J$ and $J|_{\partial U} = F|_{\partial U}$ and $J \cong F$ in $AC_{\partial U}(\overline{U}, E)$ such that $d\left((F^\star)^{-1}\,(B)\right) \neq d\left((J^\star)^{-1}\,(B)\right)$.

Out main result is the following.

Theorem 2.13. *Let E be a normed linear space, U an open subset of E, $B = \{(x,x) : \ x \in \overline{U}\}$ and d be the map defined in* (2.1)*. Suppose $F \in AC_{\partial U}(\overline{U}, E)$. Then the following are equivalent:*

(i). *$F^\star = I \times F : \overline{U} \to K(\overline{U} \times E)$ is d–inessential;*

(ii). *$d\left((F^\star)^{-1}\,(B)\right) = d(\emptyset)$ or there exists a map $G \in AC_{\partial U}(\overline{U}, E)$ with $G^\star = I \times G$ and $G \cong F$ in $AC_{\partial U}(\overline{U}, E)$ such that $d\left((F^\star)^{-1}\,(B)\right) \neq d\left((G^\star)^{-1}\,(B)\right)$.*

Proof. (i) implies (ii) is immediate. Next we prove (ii) implies (i). If $d\left((F^\star)^{-1}\,(B)\right) = d(\emptyset)$ then trivially (i) is true. Next suppose there exists a map $G \in AC_{\partial U}(\overline{U}, E)$ with $G^\star = I \times G$ and $G \cong F$ in $AC_{\partial U}(\overline{U}, E)$ such that $d\left((F^\star)^{-1}\,(B)\right) \neq d\left((G^\star)^{-1}\,(B)\right)$. Let $H : \overline{U} \times [0,1] \to K(E)$ be a upper semicontinuous compact map with $H_t \in AC_{\partial U}(\overline{U}, E)$ for each $t \in [0,1]$, $H_0 = F$ and $H_1 = G$ (here $H_t(x) = H(x,t)$). Let $H^\star : \overline{U} \times [0,1] \to K(\overline{U} \times E)$ be given by

$$H^\star(x,\lambda) = (x\,,\, H(x,\lambda)).$$

Consider

$$D = \left\{x \in \overline{U}:\ (x,x) \in H^\star(x,t) \ \text{ for some } \ t \in [0,1]\right\}.$$

If $D = \emptyset$ then in particular $(H^\star(x,0))^{-1}(B) = \emptyset$ i.e. $(F^\star)^{-1}(B) = \emptyset$ so $d\left((F^\star)^{-1}(B)\right) = d(\emptyset)$, so $F^\star$ is d–inessential. Next suppose $D \neq \emptyset$. Note D is closed in E. Also since $x \notin H_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ then $D \cap \partial U = \emptyset$. Thus there exists a continuous map $\mu : \overline{U} \to [0,1]$ with $\mu(\partial U) = 0$ and $\mu(D) = 1$. Define a map $R_\mu : \overline{U} \to K(E)$ by $R_\mu(x) = H(x, \mu(x))$ and let $R_\mu^\star = I \times R_\mu$. As in Theorem 2.5 note $R_\mu \in AC_{\partial U}(\overline{U}, E)$ with $R_\mu|_{\partial U} = H_0|_{\partial U} = F|_{\partial U}$ since $\mu(\partial U) = 0$. Note also (see Theorem 2.5) since $\mu(D) = 1$ that $\left(R_\mu^\star\right)^{-1}(B) = (G^\star)^{-1}(B)$ so $d\left(\left(R_\mu^\star\right)^{-1}(B)\right) = d\left((G^\star)^{-1}(B)\right)$. Thus $d\left((F^\star)^{-1}(B)\right) \neq d\left(\left(R_\mu^\star\right)^{-1}(B)\right)$.

We now claim

$$R_\mu \cong F \ \text{ in } \ AC_{\partial U}(\overline{U}, E). \tag{2.11}$$

Let $Q : \overline{U} \times [0,1] \to K(E)$ be given by $Q(x,t) = H(x, t\,\mu(x))$. Now $Q_0 = F$, $Q_1 = R_\mu$, $Q : \overline{U} \times [0,1] \to K(E)$ is an upper semicontinuous compact map with $Q_t \in AC(\overline{U}, E)$ for each $t \in [0,1]$. Also $x \notin Q_t(x)$ for $x \in \partial U$ and $t \in [0,1]$ since if there exists $t \in [0,1]$ and $x \in \partial U$ with $x \in Q_t(x)$ then $x \in H(x, t\,\mu(x))$ so $x \in D$ and as a result $\mu(x) = 1$ i.e. $x \in H(x,t)$, a contradiction. Thus (2.11) holds.

Consequently $F^\star$ is d-inessential (take $J = R_\mu$ in the definition of d-inessential). □

Now Theorem 2.13 immediately yields the following continuation theorem.

Theorem 2.14. *Let E be a normed linear space, U an open subset of E, $B = \{(x,x) :\ x \in \overline{U}\}$ and d be the map defined in* (2.1). *Suppose Φ and Ψ are two maps in $AC_{\partial U}(\overline{U}, E)$ with $\Phi^\star = I \times \Phi$ and $\Psi^\star = I \times \Psi$ and with $\Phi \cong \Psi$ in $AC_{\partial U}(\overline{U}, E)$. The $\Phi^\star$ is d-inessential if and only if $\Psi^\star$ is d-inessential.*

Proof. Assume $\Phi^\star$ is d-inessential. Then (see Theorem 2.13) either $d\left((\Phi^\star)^{-1}(B)\right) = d(\emptyset)$ or there exists a map $Q \in AC_{\partial U}(\overline{U}, E)$ with $Q^\star = I \times Q$ and $Q \cong \Phi$ in $AC_{\partial U}(\overline{U}, E)$ such that $d\left((\Phi^\star)^{-1}(B)\right) \neq d\left((Q^\star)^{-1}(B)\right)$.

Suppose first that $d\left((\Phi^\star)^{-1}(B)\right) = d(\emptyset)$. There are two cases to consider, either $d\left((\Psi^\star)^{-1}(B)\right) \neq d(\emptyset)$ or $d\left((\Psi^\star)^{-1}(B)\right) = d(\emptyset)$.

Case (1). Suppose $d\left((\Psi^\star)^{-1}(B)\right) \neq d(\emptyset)$.

Then $d\left((\Phi^\star)^{-1}(B)\right) \neq d\left((\Psi^\star)^{-1}(B)\right)$ and we know $\Phi \cong \Psi$ in $AC_{\partial U}(\overline{U}, E)$. Now Theorem 2.13 (with $F = \Psi$ and $G = \Phi$) guarantees that $\Psi^\star$ is d-inessential.

Case (2). Suppose $d\left((\Psi^\star)^{-1}(B)\right) = d(\emptyset)$.

Then by definition $\Psi^\star$ is d-inessential.

Next suppose there exists a map $Q \in AC_{\partial U}(\overline{U}, E)$ with $Q^\star = I \times Q$ and $Q \cong \Phi$ in $AC_{\partial U}(\overline{U}, E)$ such that $d\left(\left(\Phi^\star\right)^{-1}(B)\right) \neq d\left(\left(Q^\star\right)^{-1}(B)\right)$. Note (since $\cong$ is an equivalence relation in $AC_{\partial U}(\overline{U}, E)$) also that $Q \cong \Psi$ in $AC_{\partial U}(\overline{U}, E)$. There are two cases to consider, either $d\left(\left(Q^\star\right)^{-1}(B)\right) \neq d\left(\left(\Psi^\star\right)^{-1}(B)\right)$ or $d\left(\left(Q^\star\right)^{-1}(B)\right) = d\left(\left(\Psi^\star\right)^{-1}(B)\right)$.

Case (1). Suppose $d\left(\left(Q^\star\right)^{-1}(B)\right) \neq d\left(\left(\Psi^\star\right)^{-1}(B)\right)$.

Then Theorem 2.13 (with $F = \Psi$ and $G = Q$) guarantees that $\Psi^\star$ is d-inessential.

Case (2). Suppose $d\left(\left(Q^\star\right)^{-1}(B)\right) = d\left(\left(\Psi^\star\right)^{-1}(B)\right)$.

Then $d\left(\left(\Phi^\star\right)^{-1}(B)\right) \neq d\left(\left(\Psi^\star\right)^{-1}(B)\right)$ and we know $\Phi \cong \Psi$ in $AC_{\partial U}(\overline{U}, E)$. Now Theorem 2.13 (with $F = \Psi$ and $G = \Phi$) guarantees that $\Psi^\star$ is d-inessential.

Thus in all cases $\Psi^\star$ is d-inessential.

Similarly if $\Psi^\star$ is d-inessential then $\Phi^\star$ is d-inessential. □

Remark 2.15. An obvious question is the condition $F \cong J$ in $AC_{\partial U}(\overline{U}, E)$ automatically satisfied in Definition 2.12 i.e. if F and J are in $AC_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$ is $F \cong J$ in $AC_{\partial U}(\overline{U}, E)$? The argument in Theorem 2.9 provides a partial answer. Let E be a infinite dimensional normed linear space and U an open convex subset of E with $0 \in U$. Let F, J be in $AC_{\partial U}(\overline{U}, E)$ with $J|_{\partial U} = F|_{\partial U}$. We know there exists a continuous retraction $r : \overline{U} \to \partial U$. Let the map $F^\star$ be given by $F^\star(x) = F(r(x))$ for $x \in \overline{U}$. Of course $F^\star(x) = J(r(x))$ for $x \in \overline{U}$ since $J|_{\partial U} = F|_{\partial U}$. With

$$H(x, \lambda) = J(2\,\lambda\, r(x) + (1 - 2\,\lambda)\, x) = J \circ j\,(x, \lambda) \quad \text{for} \quad (x, \lambda) \in \overline{U} \times \left[0, \frac{1}{2}\right]$$

(here $j : \overline{U} \times \left[0, \frac{1}{2}\right] \to \overline{U}$ is given by $j(x, \lambda) = 2\,\lambda\, r(x) + (1 - 2\,\lambda)\, x)$ it is easy to see that

$$J \cong F^\star \quad \text{in} \quad AC_{\partial U}(\overline{U}, E);$$

notice if there exists $x \in \partial U$ and $\lambda \in \left[0, \frac{1}{2}\right]$ with $x \in H_\lambda(x)$ then since $r(x) = x$ we have $x \in J(2\,\lambda\, x + (1 - 2\,\lambda)\, x) = J(x)$, a contradiction. Similarly with

$$Q(x, \lambda) = F((2 - 2\,\lambda)\, r(x) + (2\,\lambda - 1)\, x) \quad \text{for} \quad (x, \lambda) \in \overline{U} \times \left[\frac{1}{2}, 1\right]$$

it is easy to see that

$$F^\star \cong F \quad \text{in} \quad AC_{\partial U}(\overline{U}, E).$$

Combining gives $J \cong F$ in $AC_{\partial U}(\overline{U}, E)$. For a certain subclass of the acyclic maps we can obtain a more complete answer. We say $F \in K(\overline{U}, E)$ if $F : \overline{U} \to CK(E)$ is a upper continuous compact map; here $CK(E)$ denotes the family of nonempty, convex, compact subsets of E. We can also write the analogue of $K_{\partial U}(\overline{U}, E)$, essential in $K_{\partial U}(\overline{U}, E)$ and $\cong$ in $K_{\partial U}(\overline{U}, E)$. Let E be a normed linear space and U an open subset of E. If the maps F and J are in $K_{\partial U}(\overline{U}, E)$ and $J|_{\partial U} = F|_{\partial U}$

then it is easy to see that

$$\Psi(x,t) = t\,F(x) + (1-t)\,J(x)$$

guarantees that $F \cong J$ in $K_{\partial U}(\overline{U}, E)$.

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