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WELL-POSEDNESS OF GENERAL MIXED IMPLICIT QUASI-VARIATIONAL INEQUALITIES, INCLUSION PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper the concept of well-posedness to a general mixed implicit quasi-variational inequality is generalized and some characterizations of its well-posedness is given. It is proven that under appropriate conditions, the well-posedness of a general mixed implicit quasi-variational inequality is equivalent to the well-posedness of a corresponding inclusion problem. We also discuss the relations between the well-posedness of a general mixed implicit quasi-variational inequality and the well-posedness of a fixed point problem. Finally, we derive some conditions under which a general mixed implicit quasi-variational inequality is well-posed.

1. Introduction

The concept of well-posedness for a minimization problem was initially introduced by Tykhonov [31] and ever since has been widely studied, since the concept of well-posedness has been applied to other contexts: variational inequality problems [7, 10, 18–21], saddle point problems [5], Nash equilibrium problems [20, 22–26, 28], inclusion problems [16, 17], and fixed point problems [16, 17, 32]. The concept of Tykhonov well-posedness in the generalized sense was introduced, which means the existence of minimizers and the convergence of some subsequence of every minimizing sequence toward a minimizer. It is clear that the concept of well-posedness is motivated by the numerical methods producing optimizing sequences. On account of its importance in optimization problems, various concepts of well-posedness have been introduced and studied for minimization problems in past decades; see, e.g., [2, 8, 14, 21, 27, 31, 37, 38].

Concerning the well-posedness of a given variational problem, it is very interesting and quite important to establish its metric characterization, to find conditions under which the problem is well-posed, and to investigate its links with the well-posedness of other related problems. Some metric characterizations of various well-posedness were established for minimization problems [8], variational inequalities [7, 10, 18, 19] and Nash equilibrium problems [25]. For the well-posedness conditions of various variational problems, we refer the readers to [7, 8, 10, 18, 19, 26, 28]. The relations between the well-posedness of variational inequalities and the well-posedness of minimization problems were discussed in [7, 19, 21]. Lemaire [16] discussed the relations among the well-posedness of minimization problems, inclusion problems

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and fixed point problems. Recently, Lemaire et al [17] further extended the result in [16] by considering perturbations.

Very recently, motivated by the mentioned work as above, Fang, Huang and Yao [11] investigated the well-posedness of a mixed variational inequality which includes as a special case the classical variational inequality. They gave some metric characterizations of its well-posedness and establish the links with the well-posedness of inclusion problems and fixed point problems. Furthermore, they proved that under suitable conditions, the well-posedness of the mixed variational inequality is equivalent to the existence and uniqueness of its solutions, and the well-posedness in the generalized sense is equivalent to the existence of solutions.

In this paper, inspired by Fang, Huang and Yao [11], we generalize the concept of well-posedness to a general mixed implicit quasi-variational inequality which includes as a special case the mixed variational inequality. We derive some metric characterizations of its well-posedness and establish the links with the well-posedness of inclusion problems and fixed point problems. Finally we also prove that under appropriate conditions, the well-posedness of the general mixed implicit quasi-variational inequality is equivalent to the existence and uniqueness of its solutions, and the well-posedness in the generalized sense is equivalent to the existence of solutions. The results presented in this paper are the improvements and extension of the corresponding ones in Fang, Huang and Yao [11].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. In order to show the main results, we need the follow concepts and results.

Definition 2.1. A mapping $g : H \to H$ is said to be

(i) monotone if

$$\langle g(x) - g(y), x - y \rangle \geq 0, \quad \forall x, y \in H;$$

(ii) δ-strongly monotone if there exists a constant $\delta > 0$ such that

$$\langle g(x) - g(y), x - y \rangle \geq \delta \|x - y\|^2, \quad \forall x, y \in H;$$

(iii) σ-Lipschitz continuous if there exists a constant $\sigma > 0$ such that

$$\|g(x) - g(y)\| \leq \sigma \|x - y\|, \quad \forall x, y \in H.$$

We remark that if mapping $g : H \to H$ is δ-strongly monotone and σ-Lipschitz continuous then g is a homeomorphism.

Let $\varphi : H \to \mathcal{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Denote by $\text{dom}\varphi$ the efficient domain of φ, i.e.,

$$\text{dom}\varphi = \{x \in H : \varphi(x) < +\infty\}.$$

Denote by $\partial\varphi$ and $\partial_\epsilon\varphi$ the subdifferential and ϵ-subdifferential of φ respectively, i.e.,

$$\partial\varphi(x) = \{x^* \in H : \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle, \forall y \in H\}, \quad \forall x \in \text{dom}\varphi$$

and

$$\partial_\epsilon\varphi(x) = \{x^* \in H : \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle - \epsilon, \forall y \in H\}, \quad \forall x \in \text{dom}\varphi.$$

It is known that $\partial_\epsilon \varphi(x) \supset \partial\varphi(x) \neq \emptyset$ for all $x \in \text{dom}\varphi$ and for all $\epsilon > 0$.

Now, let $F, g : H \to H$ be two mappings and $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y) : H \to H$ is a proper, convex and lower semicontinuous functional on H and $g(H) \cap \text{dom}\partial\phi(\cdot, y) \neq \emptyset$. Consider the following general mixed implicit quasi-variational inequality associated with (F, g, ϕ):

$$\text{GMIQVI}(F, g, \phi) : \qquad \begin{cases} \text{Find } x \in H \text{ such that } g(x) \in \text{dom}\partial\phi(\cdot, x) \text{ and} \\ \langle F(x), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \leq 0, \quad \forall y \in H, \end{cases}$$

which has been studied intensively in Ding [6].

Some special cases.

(i) If $\phi(x, y) = \varphi(x)$ for all $x, y \in H$, the GMIQVI(F, g, ϕ) reduces to the following general mixed variational inequality associated with (F, g, φ):

$$\text{GMVI}(F, g, \varphi) : \qquad \begin{cases} \text{Find } x \in H \text{ such that } g(x) \in \text{dom}\partial\varphi \text{ and} \\ \langle F(x), g(x) - y\rangle + \varphi(g(x)) - \varphi(y) \leq 0, \quad \forall y \in H, \end{cases}$$

which was considered and studied by Hassouni and Moudafi [13].

(ii) If $g = I$ the identity mapping of H, then the GMVI(F, g, φ) reduces to the following mixed variational inequality associated with (F, φ):

$$\text{MVI}(F, \varphi) : \qquad \text{Find } x \in H \text{ such that } \langle F(x), x - y\rangle + \varphi(x) - \varphi(y) \leq 0, \quad \forall y \in H,$$

which has been studied intensively (see, e.g., [3, 9, 11, 29, 35, 36]).

(iii) If $K : H \to 2^H$ is a given multifunction such that each $K(x)$ is a closed convex subset of H (or $K(x) = m(x) + K$ where $m : H \to H$ and K is a closed convex subset of H) and if $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ is defined by

$$\phi(x, y) = \delta_{K(y)}(x), \quad \forall x, y \in H,$$

where $\delta_{K(y)}(x)$ is the indicator function of $K(y)$, i.e.,

$$\delta_{K(y)}(x) = \begin{cases} 0, \text{ if } x \in K(y), \\ +\infty, \quad \text{otherwise}, \end{cases}$$

then the GMIQVI(F, g, ϕ) reduces to the following strongly nonlinear quasi-variational inequality:

$$\text{SNQVI}(F, g, K) : \qquad \begin{cases} \text{Find } x \in H \text{ such that } g(x) \in K(x) \text{ and} \\ \langle F(x), g(x) - y\rangle \leq 0, \quad \forall y \in K(x), \end{cases}$$

which includes a number of classes of variational inequalities, quasi-variational inequalities, complementarity and quasi-complementarity problems, studied previously by many authors, see, e.g., [6, 12, 30, 33, 34].

It is easy to see that $\partial_\epsilon\phi(x, y) \supset \partial\phi(x, y) \neq \emptyset$ for all $x \in \text{dom}\phi(\cdot, y)$ and for all $\epsilon > 0$. In terms of $\partial\phi(\cdot, y)$, GMIQVI(F, g, ϕ) is equivalent to the following inclusion problem associated with (F, g, ϕ):

$$\text{IP}(F, g, \phi) : \qquad \text{Find } x \in H \text{ such that } 0 \in F(x) + \partial\phi(g(x), x).$$

For each fixed $y \in H$, the resolvent operator of $\partial\phi(\cdot, y)$ is defined by

$$J_\lambda^{\partial\phi(\cdot, y)}(x) = (I + \lambda\partial\phi(\cdot, y))^{-1}(x), \quad \forall x \in H,$$

which is well-defined, single-valued and nonexpansive, where $\lambda > 0$ is a constant. Recall that a mapping $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$. In terms of $J_\lambda^{\partial\phi(\cdot,x)}$, GMIQVI$(F, g, \phi)$ is also equivalent to the following fixed point problem associated with (F, g, ϕ):

$$\text{FP}(F, g, \phi): \qquad \text{Find } x \in H \text{ such that } x = x - g(x) + J_\lambda^{\partial\phi(\cdot,x)}(g(x) - \lambda F(x)).$$

Summarizing the above results, we have the following lemma:

Lemma 2.2. *Let $F, g : H \to H$ be two mappings and $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper, convex and lower semicontinuous functional satisfying $g(H) \cap \text{dom}\partial\phi(\cdot, y) \neq \emptyset$. Then the following conclusions are equivalent:*
(i) *x solves* GMIQVI(F, g, ϕ)*;*
(ii) *x solves* IP(F, g, ϕ)*;*
(iii) *x solves* FP(F, g, ϕ)*, where $\lambda > 0$ is a constant.*

Proof. For the sake of completeness we give the proof of the lemma. Observe that for each $y \in H$,

$$\begin{aligned} & x \text{ solves GMIQVI}(F, g, \phi) \\ \Leftrightarrow\ & \langle F(x), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \le 0 \\ \Leftrightarrow\ & \phi(y, x) - \phi(g(x), x) \ge \langle -F(x), y - g(x)\rangle \\ \Leftrightarrow\ & -F(x) \in \partial\phi(g(x), x) \\ \Leftrightarrow\ & 0 \in F(x) + \partial\phi(g(x), x), \end{aligned}$$

and for some $\lambda > 0$

$$\begin{aligned} & 0 \in F(x) + \partial\phi(g(x), x) \\ \Leftrightarrow\ & g(x) - \lambda F(x) \in g(x) + \lambda\partial\phi(g(x), x) \\ \Leftrightarrow\ & g(x) = (I + \lambda\partial\phi(\cdot, x))^{-1}(g(x) - \lambda F(x)) \\ \Leftrightarrow\ & x = x - g(x) + J_\lambda^{\partial\phi(\cdot,x)}(g(x) - \lambda F(x)). \end{aligned}$$

Thus, conclusions (i)-(iii) are equivalent. □

Definition 2.3. Let $F, g : H \to H$ be two mappings. F is said to be g-hemicontinuous if for any $x, y \in H$, the function $t \mapsto \langle F(x + t(y - x)), g(y) - g(x)\rangle$ from $[0, 1]$ into $\mathcal{R}$ is continuous at 0^+.

Clearly, the continuity implies the hemicontinuity, but the converse is not true in general.

Definition 2.4. A mapping $F : H \to H$ is said to be uniformly continuous if for any neighborhood V of 0, there exists a neighborhood U of 0 such that $F(x) - F(y) \in V$ for all $x, y \in H$ with $x - y \in U$. Obviously, the uniform continuity implies the hemicontinuity.

Definition 2.5. Let $F, g : H \to H$ be two mappings.
(i) F is said to be g-monotone if

$$\langle F(x) - F(y), g(x) - g(y)\rangle \ge 0, \quad \forall x, y \in H.$$

At the same time, g is said to be F-monotone.

(ii) F is said to be g-convex if for each $\lambda \in [0,1]$ and each $x, y \in H$,

$$\langle F(w_\lambda), g(w_\lambda)\rangle \le \lambda\langle F(w_\lambda), g(x)\rangle + (1-\lambda)\langle F(w_\lambda), g(y)\rangle$$

where $w_\lambda = \lambda x + (1-\lambda)y$.

Lemma 2.6. *Let $g : H \to H$ be a homeomorphism. Let $F : H \to H$ be g-monotone, g-convex and g-hemicontinuous. Let $\phi : H \times H \to \mathcal{R}\cup\{+\infty\}$ be such that for each fixed $y \in H$, $\phi(g(\cdot), y)$ is a proper, convex and lower semicontinuous functional, and $x \in H$ a given point. Then*

$$\langle F(x), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \le 0, \quad \forall y \in H$$

if and only if

$$\langle F(y), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \le 0, \quad \forall y \in H.$$

Proof. Suppose

$$\langle F(x), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \le 0, \quad \forall y \in H. \tag{2.1}$$

Since $g : H \to H$ is a homeomorphism, we know that inequality (2.1) is equivalent to the following inequality

$$\langle F(x), g(x) - g(y)\rangle + \phi(g(x), x) - \phi(g(y), x) \le 0, \quad \forall y \in H.$$

Since F is g-monotone, we have

$$\langle F(y) - F(x), g(y) - g(x)\rangle \ge 0, \quad \forall y \in H$$

and hence

$$\begin{aligned}&\langle F(y), g(y) - g(x)\rangle + \phi(g(y), x) - \phi(g(x), x)\\ &\qquad \ge \langle F(x), g(y) - g(x)\rangle + \phi(g(y), x) - \phi(g(x), x) \ge 0,\end{aligned}$$

for all $y \in H$. Consequently,

$$\langle F(y), g(x) - g(y)\rangle + \phi(g(x), x) - \phi(g(y), x) \le 0, \quad \forall y \in H.$$

Utilizing again the condition that $g : H \to H$ is a homeomorphism, we deduce that

$$\langle F(y), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \le 0, \quad \forall y \in H. \tag{2.2}$$

Conversely, suppose inequality (2.2) is valid. Since $g : H \to H$ is a homeomorphism, we deduce that inequality (2.2) is equivalent to the following inequality

$$\langle F(y), g(y) - g(x)\rangle + \phi(g(y), x) - \phi(g(x), x) \ge 0, \quad \forall y \in H.$$

For any given $y \in H$ we define $y_t = ty + (1-t)x$ for all $t \in (0,1)$. Replacing y by y_t in the left-hand side of the last inequality, and utilizing the convexity of $\phi(g(\cdot), x)$ and the g-convexity of F, we derive for each $t \in (0,1)$,

$$\begin{aligned}0 &\le \langle F(y_t), g(y_t) - g(x)\rangle + \phi(g(y_t), x) - \phi(g(x), x)\\ &= \langle F(y_t), g(y_t)\rangle - \langle F(y_t), g(x)\rangle + \phi(g(y_t), x) - \phi(g(x), x)\\ &\le t\langle F(y_t), g(y)\rangle + (1-t)\langle F(y_t), g(x)\rangle - \langle F(y_t), g(x)\rangle\\ &\quad + t\phi(g(y), x) + (1-t)\phi(g(x), x) - \phi(g(x), x)\\ &= t[\langle F(y_t), g(y) - g(x)\rangle + \phi(g(y), x) - \phi(g(x), x)],\end{aligned}$$

which hence implies that

$$\langle F(y_t), g(x) - g(y)\rangle + \phi(g(x), x) - \phi(g(y), x) \le 0, \quad \forall t \in (0,1). \tag{2.3}$$

Since F is g-hemicontinuous, we conclude from (2.3) that as $t \to 0^+$,

$$\langle F(x), g(x) - g(y)\rangle + \phi(g(x), x) - \phi(g(y), x) \leq 0.$$

Since $g : H \to H$ is a homeomorphism, from the arbitrariness of y we obtain

$$\langle F(x), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \leq 0, \quad \forall y \in H.$$

This completes the proof. □

Definition 2.7 (See [15])**.** Let A be a nonempty subset of H. The measure of noncompactness μ of the set A is defined by

$$\mu(A) = \inf\{\epsilon > 0 : A \subset \cup_{i=1}^{n} A_i,\ \text{diam} A_i < \epsilon,\ i = 1, 2, ..., n\},$$

where diam means the diameter of a set.

Definition 2.8. Let A, B be nonempty subsets of H. The Hausdorff metric $\mathcal{H}(\cdot, \cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$. Let $\{A_n\}$ be a sequence of nonempty subsets of H. We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all section $a_n \in A_n$. For more details on this topic, we refer the readers to [1, 15].

3. Well-posedness and metric characterization

In this section we introduce sone concepts of well-posedness of the general mixed implicit quasi-variational inequality and establish their metric characterizations. Let $\alpha \geq 0$ be a given number and let H, F, g, ϕ be defined as in the previous section.

Definition 3.1. Let $g : H \to H$ be a homeomorphism. A sequence $\{x_n\} \subset H$ is said to be an α-approximating sequence for GMIQVI(F, g, ϕ) if there exists a sequence $\{\epsilon_n\}$ of nonnegative numbers with $\epsilon_n \to 0$ such that

$$g(x_n) \in \text{dom}\phi(\cdot, x_n),$$

$$\langle F(x_n), g(x_n) - y\rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \frac{\alpha}{2}\|x_n - g^{-1}(y)\|^2 + \epsilon_n, \forall y \in H$$

for all $n \in N$. If $\alpha_1 > \alpha_2 \geq 0$, then every α_2-approximating sequence is α_1-approximating. When $\alpha = 0$, we say that $\{x_n\}$ is approximating for GMIQVI(F, g, ϕ).

Definition 3.2. We say that GMIQVI(F, g, ϕ) is strongly (resp. weakly) α-well-posed if GMIQVI(F, g, ϕ) has a unique solution and every α-approximating sequence converges strongly (resp. weakly) to the unique solution. In the sequel, strong (resp. weak) 0-well-posedness is always called as strong (resp. weak) well-posedness. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp. weak) α_1-well-posedness implies strong (resp. weak) α_2-well-posedness.

Remark 3.3. When $\phi(x, y) = \delta_K(x)$ and $g = I$ where K is a closed convex subset of H, Definition 3.2 reduces to the definition of strong (resp. weak) α-well-posedness for the classical variational inequality. For details, we refer the readers to [7, 19, 20] and the references therein.

Definition 3.4. We say that GMIQVI(F, g, ϕ) is strongly (resp. weakly) α-well-posed in the generalized sense if GMIQVI(F, g, ϕ) has a nonempty solution set S and every α-approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of S. When $\alpha = 0$, we say that GMIQVI(F, g, ϕ) is strongly (resp. weakly) well-posed in the generalized sense. Clearly, if $\alpha_1 > \alpha_2 \geq 0$, then strong (resp. weak) α_1-well-posedness in the generalized sense implies strong (resp. weak) α_2-well-posedness in the generalized sense.

Remark 3.5. When $\phi(x, y) = \delta_K(x)$ and $g = I$ where K is a closed convex subset of H, Definition 3.4 reduces to the definition of strong (resp. weak) α-well-posedness in the generalized sense for the classical variational inequality. For details, we refer the readers to [7, 19, 20] and the references therein.

Let $g : H \to H$ be a homeomorphism. The α-approximating solution set of GMIQVI(F, g, ϕ) is defined by

$$\Omega_\alpha(\epsilon) = \{x \in H : \langle F(x), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \\ \leq \frac{\alpha}{2}\|x - g^{-1}(y)\|^2 + \epsilon,\ \forall y \in H\},\ \forall \epsilon \geq 0.$$

Now we give a metric characterization of strong α-well-posedness for GMIQVI(F, g, ϕ).

Theorem 3.6. *Let $g : H \to H$ be a homeomorphism. Let $F : H \to H$ be g-monotone, g-convex and g-hemicontinuous. Assume that for each fixed $y \in H$ there hold the following conditions for a proper functional $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$:*
(i) *$x \mapsto \phi(g(x), y)$ is convex;*
(ii) *$x \mapsto \phi(g(x), x) - \phi(g(y), x)$ is lower semicontinuous.*
Then GMIQVI(F, g, ϕ) *is strongly α-well-posed if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \quad \text{and} \quad \text{diam}\Omega_\alpha(\epsilon) \to 0 \text{ as } \epsilon \to 0. \tag{3.1}$$

Proof. Suppose that GMIQVI(F, g, ϕ) is strongly α-well-posed. Then GMIQVI(F, g, ϕ) has a unique solution which lies in $\Omega_\alpha(\epsilon)$ for all $\epsilon > 0$. If diam$\Omega_\alpha(\epsilon) \not\to 0$ as $\epsilon \to 0$, then there exist constant $l > 0$ and sequences $\{\epsilon_n\} \subset \mathcal{R}_+$ with $\epsilon_n \to 0$, and $\{u_n\}, \{v_n\}$ with $u_n, v_n \in \Omega_\alpha(\epsilon_n)$ such that

$$\|u_n - v_n\| > l, \quad \forall n \in N. \tag{3.2}$$

Since $u_n, v_n \in \Omega_\alpha(\epsilon_n)$, both $\{u_n\}$ and $\{v_n\}$ are α-approximating sequences for GMIQVI(F, g, ϕ). So they have to converge strongly to the unique solution of GMIQVI(F, g, ϕ), a contraction to (3.2).

Conversely, suppose that condition (3.1) holds. Let $\{x_n\} \subset H$ be an α-approximating sequence for GMIQVI(F, g, ϕ). Then there exists a sequence $\{\epsilon_n\} \subset \mathcal{R}_+$ with $\epsilon_n \to 0$ such that

$$g(x_n) \in \text{dom}\phi(\cdot, x_n),$$

$$\langle F(x_n), g(x_n) - y\rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \frac{\alpha}{2}\|x_n - g^{-1}(y)\|^2 + \epsilon_n, \forall y \in H$$

for all $n \in N$. This implies that $x_n \in \Omega_\alpha(\epsilon_n)$. From (3.1), we know that $\{x_n\}$ is a Cauchy sequence and so it converges strongly to a point $\bar{x} \in H$. Note that

$g : H \to H$ is a homeomorphism. Hence the last inequality is equivalent to the following inequality

$$\langle F(x_n), g(x_n)-g(y)\rangle+\phi(g(x_n), x_n)-\phi(g(y), x_n) \leq \frac{\alpha}{2}\|x_n-y\|^2+\epsilon_n,\ \forall y \in H. \tag{3.3}$$

Since F is g-monotone and functional $x \mapsto \phi(g(x), x) - \phi(g(y), x)$ is lower semicontinuous for each fixed $y \in H$, it follows from (3.3) that

$$\begin{aligned}
&\langle F(y), g(\bar{x}) - g(y)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y), \bar{x}) \\
&\leq \liminf_{n\to\infty}\{\langle F(y), g(x_n) - g(y)\rangle + \phi(g(x_n), x_n) - \phi(g(y), x_n)\} \\
&\leq \liminf_{n\to\infty}\{\langle F(x_n), g(x_n) - g(y)\rangle + \phi(g(x_n), x_n) - \phi(g(y), x_n)\} \\
&\leq \liminf_{n\to\infty}\{\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n\} \\
&= \tfrac{\alpha}{2}\|\bar{x} - y\|^2, \quad \forall y \in H.
\end{aligned}$$

For any $y \in H$, put $y_t = (1 - t)\bar{x} + ty,\ \forall t \in (0, 1)$. Then

$$\langle F(y_t), g(\bar{x}) - g(y_t)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y_t), \bar{x}) \leq \frac{\alpha}{2}\|\bar{x} - y_t\|^2, \quad \forall t \in (0, 1).$$

Since F is g-convex and $\phi(g(\cdot), \bar{x})$ is convex, we have

$$\langle F(y_t), g(\bar{x}) - g(y)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y), \bar{x}) \leq \frac{t\alpha}{2}\|\bar{x} - y\|^2, \quad \forall t \in (0, 1).$$

Letting $t \to 0^+$ in the last inequality, from the g-hemicontinuity of F we get

$$\langle F(\bar{x}), g(\bar{x}) - g(y)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y), \bar{x}) \leq 0$$

for each $y \in H$. Since $g : H \to H$ is a homeomorphism, we have

$$\langle F(\bar{x}), g(\bar{x}) - y\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq 0, \quad \forall y \in H.$$

This shows that $\bar{x}$ solves GMIQVI(F, g, ϕ).

To complete the proof, we need only to prove that GMIQVI(F, g, ϕ) has a unique solution. Assume by contradiction that GMIQVI(F, g, ϕ) has two distinct solutions x_1 and x_2. Then it is easy to see that $x_1, x_2 \in \Omega_\alpha(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \|x_1 - x_2\| \leq \mathrm{diam}\Omega_\alpha(\epsilon) \to 0 \quad \text{as } \epsilon \to 0,$$

a contradiction to (3.1). This completes the proof. □

Remark 3.7. Theorem 3.6 generalizes Theorem 3.1 of [11], and hence Proposition 2.2 of [7].

In terms of noncompact measure, we have the following analogous metric characterization of strong α-well-posedness in the generalized sense.

Theorem 3.8. *Let $g : H \to H$ be a homeomorphism and $F : H \to H$ be such that the functional $x \mapsto \langle F(x), g(x) - g(y)\rangle$ is lower semicontinuous for each fixed $y \in H$. Let $\phi : H\times H \to \mathcal{R}\cup\{+\infty\}$ be a proper functional such that $x \mapsto \phi(g(x), x)-\phi(g(y), x)$ is lower semicontinuous for each fixed $y \in H$. Then* GMIQVI(F, g, ϕ) *is strongly α-well-posed in the generalized sense if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \quad \text{and} \quad \mu(\Omega_\alpha(\epsilon)) \to 0 \text{ as } \epsilon \to 0. \tag{3.4}$$

Proof. Suppose that GMIQVI(F, g, ϕ) is strongly α-well-posed in the generalized sense. Let S be the solution set of GMIQVI(F, g, ϕ). Then S is nonempty and compact. Indeed, let $\{x_n\}$ be any sequence in S. Then $\{x_n\}$ is α-approximating for GMIQVI(F, g, ϕ). Since GMIQVI(F, g, ϕ) is strongly α-well-posed in the generalized sense, $\{x_n\}$ has a subsequence which converges strongly to some point of S. Thus S is compact. Clearly, $\Omega_\alpha(\epsilon) \supset S \neq \emptyset$ for all $\epsilon > 0$. Now let us show that

$$\mu(\Omega_\alpha(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0.$$

Observe that for every $\epsilon > 0$,

$$\mathcal{H}(\Omega_\alpha(\epsilon), S) = \max\{e(\Omega_\alpha(\epsilon), S), e(S, \Omega_\alpha(\epsilon))\} = e(\Omega_\alpha(\epsilon), S).$$

Taking into account the compactness of S, we get

$$\mu(\Omega_\alpha(\epsilon)) \leq 2\mathcal{H}(\Omega_\alpha(\epsilon), S) = 2e(\Omega_\alpha(\epsilon), S).$$

To prove (3.4), it is sufficient to show that

$$e(\Omega_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

If $e(\Omega_\alpha(\epsilon), S) \not\to 0$ as $\epsilon \to 0$, then there exist $l > 0$ and $\{\epsilon_n\} \subset \mathcal{R}_+$ with $\epsilon_n \to 0$, and $x_n \in \Omega_\alpha(\epsilon_n)$ such that

$$x_n \notin S + B(0, l), \quad \forall n \in N, \tag{3.5}$$

where $B(0, l)$ is the closed ball centered at 0 with radius l. Because of $x_n \in \Omega_\alpha(\epsilon_n)$, $\{x_n\}$ is an α-approximating sequence for GMIQVI(F, g, ϕ). Since GMIQVI(F, g, ϕ) is strongly α-well-posed in the generalized sense, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging strongly to some point of S. This contradicts to (3.5) and so

$$e(\Omega_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

Conversely, assume that (3.4) holds. First, let us show that $\Omega_\alpha(\epsilon)$ is closed for all $\epsilon > 0$. Let $\{x_n\} \subset \Omega_\alpha(\epsilon)$ with $x_n \to x$. Then

$$\langle F(x_n), g(x_n) - y\rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \leq \frac{\alpha}{2}\|x_n - g^{-1}(y)\|^2 + \epsilon, \quad \forall y \in H.$$

Since $g : H \to H$ is a homeomorphism, the last inequality is equivalent to the following one

$$\langle F(x_n), g(x_n) - g(y)\rangle + \phi(g(x_n), x_n) - \phi(g(y), x_n) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon, \quad \forall y \in H.$$

Since $z \mapsto \langle F(z), g(z) - g(y)\rangle$ and $z \mapsto \phi(g(z), z) - \phi(g(y), z)$ are lower semicontinuous for each fixed $y \in H$, we deduce that

$$\langle F(x), g(x) - g(y)\rangle + \phi(g(x), x) - \phi(g(y), x) \leq \frac{\alpha}{2}\|x - y\|^2 + \epsilon, \quad \forall y \in H.$$

Since $g : H \to H$ is a homeomorphism, the last inequality is equivalent to the following one

$$\langle F(x), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \leq \frac{\alpha}{2}\|x - g^{-1}(y)\|^2 + \epsilon, \quad \forall y \in H.$$

This shows that $x \in \Omega_\alpha(\epsilon)$ and so $\Omega_\alpha(\epsilon)$ is nonempty closed for all $\epsilon > 0$. Observe that

$$S = \bigcap_{\epsilon > 0} \Omega_\alpha(\epsilon).$$

Since $\mu(\Omega_\alpha(\epsilon)) \to 0$, the Theorem on page 412 of [15] can be applied and one concludes that S is nonempty and compact with

$$e(\Omega_\alpha(\epsilon), S) = \mathcal{H}(\Omega_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

Let $\{u_n\} \subset H$ be an α-approximating sequence for GMIQVI(F, g, ϕ). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$g(u_n) \in \mathrm{dom}\phi(\cdot, u_n),$$

$$\langle F(u_n), g(u_n) - y\rangle + \phi(g(u_n), u_n) - \phi(y, u_n) \le \frac{\alpha}{2}\|u_n - g^{-1}(y)\|^2 + \epsilon_n, \ \forall y \in H$$

for all $n \in N$. So $u_n \in \Omega_\alpha(\epsilon_n)$ follows from definition. It follows from (3.4) that

$$d(u_n, S) \le e(\Omega_\alpha(\epsilon_n), S) \to 0.$$

Since S is compact, there exists $\bar{x}_n \in S$ such that

$$\|u_n - \bar{x}_n\| = d(u_n, S) \to 0.$$

Again from the compactness od S, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging strongly to $\bar{x} \in S$. Hence the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges strongly to $\bar{x}$. Thus GMIQVI(F, g, ϕ) is strongly α-well-posed in the generalized sense. □

Now we give the following example as an application of Theorem 3.6.

Example 3.9. Let H be the 2-dimensional Euclidean space $\mathcal{R}^2$. As usual, its inner product and norm are defined as

$$\langle x, y\rangle = x_1y_1 + x_2y_2 \quad \text{and} \quad \|x\| = \sqrt{x_1^2 + x_2^2}, \quad \forall x, y \in H$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Define mappings $F, g : H \to H$ and functional $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ as follows

$$F(x) = g(x) = (x_1 - x_2, x_1 + x_2) \quad \text{and} \quad \phi(x, y) = \|x\|^2 + \|y\|^2, \quad \forall x, y \in H$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Clearly, g is a homeomorphism, F is g-monotone, g-convex and g-hemicontinuous, and ϕ is a proper functional such that for each fixed $y \in H$ the following conditions are satisfied:

(i) $\phi(g(x), y) = 2\|x\|^2 + \|y\|^2$ is convex in the variable x;

(ii) $\phi(g(x), x) - \phi(g(y), x) = 2(\|x\|^2 - \|y\|^2)$ is lower semicontinuous in the variable x.

Let $\alpha = 4$. Then

$$\begin{aligned}
&\Omega_4(\epsilon)\\
&= \{x \in H : \langle F(x), g(x) - y\rangle + \phi(g(x), x) - \phi(y, x) \le \frac{\alpha}{2}\|x - g^{-1}(y)\|^2 + \epsilon, \forall y \in H\}\\
&= \{x \in H : \langle F(x), g(x) - g(y)\rangle + \phi(g(x), x) - \phi(g(y), x) \le \frac{\alpha}{2}\|x - y\|^2 + \epsilon, \forall y \in H\}\\
&= \{x \in H : 2\langle x, x - y\rangle + 2(\|x\|^2 - \|y\|^2) \le 2\|x - y\|^2 + \epsilon, \forall y \in H\}\\
&= \{x \in H : \langle x, x - y\rangle + \|x\|^2 - \|y\|^2 - \|x - y\|^2 - \frac{\epsilon}{2} \le 0, \forall y \in H\}\\
&= \{x \in H : -2\|y - \frac{x}{4}\|^2 + \frac{9\|x\|^2}{8} - \frac{\epsilon}{2} \le 0, \forall y \in H\}\\
&= \{x \in H : \|x\| \le \frac{2\sqrt{\epsilon}}{3}\}.
\end{aligned}$$

By Theorem 3.6, GMIQVI(F, g, ϕ) is 4-well-posed since $\text{diam}\Omega_4(\epsilon) = \frac{4\sqrt{\epsilon}}{3} \to 0$ as $\epsilon \to 0$.

4. Links with well-posedness of inclusion problems

In this section we shall investigate the relations between the well-posedness of general mixed implicit quasi-variational inequalities and the well-posedness of inclusion problems. In what follows we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let $A : H \to 2^H$ be a set-valued mapping. The inclusion problem associated with A is defined by

$$\text{IP}(A) : \qquad \text{Find } x \in H \text{ such that } 0 \in A(x).$$

Definition 4.1 ([16, 17]). A sequence $\{x_n\} \subset H$ is called an approximating sequence for IP(A) if $d(0, A(x_n)) \to 0$, or equivalently, there exists $y_n \in A(x_n)$ such that $\|y_n\| \to 0$ as $n \to \infty$.

Definition 4.2 ([16,17]). We say that IP(A) is strongly (resp. weakly) well-posed if it has a unique solution and every approximating sequence converges strongly (resp. weakly) to the unique solution of IP(A). IP(A) is said to be strongly (resp. weakly) well-posed in the generalized sense if the solution set S of IP(A) is nonempty and every approximating sequence has a subsequence which converges strongly (resp. weakly) to a point of S.

Definition 4.3. A proper functional $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ is said to be diagonally convex if for each $(u_i, v_i) \in H \times H,\ i = 1, 2$ and each $\lambda \in [0, 1]$

$$\phi(\lambda u_1 + (1 - \lambda)u_2, \lambda v_1 + (1 - \lambda)v_2) \le \lambda\phi(u_1, v_1) + (1 - \lambda)\phi(u_2, v_2).$$

Remark 4.4. If a proper functional $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ is diagonally convex, then for each fixed $(x, y) \in H \times H$ the functionals $u \mapsto \phi(u, y)$ and $v \mapsto \phi(x, v)$ are convex. Now we illustrate the concept of diagonal convexity. Take two fixed vectors $a, b \in H$. Define a proper functional $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ as follows

$$\phi(x, y) = \|x\|^2 + \langle a, y - b\rangle, \quad \forall (x, y) \in H \times H.$$

Then it is easy to see that $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ is diagonally convex. Meantime, it is clear that $\phi(y, \cdot)$ is affine for each fixed $y \in H$.

The following theorems establish the relations between the strong (resp. weak) well-posedness of general mixed implicit quasi-variational inequalities and the strong (resp. weak) well-posedness of inclusion problems.

Theorem 4.5. *Let $g : H \to H$ be a homeomorphism which is affine and σ-Lipschitz continuous. Let $F : H \to H$ be g-monotone and g-hemicontinuous. Let $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ be a proper and diagonally convex functional such that for each fixed $y \in H$ there hold the following conditions:*
(i) *$\phi(y, \cdot)$ is affine;*
(ii) *$\phi(\cdot, y)$ is a lower semicontinuous functional satisfying $g(H) \cap \text{dom}\partial\phi(\cdot, y) \neq \emptyset$;*
(iii) *$x \mapsto \phi(g(x), x) - \phi(g(y), x)$ is weakly lower semicontinuous.*
If GMIQVI(F, g, ϕ) *is weakly well-posed, then* IP(F, g, ϕ) *is weakly well-posed.*

Proof. Suppose that GMIQVI(F, g, ϕ) is weakly well-posed. Then GMIQVI(F, g, ϕ) has a unique solution x^*. By Lemma 2.2, x^* is also the unique solution of IP(F, g, ϕ). Let $\{x_n\}$ be an approximating sequence for IP(F, g, ϕ). Then there exists $y_n \in F(x_n) + \partial\phi(g(x_n), x_n)$ such that $\|y_n\| \to 0$. It follows that

$$\phi(y, x_n) - \phi(g(x_n), x_n) \geq \langle y_n - F(x_n), y - g(x_n)\rangle, \quad \forall y \in H, n \in N.$$

Since $g : H \to H$ is a homeomorphism, the last inequality is equivalent to the following one

$$\phi(g(y), x_n) - \phi(g(x_n), x_n) \geq \langle y_n - F(x_n), g(y) - g(x_n)\rangle, \quad \forall y \in H, n \in N. \tag{4.1}$$

If $\{x_n\}$ is unbounded, without loss of generality, we may assume that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we may assume that $t_n \in (0, 1]$ and $z_n \rightharpoonup z(\neq x^*)$. For any $y \in H$, it follows from the affinity of g that

$$\begin{aligned} \langle F(y), g(z) - g(y)\rangle &= \langle F(y), g(z) - g(z_n)\rangle + \langle F(y), g(z_n) - g(x^*)\rangle \\ &\quad + \langle F(y), g(x^*) - g(y)\rangle \\ &= \langle F(y), g(z) - g(z_n)\rangle + t_n\langle F(y), g(x_n) - g(x^*)\rangle. \\ &\quad + \langle F(y), g(x^*) - g(y)\rangle \\ &= \langle F(y), g(z) - g(z_n)\rangle + t_n\langle F(y), g(x_n) - g(y)\rangle \\ &\quad + (1 - t_n)\langle F(y), g(x^*) - g(y)\rangle \end{aligned} \tag{4.2}$$

Since F is g-monotone,

$$\begin{aligned} &\langle F(y), g(x^*) - g(y)\rangle \leq \langle F(x^*), g(x^*) - g(y)\rangle \\ &\qquad \text{and} \quad \langle F(y), g(x_n) - g(y)\rangle \leq \langle F(x_n), g(x_n) - g(y)\rangle. \end{aligned} \tag{4.3}$$

Furthermore, since x^* is the unique solution of GMIQVI(F, g, ϕ), we have

$$\langle F(x^*), g(x^*) - y\rangle + \phi(g(x^*), x^*) - \phi(y, x^*) \leq 0, \quad \forall y \in H$$

which is equivalent to the following inequality

$$\langle F(x^*), g(x^*) - g(y)\rangle + \phi(g(x^*), x^*) - \phi(g(y), x^*) \leq 0, \quad \forall y \in H. \tag{4.4}$$

Also, since ϕ is diagonally convex, and both $\phi(g(y), \cdot)$ and g are affine, it follows from (4.1)-(4.4) that

$$\begin{aligned} &\langle F(y), g(z) - g(y)\rangle \\ &\leq \langle F(y), g(z) - g(z_n)\rangle + t_n\phi(g(y), x_n) - t_n\phi(g(x_n), x_n) + t_n\langle y_n, g(x_n) - g(y)\rangle \\ &\quad + (1 - t_n)[\phi(g(y), x^*) - \phi(g(x^*), x^*)] \\ &= \langle F(y), g(z) - g(z_n)\rangle + t_n\phi(g(y), x_n) + (1 - t_n)\phi(g(y), x^*) \\ &\quad - [t_n\phi(g(x_n), x_n) + (1 - t_n)\phi(g(x^*), x^*)] + \tfrac{\langle y_n, g(x_n) - g(y)\rangle}{\|x_n - x^*\|} \\ &\leq \langle F(y), g(z) - g(z_n)\rangle + \phi(g(y), z_n) - \phi(g(z_n), z_n) + \tfrac{\langle y_n, g(x_n) - g(y)\rangle}{\|x_n - x^*\|} \\ &\leq \langle F(y), g(z) - g(z_n)\rangle + \phi(g(y), z_n) - \phi(g(z_n), z_n) + \sigma\|y_n\|\tfrac{\|x_n - y\|}{\|x_n - x^*\|}. \end{aligned}$$

Note that $x \mapsto \phi(g(x), x) - \phi(g(y), x)$ is weakly lower semicontinuous. Utilizing the fact that every convex and lower semicontinuous functional has to be weakly lower semicontinuous, we deduce that

$$\begin{aligned}
&\langle F(y), g(z) - g(y)\rangle \\
&\leq \limsup_{n\to\infty}\{\langle F(y), g(z) - g(z_n)\rangle + \phi(g(y), z_n) - \phi(g(z_n), z_n) + \sigma\|y_n\|\frac{\|x_n - y\|}{\|x_n - x^*\|}\} \\
&\leq \phi(g(y), z) - \phi(g(z), z),
\end{aligned}$$

which is equivalent to the following inequality

$$\langle F(y), g(z) - y\rangle + \phi(g(z), z) - \phi(y, z) \leq 0, \quad \forall y \in H.$$

This together with Lemma 2.6 yields that z solves GMIQVI(F, g, ϕ), a contradiction. Thus, $\{x_n\}$ is bounded.

Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. It follows from (4.1) that

$$\langle F(x_{n_k}), g(x_{n_k}) - g(y)\rangle + \phi(g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k}) \leq \langle y_{n_k}, g(x_{n_k}) - g(y)\rangle, \quad \forall y \in H, n \in N.$$

Since g is affine, F is g-monotone, $x \mapsto \phi(g(x), x) - \phi(g(y), x)$ is weakly lower semicontinuous, and $\|y_n\| \to 0$, we have

$$\begin{aligned}
&\langle F(y), g(\bar{x}) - g(y)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y), \bar{x}) \\
&\leq \liminf_{n\to\infty}\{\langle F(y), g(x_{n_k}) - g(y)\rangle + \phi(g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k})\} \\
&\leq \liminf_{n\to\infty}\{\langle F(x_{n_k}), g(x_{n_k}) - g(y)\rangle + \phi(g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k})\} \\
&\leq \liminf_{n\to\infty}\langle y_{n_k}, g(x_{n_k}) - g(y)\rangle \\
&\leq \liminf_{n\to\infty} \sigma\|y_{n_k}\|\|x_{n_k} - y\| = 0, \quad \forall y \in H,
\end{aligned}$$

which is equivalent to the following inequality

$$\langle F(y), g(\bar{x}) - y\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \leq 0, \quad \forall y \in H.$$

This together with Lemma 2.6 yields that $\bar{x}$ solves GMIQVI(F, g, ϕ). We have $\bar{x} = x^*$ since GMIQVI(F, g, ϕ) has a unique solution x^*. Therefore $\{x_n\}$ converges weakly to x^* and so IP(F, g, ϕ) is weakly well-posed. □

Theorem 4.6. *Let $g : H \to H$ be a homeomorphism whose inverse g^{-1} is uniformly continuous. Let $F : H \to H$ be uniformly continuous and g-monotone, and let $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper, convex and lower semicontinuous functional satisfying $g(H) \cap \mathrm{dom}\partial\phi(\cdot, y) \neq \emptyset$. Assume that for any bounded sequences $\{x_n\}, \{y_n\}$ in H there holds the following:*

$$\|x_n - y_n\| \to 0 \ (n \to \infty) \quad \Rightarrow \quad \partial\phi(g(y_n), x_n) \subset \partial\phi(g(y_n), y_n) \text{ for } n \text{ sufficiently large.}$$

If IP(F, g, ϕ) *is strongly (resp. weakly) well-posed, then* GMIQVI(F, g, ϕ) *is strongly (resp. weakly) well-posed.*

Proof. Let $\{x_n\}$ be an approximating sequence for GMIQVI(F, g, ϕ). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\phi(g(x_n), x_n) \leq \phi(y, x_n) + \langle F(x_n), y - g(x_n)\rangle + \epsilon_n, \quad \forall y \in H, n \in N.$$

Define $\tilde{\phi}_n : H \to \mathcal{R}\cup\{+\infty\}$ as follows:

$$\tilde{\phi}_n(y) = \phi(y, x_n) + \langle F(x_n), y - g(x_n)\rangle, \quad \forall y \in H.$$

Clearly $\tilde{\phi}_n$ is proper, convex and lower semicontinuous and $0 \in \partial_{\epsilon_n}\tilde{\phi}_n(g(x_n))$ for all $n \in N$. By the Brøndsted-Rockafellar theorem [4], there exists $\bar{x}_n \in H$ and

$$x_n^* \in \partial\tilde{\phi}_n(g(\bar{x}_n)) = \partial\phi(g(\bar{x}_n), x_n) + F(x_n)$$

such that

$$\|g(x_n) - g(\bar{x}_n)\| \le \sqrt{\epsilon_n}, \quad \|x_n^*\| \le \sqrt{\epsilon_n}.$$

It follows that

$$x_n^* + F(\bar{x}_n) - F(x_n) \in F(\bar{x}_n) + \partial\phi(g(\bar{x}_n), x_n) \subset F(\bar{x}_n) + \partial\phi(g(\bar{x}_n), \bar{x}_n)$$

for n sufficiently large. Since $g^{-1}, F : H \to H$ are uniformly continuous, we deduce that

$$\|x_n - \bar{x}_n\| = \|g^{-1}(g(x_n)) - g^{-1}(g(\bar{x}_n))\| \to 0$$

and hence

$$\|x_n^* + F(\bar{x}_n) - F(x_n)\| \le \|x_n^*\| + \|F(\bar{x}_n) - F(x_n)\| \to 0.$$

So $\{\bar{x}_n\}$ is an approximating sequence for IP(F, g, ϕ).

Let x^* be the unique solution of GMIQVI(F, g, ϕ). By Lemma 2.2, x^* is also the unique solution of IP(F, g, ϕ).

If IP(F, g, ϕ) is strongly well-posed, then $\bar{x}_n \to x^*$. It follows that

$$\|x_n - x^*\| \le \|x_n - \bar{x}_n\| + \|\bar{x}_n - x^*\| \to 0$$

and so GMIQVI(F, g, ϕ) is strongly well-posed.

If IP(F, g, ϕ) is weakly well-posed, then $\bar{x}_n \rightharpoonup x^*$. For any $f \in H$, we have

$$|\langle f, x_n - x^*\rangle| \le |\langle f, x_n - \bar{x}_n\rangle| + |\langle f, \bar{x}_n - x^*\rangle| \le \|f\|\|x_n - \bar{x}_n\| + |\langle f, \bar{x}_n - x^*\rangle| \to 0.$$

Thus GMIQVI(F, g, ϕ) is weakly well-posed. □

For the well-posedness in the generalized sense, we have the following analogous results.

Theorem 4.7. *Let $g : H \to H$ be a homeomorphism which is σ-Lipschitz continuous, and $F : H \to H$ be g-hemicontinuous and g-monotone. Let $\phi : H \times H \to \mathcal{R}\cup\{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper, convex and lower semicontinuous functional satisfying $g(H) \cap \mathrm{dom}\partial\phi(\cdot, y) \neq \emptyset$. If* GMIQVI$(F, g, \phi)$ *is strongly (resp. weakly) σ^2-well-posed in the generalized sense, then* IP(F, g, ϕ) *is strongly (resp. weakly) well-posed in the generalized sense.*

Proof. Let $\{x_n\}$ be an approximating sequence for IP(F, g, ϕ). Then there exists $y_n \in F(x_n) + \partial\phi(g(x_n), x_n)$ such that $\|y_n\| \to 0$. It follows that

$$\phi(y, x_n) - \phi(g(x_n), x_n) \ge \langle y_n - F(x_n), y - g(x_n)\rangle, \quad \forall y \in H, n \in N,$$

which is equivalent to the following inequality

$$\phi(g(y), x_n) - \phi(g(x_n), x_n) \ge \langle y_n - F(x_n), g(y) - g(x_n)\rangle, \quad \forall y \in H, n \in N.$$

Hence we have

$$\begin{aligned}&\langle F(x_n), g(x_n) - g(y)\rangle + \phi(g(x_n), x_n) - \phi(g(y), x_n)\\ &\le \langle y_n, g(x_n) - g(y)\rangle\\ &\le \tfrac{1}{2}\|g(x_n) - g(y)\|^2 + \tfrac{1}{2}\|y_n\|^2\\ &\le \tfrac{\sigma^2}{2}\|x_n - y\|^2 + \tfrac{1}{2}\|y_n\|^2, \quad \forall y \in H, n \in N,\end{aligned}$$

which is equivalent to the following inequality

$$\langle F(x_n), g(x_n) - y\rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \le \frac{\sigma^2}{2}\|x_n - g^{-1}(y)\|^2 + \frac{1}{2}\|y_n\|^2, \quad \forall y \in H, n \in N.$$

This together with $\|y_n\| \to 0$ implies that $\{x_n\}$ is σ^2-approximating for GMIQVI(F, g, ϕ). Since GMIQVI(F, g, ϕ) is strongly (resp. weakly) σ^2-well-posed in the generalized sense, $\{x_n\}$ converges strongly (resp. weakly) to some solution x^* of GMIQVI(F, g, ϕ). By Lemma 2.2, x^* is also a solution of IP(F, g, ϕ). So IP(F, g, ϕ) is strongly (resp. weakly) well-posed in the generalized sense. □

Theorem 4.8. *Let $g : H \to H$ be a homeomorphism whose inverse g^{-1} is uniformly continuous. Let $F : H \to H$ be uniformly continuous and g-monotone, and let $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper, convex and lower semicontinuous functional satisfying $g(H) \cap \mathrm{dom}\partial\phi(\cdot, y) \ne \emptyset$. Assume that for any bounded sequences $\{x_n\}, \{y_n\}$ in H there holds the following:*

$$\|x_n - y_n\| \to 0 \ (n \to \infty) \quad \Rightarrow \quad \partial\phi(g(y_n), x_n) \subset \partial\phi(g(y_n), y_n) \text{ for } n \text{ sufficiently large.}$$

If IP(F, g, ϕ) *is strongly (resp. weakly) well-posed in the generalized sense, then* GMIQVI(F, g, ϕ) *is strongly (resp. weakly) well-posed in the generalized sense.*

Proof. The conclusion follows from the arguments similar to that of Theorem 4.6. □

5. Links with well-posedness of fixed point problems

In this section, we shall investigate the relations between the well-posedness of general mixed implicit quasi-variational inequalities and the well-posedness of fixed point problems. Let $T : H \to H$ be a single-valued mapping. The fixed-point problem associated with T is defined by

$$\mathrm{FP}(T): \quad \text{Find } x \in H \text{ such that } T(x) = x.$$

We first recall some concepts.

Definition 5.1 ([16, 17])**.** A sequence $\{x_n\} \subset H$ is called an approximating sequence for FP(T) if $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$.

Definition 5.2 ([16, 17])**.** We say that FP(T) is strongly (resp. weakly) well-posed if FP(T) has a unique solution and every approximating sequence for FP(T) converges strongly (resp. weakly) to the unique solution. FP(T) is said to be strongly (resp. weakly) well-posed in the generalized sense if FP(T) has a nonempty solution set S and every approximating sequence for FP(T) has a subsequence which converges strongly (resp. weakly) to some point of S.

Theorem 5.3. *Let $g : H \to H$ be a homeomorphism which is affine and σ-Lipschitz continuous. Let $F : H \to H$ be Lipschitz continuous and g-monotone. Let $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ be a proper and diagonally convex functional such that for each fixed $y \in H$ there hold the following conditions:*

(i) *$\phi(y, \cdot)$ is affine;*

(ii) *$\phi(\cdot, y)$ is a lower semicontinuous functional satisfying $g(H) \cap \mathrm{dom}\partial\phi(\cdot, y) \neq \emptyset$;*

(iii) *for any weak convergence sequence $\{z_n\} \subset H$ with $z_n \rightharpoonup z \in H$ and any sequence $\{e_n\} \subset H$ with $e_n \to 0$, there holds for each fixed $y \in H$*

$$\liminf_{n\to\infty}[\phi(g(z_n) + e_n, z_n) - \phi(g(y), z_n)] \geq \phi(g(z), z) - \phi(g(y), z);$$

(iv) *for any sequences $\{x_n\}, \{w_n\} \subset H$ with $\|w_n\| \to +\infty$ and $\|x_n - w_n\| \to 0$,*

$$\limsup_{n\to\infty} \frac{\langle F(x_n), g(w_n) - g(x_n) + x_n - w_n\rangle}{\|w_n\|} \leq 0.$$

If GMIQVI(F, g, ϕ) *is weakly well-posed, then* FP(F, g, ϕ) *is weakly well-posed, where $\lambda > 0$ is a constant.*

Remark 5.4. It is easy to see that condition (iv) of Theorem 5.3 holds if one of the following statements (a), (b) holds:

(a) $g = I$ the identity mapping of H;

(b) $F : H \to H$ is Lipschitz continuous.

In addition, if $g = I$ the identity mapping of H and $\phi(x, y) = \varphi(x)$ is a proper, convex and lower semicontinuous functional, then there is no doubt that condition (iii) in Theorem 5.3 holds.

Proof of Theorem 5.3. Suppose that GMIQVI(F, g, ϕ) is weakly well-posed. Let x^* be the unique solution of GMIQVI(F, g, ϕ). By Lemma **??**, x^* is also the unique solution of FP(F, g, ϕ). Let $\{x_n\}$ be an approximating sequence for FP(F, g, ϕ). Then $\|x_n - w_n\| \to 0$, where

$$w_n = x_n - g(x_n) + J_\lambda^{\partial\phi(\cdot,x_n)}(g(x_n) - \lambda F(x_n)).$$

By the definition of $J_\lambda^{\partial\phi(\cdot,x)}$,

$$\frac{x_n - w_n}{\lambda} - F(x_n) \in \partial\phi(w_n - x_n + g(x_n), x_n).$$

It follows that

(5.1)

$$\phi(y, x_n) - \phi(w_n - x_n + g(x_n), x_n) \geq \langle\frac{x_n - w_n}{\lambda} - F(x_n), y - (w_n - x_n + g(x_n))\rangle, \quad \forall y \in H, n \in N,$$

which is equivalent to the following inequality

(5.2)

$$\phi(g(y), x_n) - \phi(w_n - x_n + g(x_n), x_n) \geq \langle\frac{x_n - w_n}{\lambda} - F(x_n), g(y) - (w_n - x_n + g(x_n))\rangle, \quad \forall y \in H, n \in N.$$

If $\{w_n\}$ is unbounded, without loss of generality, we may assume that $\|w_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|w_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we may assume that $t_n \in (0,1)$ and $z_n \rightharpoonup z(\neq x^*)$. From the affinity of g it follows that

$$\begin{aligned} \langle F(y), g(z) - g(y)\rangle &= \langle F(y), g(z) - g(z_n)\rangle + \langle F(y), g(z_n) - g(x^*)\rangle \\ &\quad + \langle F(y), g(x^*) - g(y)\rangle \\ &= \langle F(y), g(z) - g(z_n)\rangle + t_n\langle F(y), g(x_n) - g(y)\rangle \\ &\quad + (1-t_n)\langle F(y), g(x^*) - g(y)\rangle. \end{aligned} \tag{5.3}$$

Since x^* is the unique solution of GMIQVI(F, g, ϕ), we have

$$\langle F(x^*), g(x^*) - y\rangle + \phi(g(x^*), x^*) - \phi(y, x^*) \leq 0, \quad \forall y \in H, n \in N,$$

which is equivalent to the following inequality

$$\langle F(x^*), g(x^*) - g(y)\rangle + \phi(g(x^*), x^*) - \phi(g(y), x^*) \leq 0, \quad \forall y \in H, n \in N. \tag{5.4}$$

Note that ϕ is diagonally convex and g and $\phi(g(y), \cdot)$ are affine. Hence it follows from (13)-(15) that for all $y \in H$ and $n \in N$,

$$\begin{aligned} &\langle F(y), g(z) - g(y)\rangle \\ &\leq \langle F(y), g(z) - g(z_n)\rangle + t_n\langle F(x_n), g(x_n) - g(y)\rangle + (1-t_n)\langle F(x^*), g(x^*) - g(y)\rangle \\ &\leq \langle F(y), g(z) - g(z_n)\rangle + t_n[\phi(g(y), x_n) - \phi(w_n - x_n + g(x_n), x_n) \\ &\quad + \langle \tfrac{w_n - x_n}{\lambda}, g(y) - (w_n - x_n + g(x_n))\rangle + \langle F(x_n), g(w_n) - g(x_n) + x_n - w_n\rangle] \\ &\quad + (1-t_n)[\phi(g(y), x^*) - \phi(g(x^*), x^*)] \\ &= \langle F(y), g(z) - g(z_n)\rangle + t_n\phi(g(y), x_n) + (1-t_n)\phi(g(y), x^*) \\ &\quad - [t_n\phi(w_n - x_n + g(x_n), x_n) + (1-t_n)\phi(g(x^*), x^*)] \\ &\quad + t_n[\langle \tfrac{w_n - x_n}{\lambda}, g(y) - (w_n - x_n + g(x_n))\rangle + \langle F(x_n), g(w_n) - g(x_n) + x_n - w_n\rangle] \\ &\leq \langle F(y), g(z) - g(z_n)\rangle + \phi(g(y), z_n) - \phi(g(z_n) + t_n(w_n - x_n), z_n) \\ &\quad + t_n[\langle \tfrac{w_n - x_n}{\lambda}, g(y) - (w_n - x_n + g(x_n))\rangle + \langle F(x_n), g(w_n) - g(x_n) + x_n - w_n\rangle]. \end{aligned}$$

Consequently, from conditions (i), (iii) and (iv) it follows that

$$\begin{aligned} &\langle F(y), g(z) - g(y)\rangle \\ &\leq \limsup_{n\to\infty}\{\langle F(y), g(z) - g(z_n)\rangle + \phi(g(y), z_n) - \phi(g(z_n) + t_n(w_n - x_n), z_n) \\ &\quad + t_n[\langle \tfrac{w_n - x_n}{\lambda}, g(y) - (w_n - x_n + g(x_n))\rangle + \langle F(x_n), g(w_n) - g(x_n) + x_n - w_n\rangle]\} \\ &\leq \phi(g(y), z) - \phi(g(z), z), \quad \forall y \in H. \end{aligned}$$

This together with Lemma 2.6 implies that z solves GMIQVI(F, g, ϕ), a contradiction. Thus, $\{w_n\}$ is bounded.

Let $\{w_{n_k}\}$ be any subsequence of $\{w_n\}$ such that $w_{n_k} \rightharpoonup \bar{w}$ as $k \to \infty$. From (5.2) we have

$$\begin{aligned} \langle F(w_{n_k}), (w_{n_k} - x_{n_k} + g(x_{n_k})) - g(y)\rangle + \phi(w_{n_k} - x_{n_k} + g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k}) \\ \leq \langle \frac{x_{n_k} - w_{n_k}}{\lambda}, (w_{n_k} - x_{n_k} + g(x_{n_k})) - g(y)\rangle \\ + \langle F(w_{n_k}) - F(x_{n_k}), (w_{n_k} - x_{n_k} + g(x_{n_k})) - g(y)\rangle, \end{aligned}$$

for all $y \in H$. Since F is g-monotone and uniformly continuous, from condition (iii) and the affinity of g we obtain

$$
\begin{aligned}
&\langle F(y), g(\bar{w}) - g(y)\rangle + \phi(g(\bar{w}), \bar{w}) - \phi(g(y), \bar{w}) \\
&\le \liminf_{n\to\infty}\{\langle F(y), g(w_{n_k}) - g(y)\rangle + \phi(w_{n_k} - x_{n_k} + g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k})\} \\
&\le \liminf_{n\to\infty}\{\langle F(w_{n_k}), g(w_{n_k}) - g(y)\rangle + \langle F(w_{n_k}), g(x_{n_k}) - g(w_{n_k}) + w_{n_k} - x_{n_k}\rangle \\
&\ +\phi(w_{n_k} - x_{n_k} + g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k})\} \\
&\le \liminf_{n\to\infty}\{\langle \frac{x_{n_k} - w_{n_k}}{\lambda}, (w_{n_k} - x_{n_k} + g(x_{n_k})) - g(y)\rangle \\
&\ +\langle F(w_{n_k}) - F(x_{n_k}), (w_{n_k} - x_{n_k} + g(x_{n_k})) - g(y)\rangle\} \\
&= 0, \quad \forall y \in H.
\end{aligned}
$$

This together with Lemma 2.6 yields that $\bar{w}$ solves GMIQVI(F, g, ϕ). We have $w_n \rightharpoonup x^*$ since GMIQVI(F, g, ϕ) has a unique solution x^*. For any $f \in H$, it follows that

$$
\begin{aligned}
|\langle f, x_n - x^*\rangle| &\le |\langle f, x_n - w_n\rangle| + |\langle f, w_n - x^*\rangle| \\
&\le \|f\|\|x_n - w_n\| + |\langle f, w_n - x^*\rangle| \to 0.
\end{aligned}
$$

Therefore, $x \rightharpoonup x^*$ and so FP(F, g, ϕ) is weakly well-posed. □

Theorem 5.5. *Let $g : H \to H$ be a homeomorphism whose inverse g^{-1} is uniformly continuous. Let $F : H \to H$ be uniformly continuous and g-monotone, and let $\phi : H \times H \to \mathcal{R}\cup\{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper, convex and lower semicontinuous functional satisfying $g(H) \cap \mathrm{dom}\partial\phi(\cdot, y) \neq \emptyset$. Assume that for any bounded sequences $\{x_n\}, \{y_n\}$ in H there holds the following:*

$\|x_n - y_n\| \to 0\ (n \to \infty) \quad \Rightarrow \quad \partial\phi(g(y_n), x_n) \subset \partial\phi(g(y_n), y_n)$ for n sufficiently large.

If FP(F, g, ϕ) *is strongly (resp. weakly) well-posed, then* GMIQVI(F, g, ϕ) *is strongly (resp. weakly) well-posed.*

Proof. Let $\{x_n\}$ be an approximating sequence for GMIQVI(F, g, ϕ). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\phi(g(x_n), x_n) \le \phi(y, x_n) + \langle F(x_n), y - g(x_n)\rangle + \epsilon_n, \quad \forall y \in H, n \in N.$$

Define $\tilde{\phi}_n : H \to \mathcal{R}\cup\{+\infty\}$ as follows:

$$\tilde{\phi}_n(y) = \phi(y, x_n) + \langle F(x_n), y - g(x_n)\rangle, \quad \forall y \in H.$$

Clearly $\tilde{\phi}_n$ is proper, convex and lower semicontinuous and $0 \in \partial_{\epsilon_n}\tilde{\phi}_n(g(x_n))$ for all $n \in N$. By the Brøndsted-Rockafellar theorem [4], there exists $\bar{x}_n \in H$ and

$$x_n^* \in \partial\tilde{\phi}_n(g(\bar{x}_n)) = \partial\phi(g(\bar{x}_n), x_n) + F(x_n) \tag{5.5}$$

such that

$$\|g(x_n) - g(\bar{x}_n)\| \le \sqrt{\epsilon_n}, \quad \|x_n^*\| \le \sqrt{\epsilon_n}. \tag{5.6}$$

It follows that

$$
\begin{aligned}
g(\bar{x}_n) + \lambda(x_n^* - F(x_n)) &\in \partial\phi(g(\bar{x}_n), x_n) + g(\bar{x}_n) \\
&\subset g(\bar{x}_n) + \lambda\partial\phi(g(\bar{x}_n), \bar{x}_n) + g(\bar{x}_n) \\
&= (I + \lambda\partial\phi(\cdot, \bar{x}_n))(g(\bar{x}_n)),
\end{aligned}
$$

and hence

$$\bar{x}_n = \bar{x}_n - g(\bar{x}_n) + J_\lambda^{\partial\phi(\cdot,\bar{x}_n)}(g(\bar{x}_n) + \lambda(x_n^* - F(x_n))). \tag{5.7}$$

It follows from (5.5)-(5.7) that

$$\begin{aligned}
&\|\bar{x}_n - (\bar{x}_n - g(\bar{x}_n) + J_\lambda^{\partial\phi(\cdot,\bar{x}_n)}(g(\bar{x}_n) - \lambda F(\bar{x}_n)))\| \\
&= \|J_\lambda^{\partial\phi(\cdot,\bar{x}_n)}(g(\bar{x}_n) + \lambda(x_n^* - F(x_n))) - J_\lambda^{\partial\phi(\cdot,\bar{x}_n)}(g(\bar{x}_n) - \lambda F(\bar{x}_n))\| \\
&\le \|(g(\bar{x}_n) + \lambda(x_n^* - F(x_n))) - (g(\bar{x}_n) - \lambda F(\bar{x}_n))\| \\
&\le \lambda\|x_n^*\| + \lambda\|F(\bar{x}_n) - F(x_n)\| \to 0
\end{aligned}$$

and so $\{\bar{x}_n\}$ is an approximating sequence for $\mathrm{FP}(F, g, \phi)$.

Let x^* be the unique solution of $\mathrm{FP}(F, g, \phi)$. By Lemma 2.2, x^* is also the unique solution of $\mathrm{GMIQVI}(F, g, \phi)$.

If $\mathrm{FP}(F, g, \phi)$ is strongly well-posed, then $\bar{x}_n \to x^*$. It follows that

$$\|x_n - x^*\| \le \|x_n - \bar{x}_n\| + \|\bar{x}_n - x^*\| \to 0.$$

Thus $\mathrm{GMIQVI}(F, g, \phi)$ is strongly well-posed.

If $\mathrm{FP}(F, g, \phi)$ is weakly well-posed, then $\bar{x}_n \rightharpoonup x^*$. For any $f \in H$, we have

$$|\langle f, x_n - x^*\rangle| \le |\langle f, x_n - \bar{x}_n\rangle| + |\langle f, \bar{x}_n - x^*\rangle| \le \|f\|\sqrt{\epsilon_n} + |\langle f, \bar{x}_n - x^*\rangle| \to 0$$

and so $\mathrm{GMIQVI}(F, g, \phi)$ is weakly well-posed. □

Theorem 5.6. *Let $g : H \to H$ be a homeomorphism which is σ-Lipschitz continuous, and $F : H \to H$ be uniformly continuous and g-monotone. Let $\phi : H \times H \to \mathcal{R}\cup\{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot, y)$ is a proper, convex and lower semicontinuous functional satisfying $g(H) \cap \mathrm{dom}\partial\phi(\cdot, y) \ne \emptyset$. For any approximating sequence $\{x_n\} \subset H$ of* $\mathrm{FP}(F, g, \phi)$*, suppose for each $n \in N$ there exists $\delta_n > 0$ such that*

$$|\phi(g(w_n), w_n) - \phi(w_n - x_n + g(x_n), x_n)| \le \delta_n \text{ and } |\phi(y, x_n) - \phi(y, w_n)| \le \delta_n, \quad \forall y \in H,$$

where $\delta_n \to 0$ and $w_n = x_n - g(x_n) + J_\lambda^{\partial\phi(\cdot,x_n)}(g(x_n) - \lambda F(x_n))$. If $\mathrm{GMIQVI}(F, g, \phi)$ *is strongly (resp. weakly) $(1+\frac{1}{\lambda})$-well-posed in the generalized sense, then* $\mathrm{FP}(F, g, \phi)$ *is strongly (resp. weakly) well-posed in the generalized sense, where $\lambda > 0$ is a constant.*

Proof. Let $\{x_n\}$ be an approximating sequence for $\mathrm{FP}(F, g, \phi)$. Then $\|x_n - w_n\| \to 0$, where

$$w_n = x_n - g(x_n) + J_\lambda^{\partial\phi(\cdot,x_n)}(g(x_n) - \lambda F(x_n)).$$

By the definition of $J^{\partial\phi(\cdot,x_n)}$,

$$\frac{x_n - w_n}{\lambda} - F(x_n) \in \partial\phi(w_n - x_n + g(x_n), x_n).$$

From the definition of subdifferential, we get

$$\phi(y, x_n) - \phi(w_n - x_n + g(x_n), x_n) \ge \langle \frac{x_n - w_n}{\lambda} - F(x_n), y - (w_n - x_n + g(x_n))\rangle, \quad \forall y \in H, n \in N,$$

which is equivalent to the following inequality

$$\phi(g(y),x_n)-\phi(w_n-x_n+g(x_n),x_n) \geq \langle \frac{x_n-w_n}{\lambda}-F(x_n), g(y)-(w_n-x_n+g(x_n))\rangle, \quad \forall y \in H,$$

for all $n \in N$. It follows from g-monotonicity that

$$\begin{aligned}
&\langle F(w_n),(w_n-x_n+g(x_n))-g(y)\rangle+\phi(w_n-x_n+g(x_n),x_n)-\phi(g(y),x_n)\\
&\leq \langle F(w_n)-F(x_n),(w_n-x_n+g(x_n))-g(y)\rangle\\
&\quad+\tfrac{1}{\lambda}\langle x_n-w_n,(w_n-x_n+g(x_n))-g(y)\rangle\\
&\leq \langle F(w_n)-F(x_n),w_n-x_n+g(w_n)-g(y)\rangle\\
&\quad+\tfrac{1}{\lambda}\langle x_n-w_n,g(x_n)-g(w_n)+g(w_n)-g(y)\rangle\\
&= \langle F(w_n)-F(x_n),w_n-x_n\rangle+\langle F(w_n)-F(x_n),g(w_n)-g(y)\rangle\\
&\quad+\tfrac{1}{\lambda}\langle x_n-w_n,g(x_n)-g(w_n)\rangle+\tfrac{1}{\lambda}\langle x_n-w_n,g(w_n)-g(y)\rangle\\
&\leq \tfrac{1}{2}(1+\tfrac{1}{\lambda})\|g(w_n)-g(y)\|^2+(\tfrac{1}{2}\|F(w_n)-F(x_n)\|^2+\tfrac{1}{2\lambda}\|x_n-w_n\|^2)\\
&\quad+\langle F(w_n)-F(x_n),w_n-x_n\rangle+\tfrac{1}{\lambda}\langle x_n-w_n,g(x_n)-g(w_n)\rangle, \quad \forall y \in H,
\end{aligned}$$

for all $n \in N$. Thus from the last inequality we get

$$\begin{aligned}
&\langle F(w_n),g(w_n)-g(y)\rangle+\phi(g(w_n),w_n)-\phi(g(y),w_n)\\
&\leq \tfrac{1}{2}(1+\tfrac{1}{\lambda})\|g(w_n)-g(y)\|^2+(\tfrac{1}{2}\|F(w_n)-F(x_n)\|^2+\tfrac{1}{2\lambda}\|x_n-w_n\|^2)\\
&\quad+\langle F(w_n)-F(x_n),w_n-x_n\rangle+\tfrac{1}{\lambda}\langle x_n-w_n,g(x_n)-g(w_n)\rangle\\
&\quad+\langle F(w_n),x_n-w_n+g(w_n)-g(x_n)\rangle+\phi(g(w_n),w_n)-\phi(w_n-x_n+g(x_n),x_n)\\
&\quad+\phi(g(y),x_n)-\phi(g(y),w_n)\\
&\leq \tfrac{1}{2}(1+\tfrac{1}{\lambda})\|g(w_n)-g(y)\|^2+(\tfrac{1}{2}\|F(w_n)-F(x_n)\|^2+\tfrac{1}{2\lambda}\|x_n-w_n\|^2)\\
&\quad+|\langle F(w_n)-F(x_n),w_n-x_n\rangle|+\tfrac{1}{\lambda}|\langle x_n-w_n,g(x_n)-g(w_n)\rangle|\\
&\quad+|\langle F(w_n),x_n-w_n+g(w_n)-g(x_n)\rangle|+2\delta_n, \quad \forall y \in H, n \in N.
\end{aligned}$$

Since F and g are uniformly continuous and $\|w_n-x_n\| \to 0$, we deduce that $\{w_n\}$ is $(1+\frac{1}{\lambda})$-approximating for GMIQVI(F,g,ϕ).

If GMIQVI(F,g,ϕ) is strongly $(1+\frac{1}{\lambda})$-well-posed in the generalized sense, then $\{w_n\}$ has a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \to x^*$ as $k \to \infty$, where x^* is a solution of GMIQVI(F,g,ϕ). By Lemma 2.2, x^* is also a solution of FP(F,g,ϕ). It follows that

$$\|x_{n_k}-x^*\| \leq \|x_{n_k}-w_{n_k}\|+\|w_{n_k}-x^*\| \to 0$$

as $k \to \infty$. Thus FP(F,g,ϕ) is strongly well-posed in the generalized sense.

If GMIQVI(F,g,ϕ) is weakly $(1+\frac{1}{\lambda})$-well-posed in the generalized sense, then $\{w_n\}$ has a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \rightharpoonup x^*$ as $k \to \infty$, where x^* is a solution of GMIQVI(F,g,ϕ). By Lemma 2.2, x^* is also a solution of FP(F,g,ϕ). For any $f \in H$, it follows that, as $k \to \infty$,

$$\begin{aligned}
|\langle f,x_{n_k}-x^*\rangle| &\leq |\langle f,x_{n_k}-w_{n_k}\rangle|+|\langle f,w_{n_k}-x^*\rangle|\\
&\leq \|f\|\cdot\|x_{n_k}-w_{n_k}\|+|\langle f,w_{n_k}-x^*\rangle| \to 0.
\end{aligned}$$

Thus FP(F,g,ϕ) is weakly well-posed in the generalized sense. □

Theorem 5.7. *Let $g: H \to H$ be a homeomorphism whose inverse g^{-1} is uniformly continuous. Let $F: H \to H$ be uniformly continuous and g-monotone, and let $\phi: H \times H \to \mathcal{R} \cup \{+\infty\}$ be such that for each fixed $y \in H$, $\phi(\cdot,y)$ is a proper, convex and lower semicontinuous functional satisfying $g(H) \cap \mathrm{dom}\partial\phi(\cdot,y) \neq \emptyset$. Assume that for any bounded sequences $\{x_n\},\{y_n\}$ in H there holds the following:*

$$\|x_n-y_n\| \to 0 \ (n \to \infty) \quad \Rightarrow \quad \partial\phi(g(y_n),x_n) \subset \partial\phi(g(y_n),y_n) \text{ for } n \text{ sufficiently large.}$$

If $\mathrm{FP}(F, g, \phi)$ *is strongly (resp. weakly) well-posed in the generalized sense, then* $\mathrm{GMIQVI}(F, g, \phi)$ *is strongly (resp. weakly) well-posed in the generalized sense.*

Proof. The conclusion follows from the arguments similar to that of Theorem 5.5. □

6. Conclusions for well-posedness

In this section we shall prove that under suitable conditions the well-posedness of the general mixed implicit quasi-variational inequality is equivalent to the existence and uniqueness of its solutions, and the well-posedness in the generalized sense is equivalent to the existence of its solutions.

Theorem 6.1. *Let* $g : H \to H$ *be a homeomorphism which is affine and* σ*-Lipschitz continuous. Let* $F : H \to H$ *be* g*-monotone and* g*-hemicontinuous. Let* $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ *be a proper and diagonally convex functional such that for each fixed* $y \in H$ *there hold the following conditions:*
(i) $\phi(y, \cdot)$ *is affine;*
(ii) $\phi(\cdot, y)$ *is a lower semicontinuous functional satisfying* $g(H) \cap \mathrm{dom}\partial\phi(\cdot, y) \neq \emptyset$*;*
(iii) $x \mapsto \phi(g(x), x) - \phi(g(y), x)$ *is weakly lower semicontinuous.*
If $\mathrm{GMIQVI}(F, g, \phi)$ *is weakly well-posed if and only if it has a unique solution.*

Proof. The necessity is obvious. For the sufficiency, suppose that $\mathrm{GMIQVI}(F, g, \phi)$ has a unique solution x^*. If $\mathrm{GMIQVI}(F, g, \phi)$ is not weakly well-posed, then there exists an approximating sequence $\{x_n\}$ for $\mathrm{GMIQVI}(F, g, \phi)$ such that $x_n \not\rightharpoonup x^*$. Thus, there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\langle F(x_n), g(x_n) - y\rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \le \epsilon_n, \quad \forall y \in H, n \in N,$$

which is equivalent to the following inequality

$$\langle F(x_n), g(x_n) - g(y)\rangle + \phi(g(x_n), x_n) - \phi(g(y), x_n) \le \epsilon_n, \quad \forall y \in H, n \in N. \tag{6.1}$$

If $\{x_n\}$ is unbounded, without loss of generality, we may assume that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we may assume that $t_n \in (0, 1]$ and $z_n \rightharpoonup z (\neq x^*)$. By the arguments similar to that of Theorem 4.5, we have

$$\langle F(y), g(z) - g(y)\rangle \le \langle F(y), g(z) - g(z_n)\rangle + \phi(g(y), z_n) - \phi(g(z_n), z_n) + t_n\epsilon_n, \quad \forall y \in H, n \in N.$$

It follows that

$$\begin{aligned} \langle F(y), g(z) - g(y)\rangle &\le \limsup_{n\to\infty}\{\langle F(y), g(z) - g(z_n)\rangle \\ &\quad + \phi(g(y), z_n) - \phi(g(z_n), z_n) + t_n\epsilon_n\} \\ &\le \phi(g(y), z) - \phi(g(z), z), \quad \forall y \in H, \end{aligned}$$

which is equivalent to the following inequality

$$\langle F(y), g(z) - y\rangle \le \phi(y, z) - \phi(g(z), z), \quad \forall y \in H.$$

This together with Lemma 2.6 yields that z solves GMIQVI(F, g, ϕ), a contradiction. Thus, $\{x_n\}$ is bounded.

Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. It follows from (6.1) that

$$\langle F(x_{n_k}), g(x_{n_k}) - g(y)\rangle + \phi(g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k}) \le \epsilon_{n_k}, \quad \forall y \in H, k \in N.$$

Since F is g-monotone, from condition (iii) we conclude that

$$\begin{aligned} &\langle F(y), g(\bar{x}) - g(y)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y), \bar{x}) \\ &\le \liminf n \to \infty \{\langle F(y), g(x_{n_k}) - g(y)\rangle + \phi(g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k})\} \\ &\le \liminf n \to \infty \{\langle F(x_{n_k}), g(x_{n_k}) - g(y)\rangle + \phi(g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k})\} \\ &\le \liminf n \to \infty \epsilon_n = 0, \quad \forall y \in H. \end{aligned}$$

This together with Lemma 2.6 yields that $\bar{x}$ solves GMIQVI(F, g, ϕ). We have $\bar{x} = x^*$ since GMIQVI(F, g, ϕ) has a unique solution x^*. Thus $\{x_n\}$ converges weakly to x^*, a contradiction. So GMIQVI(F, g, ϕ) is weakly well-posed. □

Example 6.2. Let H, F, g be as in Example 3.9 and ϕ be as in Remark 4.4. Clearly, $g : H \to H$ be a homeomorphism which is affine and σ-Lipschitz continuous. Moreover, $F : H \to H$ be g-monotone and g-hemicontinuous. On the other hand, $\phi : H \times H \to \mathcal{R} \cup \{+\infty\}$ is obviously a proper and diagonally convex functional such that for each fixed $y \in H$ there hold the following conditions:

(i) $\phi(y, \cdot)$ is affine;

(ii) $\phi(\cdot, y)$ is a lower semicontinuous functional satisfying $g(H) \cap \text{dom}\partial\phi(\cdot, y) \neq \emptyset$;

(iii) $x \mapsto \phi(g(x), x) - \phi(g(y), x)$ is weakly lower semicontinuous.

Furthermore, it is easy to see that GMIQVI(F, g, ϕ) has a unique solution $x^* = 0$. By Theorem 6.1, GMIQVI(F, g, ϕ) is well-posed.

Theorem 6.3. *Let $g : \mathcal{R}^m \to \mathcal{R}^m$ be a homeomorphism which is affine and σ-Lipschitz continuous. Let $F : H \to H$ be g-monotone and g-hemicontinuous. Let $\phi : \mathcal{R}^m \times \mathcal{R}^m \to \mathcal{R} \cup \{+\infty\}$ be a proper and diagonally convex functional such that for each fixed $y \in \mathcal{R}^m$ there hold the following conditions:*

(i) *$\phi(y, \cdot)$ is affine;*

(ii) *$\phi(\cdot, y)$ is a lower semicontinuous functional satisfying $g(\mathcal{R}^m) \cap \text{dom}\partial\phi(\cdot, y) \neq \emptyset$;*

(iii) *$x \mapsto \phi(g(x), x) - \phi(g(y), x)$ is lower semicontinuous.*

If there exists some $\epsilon > 0$ such that $\Omega_\alpha(\epsilon)$ is nonempty bounded, then GMIQVI(F, g, ϕ) *is α-well-posed in the generalized sense.*

Proof. Let $\{x_n\}$ be an α-approximating sequence for GMIQVI(F, g, ϕ). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that for all $y \in \mathcal{R}^m$ and all $n \in N$

$$\langle F(x_n), g(x_n) - y\rangle + \phi(g(x_n), x_n) - \phi(y, x_n) \le \frac{\alpha}{2}\|x_n - g^{-1}(y)\|^2 + \epsilon_n,$$

which is equivalent to the following inequality

$$\langle F(x_n), g(x_n) - g(y)\rangle + \phi(g(x_n), x_n) - \phi(g(y), x_n) \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n. \tag{6.2}$$

Let $\epsilon > 0$ be such that $\Omega_\alpha(\epsilon)$ is nonempty bounded. Then there exists n_0 such that $x_n \in \Omega_\alpha(\epsilon)$ for all $n > n_0$. This implies that $\{x_n\}$ is bounded and so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. Since g is affine and F is g-monotone, it follows from (6.2) and condition (iii) that for all $y \in \mathcal{R}^m$

$$\begin{aligned}
&\langle F(y), g(\bar{x}) - g(y)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y), \bar{x})\\
&\le \liminf_{k\to\infty}\{\langle F(y), g(x_{n_k}) - g(y)\rangle + \phi(g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k})\}\\
&\le \liminf_{k\to\infty}\{\langle F(x_{n_k}), g(x_{n_k}) - g(y)\rangle + \phi(g(x_{n_k}), x_{n_k}) - \phi(g(y), x_{n_k})\}\\
&\le \liminf_{k\to\infty}\{\frac{\alpha}{2}\|x_{n_k} - y\|^2 + \epsilon_{n_k}\}\\
&= \tfrac{\alpha}{2}\|\bar{x} - y\|^2.
\end{aligned}$$

For any $y \in \mathcal{R}^m$, let $y_t = \bar{x} + t(y - \bar{x})$ for all $t \in (0,1)$. Then

$$\langle F(y_t), g(\bar{x}) - g(y_t)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y_t), \bar{x}) \le \frac{\alpha}{2}\|\bar{x} - y_t\|^2.$$

By the convexity of $\phi(\cdot, \bar{x})$ and the affinity of g, we conclude that for all $y \in \mathcal{R}^m$ and all $t \in (0,1)$

$$\langle F(y_t), g(\bar{x}) - g(y)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y), \bar{x}) \le \frac{t\alpha}{2}\|\bar{x} - y\|^2.$$

Letting $t \to 0^+$ in the above inequality, we have for all $y \in \mathcal{R}^m$

$$\langle F(y), g(\bar{x}) - g(y)\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(g(y), \bar{x}) \le 0,$$

which is equivalent to the following inequality

$$\langle F(y), g(\bar{x}) - y\rangle + \phi(g(\bar{x}), \bar{x}) - \phi(y, \bar{x}) \le 0.$$

This together with Lemma 2.6 implies that $\bar{x}$ solves GMIQVI(F, g, ϕ). Thus GMIQVI(F, g, ϕ) is α-well-posed in the generalized sense. □

Theorem 6.3 says nothing but that, under suitable conditions, the α-well-posedness in the generalized sense is equivalent to the existence of solutions.

The following example shows the assumption that $\Omega_\alpha(\epsilon)$ is nonempty bounded for some $\epsilon > 0$ is essential in Theorem 6.3.

Example 6.4. Let $m = 2$, $F(x) = 0$ and $\phi(x, y) = \delta_K$, where $K = [0, +\infty) \times [0, +\infty)$. Let g be as in Example 3.9. Then, clearly, F is g-monotone and g-hemicontinuous, and there hold conditions (i)-(iii) for the proper and diagonally convex functional ϕ. For any $\epsilon > 0$, we have $\Omega_\alpha(\epsilon) = [0, +\infty) \times [0, +\infty)$. By Theorem 3.8, GMIQVI$(F, g, \phi)$ is not α-well-posed in the generalized sense.

7. Conclusions

In this paper we introduce some concepts of well-posedness for general mixed implicit quasi-variational inequalities. In Section 3, we establish some metric characterizations of strong α-well-posedness. In Section 4, we discuss the connections between the strong (weak) well-posedness of general mixed implicit quasi-variational inequalities and strong (weak) well-posedness of inclusion problems. In Section 5, we further investigate the relationships between the strong (weak) well-posedness of general mixed implicit quasi-variational inequalities and the strong (weak) well-posedness of fixed point problems. In Section 6, we prove that under suitable conditions, the well-posedness of general mixed implicit quasi-variational inequalities is equivalent to the existence and uniqueness of solutions, and that the well-posedness in the generalized sense is equivalent to the existence of solutions. Our results are the improvements and extension of the corresponding ones in [11].

It is known that the concept of α-well-posedness has been introduced and considered for optimization problems [7], variational inequalities [7, 20] and Nash equilibrium problems [20], mixed variational inequalities [11]. It is worth pointing out that in [11] there are two unsolved open problems arise in nature way.

Q1 Is it possible to consider the concept of α-well-posedness for the inclusion problems ?

Q2 Is it possible to give a metric characterization only for weak well-posedness ?

The above two unsolved open problems are interesting and important which deserve investigation in the future.

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