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SEQUENTIAL FORMULA FOR SUBDIFFERENTIAL OF UPPER ENVELOPE OF CONVEX FUNCTIONS

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Abstract. The paper provides a general description of the subdifferential of the upper envelope of a family of convex functions on a Banach space. Without any qualification condition, general sequential formulas are proved when the Banach space is finite dimensional or not. It is also shown how results under qualification condition in the literature can be derived from sequential ones.

1. Introduction

The paper is devoted to the study of the subdifferential of the upper envelope of convex functions, say

$$f(x) = \sup_{t\in T} f_t(x)\ , \tag{1.1}$$

where $(f_t)_{t\in T}$, $f_t : X \to \mathrm{R} \cup \{+\infty\}$, is a family of proper lower semicontinuous convex functions defined on a real Banach space X. Depending on whether X is finite dimensional or not and whether T have algebraic or topological structure or not, the established results are different.

It is known that under qualification condition and when T is finite, one has (see [22, 23]) the equality

$$\partial f(\bar{x}) = \mathrm{co}\{\partial f_k(\{\bar{x}) : f_k(\bar{x}) = f(\bar{x})\} + \sum_{i=1}^{n} N(\mathrm{dom}\, f_i, \bar{x}). \tag{1.2}$$

Such a reference point formula does not hold without qualification condition for a general set T. If no qualification condition is assumed, T is a general set, and X is finite (resp. infinite) dimensional, it is natural, like for the finite sum or for the continuous sum (see [6, 12, 17, 18]), to investigate whether any continuous functional x^* of the subdifferential $\partial f(\bar{x})$ can be approximated by an appropriate sequence $(x^*_{k,m})_{(k,m)\in \mathrm{N}\times\mathrm{N}}$ (resp. a net $(x^*_{k,j})_{(k,j)\in\mathrm{N}\times J}$). In other words, do there exist appropriate sequences $(x_{k,m})_{(k,m)\in\mathrm{N}\times\mathrm{N}}$, $(x^*_{k,m})_{(k,m)\in\mathrm{N}\times\mathrm{N}}$, $(w^*_m)_{m\in\mathrm{N}}$, $(\lambda_{k,m})_{(k,m)\in\mathrm{N}\times\mathrm{N}}$ and $(t_{k,m})_{(k,m)\in\mathrm{N}\times\mathrm{N}}$ (resp. nets $(x_{k,j})_{(k,j)\in\mathrm{N}\times J}$, $(x^*_{k,j})_{(k,j)\in\mathrm{N}\times J}$, $(w^*_j)_{j\in J}$, $(\lambda_{k,j})_{(k,j)\in\mathrm{N}\times J}$ and $(t^*_{k,j})_{(k,j)\in\mathrm{N}\times J}$, where J is some directed set of the form $I\times\mathcal{F}_{\bar{x}}$ with $\mathcal{F}_{\bar{x}}$ denoting the set of all finite dimensional subspaces of X containing $\bar{x}$) such that

$$x^* = \lim_{m\to+\infty}(w^*_m + \sum_{k\in\mathrm{N}} \lambda_{k,m}x^*_{k,m}) \quad \text{in} \quad (\mathrm{X}^*, \|\ \|_*),$$

and

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$$(\bar{x}, f(\bar{x})) = \lim_{m\to+\infty}(x_{k,m}, f_{t_{k,m}}(x_{k,m})),$$

$$x^*_{k,m} \in \partial f_{t_{k,m}}(x_{k,m}) \quad \text{and} \quad w^*_m \in N(\operatorname{dom} f, \bar{x}).$$

(resp.

$$x^* = \lim_{j\in J}(w^*_j + \sum_{k\in \mathrm{N}} \lambda_{k,j} x^*_{k,j}) \quad \text{in} \quad (\mathrm{X}^*, \mathrm{w}(\mathrm{X}^*, \mathrm{X})), \text{ and } \quad (\bar{\mathrm{x}}, \mathrm{f}(\bar{\mathrm{x}})) = \lim_{\mathrm{j}\in\mathrm{J}}(\mathrm{x}_{\mathrm{k,j}}, \mathrm{f}_{\mathrm{t}_{\mathrm{k,j}}}(\mathrm{x}_{\mathrm{k,j}})),$$

$$x^*_{k,j} \in \partial f_{t_{k,j}}(x_{k,j}) \quad \text{and} \quad w^*_j \in N(L \cap \operatorname{dom} f, \bar{x}) \quad \text{for } j = (i, L).$$

A. Hantoute and M. Lopez provided in ([3]) a general formula without qualification condition for the supremum of a family of proper lower semicontinuous convex functions defined on a finite dimensional vector space. An extension to any locally convex vector space X has been established in ([4]). This formula is in terms of ε-approximate subdifferentials of the functions at the fixed point of reference. In the case of finite (resp. infinite) dimensional Banach space, using this formula and a version of the Brøndsted-Rockafellar theorem, we prove the existence of the sequences (resp. nets) considered above. We refer to ([17, 18, 19, 14, 9, 2, 10, 6, 12]) and the references therein for sequential formulas concerning finite sum, chain rule, and integral sum.

In Section 1 we will recall the supremum formula proved in ([3]) and ([4]) along with the Brøndsted-Rockafellar theorem. Section 2 is devoted to proving our sequential formula relative to the supremum function defined in (1.1). We will also show in Section 3 how the general formula in [8, 20] under a general qualification condition can be derived from the sequential formula.

2. Preliminary results

We start this section by recalling the main results of Hantoute and Lopez [3], and of Hantoute, Lopez and Zalinescu [4]. We will also give a version of the Brønsted-Rockafellar theorem which will be used in Section 2. Before stating these theorems, let us recall that, for any convex function f from a (Hausdorff) topological locally convex vector space X into $\mathrm{R} \cup \{+\infty\}$ and for any real number $\varepsilon \geq 0$, the $\varepsilon-$subdifferential of f at any $x \in \operatorname{dom} f := \{u \in X : f(u) < \infty\}$ is defined by

$$\partial f_\varepsilon(x) = \{x^* \in X^* : \langle x^*, u - x\rangle \leq f(u) - f(x) + \varepsilon \quad \text{for all } u \in X\}.$$

If $\varepsilon = 0$, $\partial_\varepsilon f(x)$ corresponds to what is called the subdifferential $\partial f(x)$ of f at x. When $\operatorname{dom} f \neq \emptyset$ one says that f is proper and for $x \notin \operatorname{dom} f$ one puts $\partial f_\varepsilon(x) = \emptyset$. For $x \in \operatorname{dom} f$ the ε-subdifferential is known to be described also by

$$\partial_\varepsilon f(x) = \{x^* \in X^* : \langle x^*, u\rangle \leq f'_\varepsilon(x; u)\},$$

where f'_ε is the directional ε-derivative of f at x in the direction u defined by

$$(2.1) \qquad f'_\varepsilon(x; u) := \inf_{\theta>0} \theta^{-1}[f(x + \theta u) - f(x) + \varepsilon] = \lim_{\theta\downarrow 0} \theta^{-1}[f(x + \theta u) - f(x) + \varepsilon].$$

It will be sometimes convenient in the paper to use the graph

$$\operatorname{gph} \partial_\varepsilon f := \{(x, x^*) \in X \times X^* : x^* \in \partial_\varepsilon f(x)\}$$

of the set-valued mapping $\partial_\varepsilon f$.

We also recall that the indicator function δ_C of a convex subset C of X is defined by $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ if $x \in X \setminus C$. The subdifferential $\partial\delta_C(x)$

for $x \in C$ is called the normal cone of C at x and it is denoted by $N(C, x)$. The weak star topology of the topological dual space X^* will be denoted by $\sigma(X^*, X)$. When X is a normed space, B_X and B_{X^*} will denote the closed unit ball of X and X^* centered at the origin.

Theorem 2.1 (see Hantoute and Lopez [3], Hantoute, Lopez and Zalinescu [4])**.** *Let X be a (Hausdorff) locally convex vector space and $(f_t)_{t\in T}$ be a family of proper lower semicontinuous convex functions. Consider the pointwise supremum function f defined in (1.1), $\bar{x} \in \operatorname{dom} f$ and for each $\varepsilon > 0$ consider also the set*

$$T_\varepsilon(\bar{x}) := \{t \in T : f_t(\bar{x}) \geq f(\bar{x}) - \varepsilon\}.$$

Then the following assertions hold.

(a) *If X is finite dimensional, then*

$$\partial f(\bar{x}) = \bigcap_{\varepsilon>0} \operatorname{cl}\big(\operatorname{co}\{\bigcup_{t\in T_\varepsilon(\bar{x})} \partial_\varepsilon f_t(\bar{x})\} + N(\operatorname{dom} f, \bar{x})\big).$$

(b) *If X is infinite dimensional, then*

$$\partial f(\bar{x}) = \bigcap_{L\in\mathcal{F}_{\bar{x}},\varepsilon>0} \operatorname{cl}_*\big(\operatorname{co}\{\bigcup_{t\in T_\varepsilon(\bar{x})} \partial_\varepsilon f_t(\bar{x})\} + N(L \cap \operatorname{dom} f, \bar{x})\big),$$

where $\mathcal{F}_{\bar{x}}$ denotes the set of all finite dimensional vector subspaces of X containing $\bar{x}$, and cl_* denotes the closure with respect to the weak-star topology on X^*.

The assertion of (a) has been proved in [3] and that of (b) in [4].

Theorem 2.2 (A version of the Brøndsted-Rockafellar theorem, see [1, 18])**.** *Let X be a Banach space and $f : X \to \mathrm{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then for any real number $\varepsilon > 0$ and $x^* \in \partial_\varepsilon f(\bar{x})$ there exists $(x_\varepsilon, x_\varepsilon^*) \in X \times X^*$ such that*

(a) $x_\varepsilon^* \in \partial f(x_\varepsilon)$;
(b) $\|x_\varepsilon - \bar{x}\| \leq \sqrt{\varepsilon}$;
(c) $\|x^* - x_\varepsilon^*\| \leq \sqrt{\varepsilon}$;
(d) $|f(x_\varepsilon) - \langle x_\varepsilon^*, x_\varepsilon - \bar{x}\rangle - f(\bar{x})| \leq 2\varepsilon$.

3. Sequential calculus for the supremum function

Throughout this section we assume that the function f is given by (1.1) and the functions $f_t : X \to \mathrm{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex on the Banach space X.

Theorem 3.1. *Let X be a finite dimensional vector space.*
For any $\bar{x} \in$ dom f, one has $x^* \in \partial f(\bar{x})$ if and only if there are sequences $(\varepsilon_n)_{n\in\mathrm{N}}$ in $]0, +\infty[$ with $\varepsilon_n \downarrow 0$, $(\lambda_{k,n})_{(k,n)\in\mathrm{N}\times\mathrm{N}}$ in $[0,1]$, $(t_{k,n})_{(k,n)\in\mathrm{N}\times\mathrm{N}}$ in T, $(x_{k,n})_{(k,n)\in\mathrm{N}\times\mathrm{N}}$ in X, $(x_{k,n}^*)_{(k,n)\in\mathrm{N}\times\mathrm{N}}$ in X, $(w_n^*)_{n\in\mathrm{N}}$ in X such that, for each $n \in \mathrm{N}$, the sequence $(\lambda_{k,n})_{k\in\mathrm{N}}$ has a finite support and such that

(a) $t_{k,n} \in T_{\varepsilon_n}(\bar{x})$ *where* $T_{\varepsilon_n}(\bar{x}) := \{t \in T : f_t(\bar{x}) \geq f(\bar{x}) - \varepsilon_n\}$;
(b) $\sum_{k\in\mathrm{N}} \lambda_{k,n} = 1$;
(c) $w_n^* \in N(\operatorname{dom} f, \bar{x})$;

(d) $x^*_{k,n} \in \partial f_{t_{k,n}}(x_{k,n})$;
(e) $x^* = \lim\limits_{n\to\infty} (w^*_n + \sum\limits_{k\in \mathrm{N}} \lambda_{k,n} x^*_{k,n})$;
(f) $\bar{x} = \lim\limits_{n\to\infty} x_{k,n}$, *the limit being uniform with respect to the integer* $k \in \mathrm{N}$;
(g) $\lim\limits_{n\to\infty} \big(f_{t_{k,n}}(x_{k,n}) - \langle x^*_{k,n}, x_{k,n} - \bar{x}\rangle\big) = f(\bar{x})$, *the limit being uniform with respect to the integer* $k \in \mathrm{N}$.

Proof. • Suppose that $x^* \in \partial f(\bar{x})$.
Using (a) of Theorem 2.1 with $\varepsilon = \frac{1}{n}$, we have for every integer $n \in \mathrm{N}$,

$$x^* \in \mathrm{co}\{\bigcup_{t\in T_{\frac{1}{n}}(\bar{x})} \partial_{\frac{1}{n}} f_t(\bar{x})\} + N(\mathrm{dom}\, f, \bar{x}) + \frac{1}{n}\mathrm{B}_X.$$

Then, by definition of the convex hull of a set, for each $n \in \mathrm{N}$ there exist m_n in N, $t_{1,n}, \ldots, t_{m_n,n}$ in $T_{\frac{1}{n}}(\bar{x})$, $\lambda_{1,n}, \ldots, \lambda_{m_n,n}$ in $[0,1]$, $u^*_{1,n}, \ldots, u^*_{m_n,n}, w^*_n, b^*_n$ in X, such that $u^*_{k,n} \in \partial_{\frac{1}{n}} f_{t_{k,n}}(\bar{x})$, $b^*_n \in \mathrm{B}_X$, $w^*_n \in N(\mathrm{dom}\, f, \bar{x})$, $x^* = \sum\limits_{k=1}^{m_n} \lambda_{k,m} u^*_{k,n} + w^*_n + \frac{1}{n} b^*_n$, $\sum\limits_{k=1}^{m_n} \lambda_{k,m} = 1$. Consequently, for each $n \in \mathrm{N}$, we have a sequence $(\lambda_{k,n})_{k\in\mathrm{N}}$ with a finite support, $t_{k,n} \in T_{\frac{1}{n}}(\bar{x})$ for all $k \in \mathrm{N}$ and

$$u^*_{k,n} \in \partial_{\frac{1}{n}} f_{t_{k,n}}(\bar{x}) \quad , \quad b^*_n \in \mathrm{B}_X, \quad w^*_n \in N(\mathrm{dom}\, f, \bar{x}), \tag{3.1}$$

along with

$$x^* = \sum_{k\in\mathrm{N}} \lambda_{k,n} u^*_{k,n} + w^*_n + \frac{1}{n} b^*_n, \quad \sum_{k\in\mathrm{N}} \lambda_{k,n} = 1. \tag{3.2}$$

For each $k \in \mathrm{N}$ and $\varepsilon = \frac{1}{2n}$, we now apply Theorem 2.2 to get $(x_{k,n}, x^*_{k,n}) \in X^2$ such that

(i) $x^*_{k,n} \in \partial f_{t_{k,n}}(x_{k,n})$,
(ii) $\|x^*_{k,n} - u^*_{k,n}\| \le \frac{1}{\sqrt{2n}}$,
(iii) $\|x_{k,n} - \bar{x}\| \le \frac{1}{\sqrt{2n}}$,
(iv) $|f_{t_{k,n}}(x_{k,n}) - \langle x^*_{k,n}, x_{k,n} - \bar{x}\rangle - f_{t_{k,n}}(\bar{x})| \le \frac{1}{n}$.

Since $t_{k,n} \in T_{\frac{1}{n}}(\bar{x})$, we have $|f_{t_{k,n}}(\bar{x}) - f(\bar{x})| \le \frac{1}{n}$ and then

$$\begin{aligned} & |f_{t_{k,n}}(x_{k,n}) - \langle x^*_{k,n}, x_{k,n} - \bar{x}\rangle - f(\bar{x})| \\ \le\; & |f_{t_{k,n}}(\bar{x}) - f(\bar{x})| + |f_{t_{k,n}}(x_{k,n}) - \langle x^*_{k,n}, x_{k,n} - \bar{x}\rangle - f_{t_{k,n}}(\bar{x})| \\ \le\; & \frac{2}{n}, \end{aligned}$$

which ensures that

$$\sup_{k\in\mathrm{N}} |f_{t_{k,n}}(x_{k,n}) - \langle x^*_{k,n}, x_{k,n} - \bar{x}\rangle - f(\bar{x})| \underset{n\to\infty}{\longrightarrow} 0.$$

It remains to prove the assertion (e) of the theorem. By (3.2) we have

$$x^* = w^*_n + \sum_{k\in\mathrm{N}} \lambda_{k,n} x^*_{k,n} + \sum_{k\in\mathrm{N}} \lambda_{k,n}(u^*_{k,n} - x^*_{k,n}) + \frac{1}{n} b^*_n,$$

which entails by (3.1), (3.2) and (ii) that

$$\begin{aligned} \|x^* - (w_n^* + \sum_{k\in\mathrm{N}} \lambda_{k,n} x_{k,n}^*)\| &\le \sum_{k\in\mathrm{N}} \lambda_{k,n} \|x_{k,n}^* - u_{k,n}^*\| + \frac{1}{n}\|b_n^*\| \\ &\le \frac{1}{\sqrt{2n}} + \frac{1}{n}, \end{aligned}$$

and hence

$$(w_n^* + \sum_{k\in\mathrm{N}} \lambda_{k,n} x_{k,n}^*) \underset{n\to\infty}{\longrightarrow} x^*.$$

So the implication $\Rightarrow$ of the theorem is etablished.

• Now suppose that, for $(\bar{x}, x^*) \in X \times X$, there exist sequences $(\varepsilon_n)_n$ in $]0, +\infty[$, $(\lambda_{k,n})_{(k,n)\in\mathrm{N}\times\mathrm{N}}$ in $[0,1]$, $(t_{k,n})_{(k,n)\in\mathrm{N}\times\mathrm{N}}$ in T, $(x_{k,n})_{(k,n)\in\mathrm{N}\times\mathrm{N}}$ in X, $(x_{k,n}^*)_{(k,n)\in\mathrm{N}\times\mathrm{N}}$ in X, and $(w_n^*)_{n\in\mathrm{N}}$ in X satisfying the assertions of the theorem.

Let $x \in \operatorname{dom} \mathrm{f}$. According to (g) we have

$$\varepsilon'_n := \sup_{k\in\mathrm{N}} |f_{t_{k,n}}(x_{k,n}) - \langle x_{k,n}^*, x_{k,n} - \bar{x}\rangle - f(\bar{x})| \underset{n\to\infty}{\longrightarrow} 0$$

and taking (d) and (a) into account we also have

$$\begin{aligned} \langle x_{k,n}^*, x - \bar{x}\rangle &= \langle x_{k,n}^*, x - x_{k,n}\rangle + \langle x_{k,n}^*, x_{k,n} - \bar{x}\rangle \\ &\le f_{t_{k,n}}(x) - f_{t_{k,n}}(x_{k,n}) + \langle x_{k,n}^*, x_{k,n} - \bar{x}\rangle, \\ &\le f_{t_{k,n}}(x) - f(\bar{x}) - (f_{t_{k,n}}(x_{k,n}) - \langle x_{k,n}^*, x_{k,n} - \bar{x}\rangle - f(\bar{x})), \\ &\le f(x) - f(\bar{x}) - (f_{t_{k,n}}(x_{k,n}) - \langle x_{k,n}^*, x_{k,n} - \bar{x}\rangle - f(\bar{x})). \end{aligned}$$

We deduce that

$$\langle x_{k,n}^*, x - \bar{x}\rangle \le f(x) - f(\bar{x}) + \varepsilon'_n.$$

Further, by (b) we may write that

$$\langle \sum_{k\in\mathrm{N}} \lambda_{k,n} x_{k,n}^*, x - \bar{x}\rangle \le f(x) - f(\bar{x}) + \varepsilon'_n,$$

and this combined with (c) gives

$$\langle w_n^* + \sum_{k\in\mathrm{N}} \lambda_{k,n} x_{k,n}^*, x - \bar{x}\rangle \le f(x) - f(\bar{x}) + \varepsilon'_n.$$

Using (e) and taking the limit as $n \to \infty$ in the previous inequality we obtain

$$\langle x^*, x - \bar{x}\rangle \le f(x) - f(\bar{x}) \quad \text{for all } x \in \operatorname{dom} f,$$

which entails that $x^* \in \partial f(\bar{x})$. The proof is then complete. □

In the case of a Banach space we have :

Theorem 3.2. *Let X be a Banach space.*
For any $\bar{x} \in$ dom f, one has $x^* \in \partial f(\bar{x})$ if and only if there are nets $(w_{i,L}^*)$ in X^*, $(\varepsilon_{i,L}) \in]0, +\infty[$ indexed by $(i, L) \in I \times \mathcal{F}_{\bar{x}}$ and which do not depend on k, and simultaneously for each $k \in \mathrm{N}$ there are nets $(\lambda_{k,i,L})$ in $[0, 1]$, $(t_{k,i,L})$ in T, $(x_{k,i,L})$ in X, $(x_{k,i,L}^*)$ in X^* (indexed by $(i, L) \in I \times \mathcal{F}_{\bar{x}}$) such that for each $(i, L) \in I \times \mathcal{F}_{\bar{x}}$ the sequence $(\lambda_{k,i,L})_{k\in\mathrm{N}}$ has a finite support and such that

(a) $t_{k,i,L} \in T_{\varepsilon_{i,L}}(\bar{x})$ *where*

$$T_{\varepsilon_{i,L}}(\bar{x}) := \{t \in T : f_t(\bar{x}) \geq f(\bar{x}) - \varepsilon_{i,L}\}, \qquad \lim_{(i,L)\in I\times\mathcal{F}_{\bar{x}}} \varepsilon_{i,L} = 0\,;$$

(b) $\sum_{k\in \mathrm{N}} \lambda_{k,i,L} = 1$ *;*
(c) $w^*_{i,L} \in N(L \cap \operatorname{dom} f, \bar{x})$ *;*
(d) $x^*_{k,i,L} \in \partial f_{t_{k,i,L}}(x_{k,i,L})$ *;*
(e) $x^* = \lim_{(i,L)\in L\times\mathcal{F}_{\bar{x}}} (w^*_{i,L} + \sum_{k\in\mathrm{N}} \lambda_{k,i,L} x^*_{k,i,L})$ *in* $(X^*, \sigma(X^*, X))$ *;*
(f) $\bar{x} = \lim_{(i,L)\in I\times\mathcal{F}_{\bar{x}}} x_{k,i,L}$ *in* $(X, \|\,\|)$*, the limit being uniform with respect to the integer* $k \in \mathrm{N}$ *;*
(g) $\lim_{(i,L)\in I\times\mathcal{F}_{\bar{z}}} \big(f_{t_{k,i,L}}(x_{k,i,L}) - \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle\big) = f(\bar{x})$*, the limit being uniform with respect to the integer* $k \in \mathrm{N}$.

When T *is finite, say* $T = \{1, \ldots, p\}$*, in assertions (b) and (e) the sum over* N *has to be replaced by the sum with* $k \in \{1, \ldots, p\}$ *and the assertion (a) becomes*

(a) $t_{k,i,L} \in T(\bar{x})$ *where* $T(\bar{x}) := \{t \in T : f_t(\bar{x}) = f(\bar{x})\}$.

Proof. • Suppose that $x^* \in \partial f(\bar{x})$ and denote by $\mathcal{V}_{X^*}(0)$ the set of weak star neighborhoods of zero in X^*. Using (b) of Theorem 2.1 with $\varepsilon = \frac{1}{n}$, we have for all $(n, V, L) \in \mathrm{N} \times \mathcal{V}_{X^*}(0) \times \mathcal{F}_{\bar{x}}$

$$x^* \in \operatorname{co}\{\bigcup_{t\in T_{\frac{1}{n}}(\bar{x})} \partial_{\frac{1}{n}} f_t(\bar{x})\} + N(L \cap \operatorname{dom} f, \bar{x}) + V.$$

Put $I := \mathrm{N} \times \mathcal{V}_{X^*}(0)$, $\varepsilon_{i,L} := \frac{1}{n}$ if $i = (n, V)$ and consider the following preorder

$$(n, V, L) \preceq (n', V', L') \quad \text{if} \quad n \leq n', \quad V' \subseteq V, \quad \text{and} \quad L \subset L'.$$

Obviously $(I \times \mathcal{F}_{\bar{z}}, \preceq)$ is a directed set and $\lim_{(i,L)\in I\times\mathcal{F}_{\bar{z}}} \varepsilon_{i,L} = 0$. To complete the proof of the implication $\Rightarrow$, it is sufficient to proceed like in the proof of Theorem 3.1.

• Suppose conversely that the properties of theorem hold.
Then, there exist nets $(\varepsilon_{i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, $(\lambda_{k,i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, $(t_{k,i,L})_{(i,L)\in\times I\times\mathcal{F}_{\bar{x}}}$, $(x_{k,i,L})_{(i,L)\in\times I\times\mathcal{F}_{\bar{x}}}$, $(x^*_{k,i,L})_{(i,L)\in\times I\times\mathcal{F}_{\bar{z}}}$, $(w^*_{i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, satisfying assertions (a)-(g) of the theorem.

Let $x \in \operatorname{dom} \mathrm{f}$ and $\mathcal{F}_{\bar{x},x}$ be the collection of vector subspaces $L \in \mathcal{F}_{\bar{x}}$ containing x. Note that the set $I \times \mathcal{F}_{\bar{x},x}$ endowed with the induced preorder (still denoted by $\preceq$) is a directed set. Let $(i, L) \in I \times \mathcal{F}_{\bar{x},x}$. By (d), (a) one has

$$\begin{aligned}
\langle x^*_{k,i,L}, x - \bar{x}\rangle &= \langle x^*_{k,i,L}, x - x_{k,i,L}\rangle + \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle \\
&\leq f_{t_{k,i,L}}(x) - f_{t_{k,i,L}}(x_{k,i,L}) + \langle x^*_{k,i,L}\,, x_{k,i,L} - \bar{x}\rangle, \\
&\leq f_{t_{k,i,L}}(x) - f(\bar{x}) - (f_{t_{k,i,L}}(x_{k,i,L}) - \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle - f(\bar{x})), \\
&\leq f(x) - f(\bar{x}) - [f_{t_{k,i,L}}(x_{k,i,L}) - \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle - f(\bar{x})]. \qquad (3.3)
\end{aligned}$$

If we put $\eta_{i,L} := \sup_{k\in\mathrm{N}} |f_{t_{k,i,L}}(x_{k,i,L}) - \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle - f(\bar{x})|$, by (g) we have $\eta_{i,L} \underset{i,L}{\rightarrow} 0$ and the last inequality in (3.3) becomes

$$\langle x^*_{k,i,L}, x - \bar{x}\rangle \leq f(x) - f(\bar{x}) + \eta_{i,L}. \qquad (3.4)$$

Then, by (b) we may write

$$\langle \sum_{k \in \mathrm{N}} \lambda_{k,i,L} x^*_{k,i,L}, x - \bar{x} \rangle \le f(x) - f(\bar{x}) + \eta_{i,L},$$

and it follows from (c) that

$$\langle w^*_{i,L} + \sum_{k \in \mathrm{N}} \lambda_{k,i,L} x^*_{k,i,L}, x - \bar{x} \rangle \le f(x) - f(\bar{x}) + \eta_{i,L}. \tag{3.5}$$

However, $(\eta_{i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x},x}}$, $(\varepsilon_{i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x},x}}$, $(\lambda_{k,i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x},x}}$, $(t_{k,i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x},x}}$, $(x_{k,i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x},x}}$, $(x^*_{k,i,L})_{(i,L)\in\times I\times\mathcal{F}_{\bar{x},x}}$, $(w^*_{i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x},x}}$ are subnets of corresponding nets, then we can use (e) to get through (3.5)

$$\langle x^*, x - \bar{x} \rangle \le f(x) - f(\bar{x}) \quad \text{for all } x \in \operatorname{dom} f,$$

which entails that $x^* \in \partial f(\bar{x})$. The proof is then complete. □

4. Application to some basic results

As a first application, it is not difficult through the qualification (QC) below to deduce the following theorem of M. Volle [23] from Theorem 3.2.

Theorem 4.1 (see [23]). *Assume here that $T = \{1, \dots, p\}$ and that $f_1, \dots, f_p : X \to \mathrm{R} \cup \{+\infty\}$ are proper convex lower semicontinuous functions on the Banach space X. Assume also that the following qualification condition holds :*

(QC) *there exist some $x_0 \in \operatorname{dom f_p}$ such that $f_1, \dots, f_{p-1}$ are finite at x_0 and continuous at x_0.*

Then for any $\bar{x} \in \operatorname{dom} f$ (where f is given by (1.1)) one has

$$\partial f(\bar{x}) = \operatorname{co}\{\partial f_t(\bar{x}) : f_t(\bar{x}) = f(\bar{x})\} + \sum_{t=1}^{p} N(\operatorname{dom} f_t, \bar{x}).$$

□

Remark 4.2. 1) In his theorem, Volle does not assume the lower semicontinuity of functions f_k. Nevertheless, using the fact that $\partial f(\bar{x}) = \partial(\operatorname{cl} f)(\bar{x})$ and $f(\bar{x}) = \operatorname{cl} f(\bar{x})$ whenever $\partial f(\bar{x}) \ne \emptyset$, the desired equality is reduced to the above case. Here $\operatorname{cl} f$ means the closure hull of the function f (see [13]).

2) Observe that the normal cone cannot be removed in the formula of Theorem 3.2. As a simple example, consider for all $x \in X$

$$f_1(x) := \delta_{[0,+\infty)}(x), \; f_2(x) = 1.$$

The functions f_1 and f_2 are clearly proper, convex and lower semicontinuous. The qualification condition (QC) is obviously satisfied and

$$f(x) := \max\{f_1(x), f_2(x)\} = 1 + \delta_{[0,+\infty)}(x).$$

Then, an easy subdifferential calculus gives

$$\partial f(0) =]-\infty; 0] \text{ and } \operatorname{co}(\bigcup_{k=1}^{2} \{\partial f_k(0) : \; f_k(0) = f(0)\}) = \{0\}.$$

The next theorem gives a description of the subdifferential of $f = \sup_{t\in T} f_t$ at a continuity point $\bar{x}$ and in the case where T is a compact topological space. It was demonstrated by B.N. Psenichnyi ([15]) when X is a normed space assuming that functions f_t are Gâteaux differentiable and by M. Valadier ([20]) when X is a topological vector space assuming that the function $(t,x) \mapsto f_t(x)$ is finite and continuous on $T\times U$ (where U is a neighbourhood of $\bar{x}$). The case where the functions f_t are finite and continuous at $\bar{x}$ for all $t \in T$ and where, for some neighbourhood V of $\bar{x}$, the functions $t \mapsto f_t(x)$ are finite and continuous on T for all $x \in V$ is due R.T. Rockafellar as said in Valadier's thesis [21]. A.D. Ioffe and V.M. Tikhomirov ([8]) gave a version of this result in the case where X is a locally convex topological vector space with weaker assumptions than those recalled above. Since our result is valid only in a Banach space, we prove the Ioffe and Tikhomirov version in the context of a Banach space.

Theorem 4.3 (see [8])**.** *Let X be a Banach space, T be a compact topological space, $f_t : X \to \mathrm{R}\cup\{+\infty\}$ be a proper lower semicontinuous convex function for each $t \in T$ and let f be given by* (1.1)*. Assume that the following properties hold for $\bar{x} \in X$ and a neighbourhood V of $\bar{x}$:*

(i) *for each $x \in V$, $t \mapsto f_t(x)$ is upper semicontinuous (usc);*
(ii) *for each $t \in T$, the convex function $x \mapsto f_t(x)$ is finite at $\bar{x}$ and continuous at $\bar{x}$.*

Then

$$\partial f(\bar{x}) = \overline{\mathrm{co}}^*(\{\bigcup_{t\in T(\bar{x})} \partial f_t(\bar{x})\}) ,$$

where $T(\bar{x}) := \{t \in T : f_t(\bar{x}) = f(\bar{x})\}$ *and* $\overline{\mathrm{co}}^*$ *denotes the weak star closed convex hull.*

Proof. We only need to prove that the first member is included in the second one since the reverse inclusion is easily seen. Let $x^* \in \partial f(\bar{x})$.

• **Step 1.** Recall that $\mathcal{F}_{\bar{x}}$ is the set of finite dimensional vector subspaces containing $\bar{x}$ and, for $u \in X$, denote by $\mathcal{F}_{\bar{x},u}$ the set of $L \in \mathcal{F}_{\bar{x}}$ with $u \in L$. By Theorem 3.2, there exist nets $(\varepsilon_{i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, $(\lambda_{k,i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, $(t_{k,i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, $(x_{k,i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, $(x^*_{k,i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, $(w^*_{i,L})_{(i,L)\in I\times\mathcal{F}_{\bar{x}}}$, satisfying assertions (a)-(g), where (g) means $\lim_{i,L} \eta_{i,L} = 0$, for

$$\eta_{i,L} := \sup_{k\in\mathrm{N}} |f_{t_{k,i,L}}(x_{k,i,l}) - \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle - f(\bar{x})|.$$

For each $(i,L) \in I \times \mathcal{F}_{\bar{x}}$ we put $K_{i,L} := \{k \in \mathrm{N} : \lambda_{k,i,L} \neq 0\}$.

Fix any $u \in X$ and choose $\rho > 0$ such that $\bar{x} + [0,\rho]\,u \subset V$. For each $\tau \in T$, by the continuity of f_τ at $\bar{x}$, there exists some $\rho(\tau) \in]0,\rho]$ such that

$$f_\tau(\bar{x} + \rho(\tau)u) < f_\tau(\bar{x}) + 1 \le f(\bar{x}) + 1.$$

So, the set $W_\tau := \{t \in T : f_t(\bar{x} + \rho(\tau)u) < f(\bar{x}) + 1\}$ is nonempty, and it is also open in T according to the upper semicontinuity of $t \mapsto f_t(\bar{x} + \rho(\tau)u)$. Since $f_t(\bar{x}) < f(\bar{x})+1$ and f_t is convex, we note, for each $t \in W_\tau$, that $f_t(\bar{x}+\theta u) < f(\bar{x})+1$ for all $\theta \in [0, \rho(\tau)]$. From the open cover $(W_\tau)_{\tau\in T}$ of T, taking as in [8] a finite cover $(W_\tau)_{\tau\in S}$ of the compact set T and putting $\theta_0(u) := \min_{\tau\in S} \rho(\tau)$ yields, for every

$\theta \in [0, \theta_0(u)]$, that $f_t(\bar{x} + \theta\, u) < f(\bar{x}) + 1$ for all $t \in T$ and hence $\bar{x} + \theta\, u \in \operatorname{dom} f$. Consequently, for each $L \in \mathcal{F}_{\bar{x}}$, taking any $u \in L$ we have $\bar{x} + \theta u \in L \cap \operatorname{dom} f$ for all $\theta \in]0, \theta_0(u)]$, which in particular gives, according to (c) of Theorem 3.2,

$$w^*_{i,L} \in L^{\perp} \quad \text{for all } i \in I. \tag{4.1}$$

• **Step 2.** We prove that $x^* \in \partial\varphi(\bar{x})$ where $\varphi(y) := \sup_{t \in T(\bar{x})} f_t(y)$ for all $y \in X$.

Observe first, for any $x \in X$, we have by (d), for all $(i, L) \in I \times \mathcal{F}_{\bar{x}}$,

$$\begin{aligned}\langle x^*_{k,i,L}, x - \bar{x}\rangle &= \langle x^*_{k,i,L}, x - x_{k,i,L}\rangle + \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle \\ &\le f_{t_{k,i,L}}(x) - f_{t_{k,i,L}}(x_{k,i,L}) + \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle \\ &= f_{t_{k,i,L}}(x) - f(\bar{x}) - [f_{t_{k,i,L}}(x_{k,i,L}) - \langle x^*_{k,i,L}, x_{k,i,L} - \bar{x}\rangle - f(\bar{x})] \\ &\le f_{k,i,L}(x) - f(\bar{x}) + \eta_{i,L}. \end{aligned} \tag{4.2}$$

Fix any $u \in X$, $\theta \in]0, \theta_0(u)]$ and $L \in \mathcal{F}_{\bar{x},u}$. We note that $\bar{x} + \theta u \in L$, that is, $L \in \mathcal{F}_{\bar{x}+\theta u}$. By (4.2) and by (b) of Theorem 3.2 we have

$$\langle w^*_{i,L} + \sum_{k \in K_{i,L}} \lambda_{k,i,L} x^*_{k,i,L}, \theta u\rangle \le \sum_{k \in K_{i,L}} \lambda_{k,i,L} f_{t_{k,i,L}}(\bar{x} + \theta u) - f(\bar{x}) + \eta_{i,L}. \tag{4.3}$$

By the compactness of $T_{\varepsilon_{i,L}}(\bar{x})$ (keep in mind (i) and the compactness of T) and by (4.3) and the assumption (i), there exists $\bar{t}_{i,L} \in T_{\varepsilon_{i,L}}(\bar{x})$ (depending on θ and u) such that $f_{\bar{t}_{i,L}}(\bar{x} + \theta u) = \max_{t \in T_{\varepsilon_{i,L}}} f_t(\bar{x} + \theta u)$, thus

$$\langle w^*_{i,L} + \sum_{k \in K_{i,L}} \lambda_{k,i,L} x^*_{k,i,L}, \theta u\rangle \le f_{\bar{t}_{i,L}}(\bar{x} + \theta u) - f(\bar{x}) + \eta_{i,L}.$$

By the compactness of T there exist $\bar{t} \in T$, a directed set J and a directed mapping $s : J \to I \times \mathcal{F}_{\bar{x},u}$ such that $\lim_{j \in J} \bar{t}_{s(j)} = \bar{t}$ with $\bar{t} \in T(\bar{x})$. Further, for all $j \in J$, we have

$$\langle w^*_{s(j)} + \sum_{k \in K_{s(j)}} \lambda_{k,s(j)} x^*_{k,s(j)}, \theta u\rangle \le f_{\bar{t}_{s(j)}}(\bar{x} + \theta u) - f(\bar{x}) + \eta_{s(j)}.$$

Then, using (e) of Theorem 3.2, the upper semicontinuity of $t \mapsto f_t(\bar{x} + \theta u)$ and taking the limit superior in both members of the last inequality, we obtain

$$\begin{aligned}\langle x^*\,,\, \theta u\rangle &\le f_{\bar{t}}(\bar{x} + \theta u) - f(\bar{x}) \\ &\le \sup_{t \in T(\bar{x})} f_t(\bar{x} + \theta u) - f(\bar{x}), \text{ because } \bar{t} \in T(\bar{x}) \\ &= \varphi(\bar{x} + \theta u) - \varphi(\bar{x}),\end{aligned}$$

which implies $\langle x^*, u\rangle \le \varphi'(\bar{x}; u)$ and completes this step.

• **Step 3.** We claim that we may suppose : For each $(i, L) \in I \times \mathcal{F}_{\bar{x}}$ and $k \in K_{i,L}$,

$$x^*_{k,i,L} \in \partial_{\eta_{i,L}} f_{t_{k,i,L}}(\bar{x}) \quad \text{and} \quad t_{k,i,L} \in T(\bar{x}). \tag{4.4}$$

To see that we apply Theorem 3.2 to the function $x \mapsto \sup_{t \in T(\bar{x})} f_t(x)$. As previously we obtain the existence of some nets satisfying assertions (a)-(e). The difference here is that $t_{k,i,L} \in T(\bar{x})$. Indeed, if we keep notation in Theorem 3.2, we have to consider the sets $(T(\bar{x}))_{\varepsilon_{i,L}}(\bar{x})$ which clearly coincide with $T(\bar{x})$.

Now, observe, for any $x \in X$, that we have by (4.2)

$$\begin{aligned}\langle x^*_{k,i,L}, x - \bar{x}\rangle &\le f_{t_{k,i,L}}(x) - f(\bar{x}) + \eta_{i,L},\\ &\le f_{t_{k,i,L}}(x) - f_{t_{k,i,L}}(\bar{x}) + \eta_{i,L}, \text{ because } t_{k,i,L} \in T(\bar{x}).\end{aligned}$$

Consequently, $x^*_{k,i,L} \in \partial_{\eta_{i,L}} f_{t_{k,i,L}}(\bar{x})$. Fix any $u \in X$. The latter inclusion ensures $\langle x^*_{k,i,L}\,,\, u\rangle \le (f_{t_{k,i,L}})'_{\eta_{i,k,L}}(\bar{x}; u)$, which combined with (4.1) entails, for each $L \in \mathcal{F}_{\bar{x},u}$,

$$\langle x^*_{k,i,L} + w^*_{i,L}\,,\, u\rangle \le (f_{t_{k,i,L}})'_{\eta_{i,L}}(\bar{x}; u).$$

It follows that

$$\begin{aligned}\langle w^*_{i,L} + \sum_{k\in K_{i,L}} \lambda_{k,i,L} x^*_{k,i,L}, u\rangle &\le \sum_{k\in K_{i,L}} \lambda_{k,i,L} \sup_{t\in T(\bar{x})} (f_t)'_{\eta_{i,L}}(\bar{x}; u)\\ &\le \sup_{t\in T(\bar{x})} (f_t)'_{\eta_{i,L}}(\bar{x}; u).\end{aligned}$$

By definition of $T(\bar{x})$ one has

(4.5)

$$(f_t)'_{\eta_{i,L}}(\bar{x}; u) := \inf_{\theta>0} \theta^{-1}[f_t(\bar{x}+\theta u) - f_t(\bar{x}) + \eta_{i,L}] = \inf_{0<\theta<\theta_0(u)} \theta^{-1}[f_t(\bar{x}+\theta u) - f(\bar{x}) + \eta_{i,L}].$$

Then, by the assumption (i) the function $t \mapsto (f_t)'_{\eta_{i,L}}(\bar{x}; u)$ is upper semicontinuous, thus by compactness of $T(\bar{x})$ there exists $t_{i,L} \in T(\bar{x})$ such that

$$\sup_{t\in T(\bar{x})} (f_t)'_{\eta_{i,L}}(\bar{x}; u) = (f_{t_{i,L}})'_{\eta_{i,L}}(\bar{x}; u).$$

We deduce that for each $(i, L) \in I \times \mathcal{F}_{\bar{x},u}$

$$\langle w^*_{i,L} + \sum_{k\in K_{i,L}} \lambda_{k,i,L} x^*_{k,i,L}, u\rangle \le (f_{t_{i,L}})'_{\eta_{i,L}}(\bar{x}; u). \tag{4.6}$$

Using again the compactness of $T(\bar{x})$, there exist $\bar{t} \in T(\bar{x})$, a directed set J, and a directed mapping $s : J \to I \times \mathcal{F}_{\bar{x},u}$ such that $\lim_{j\in J} t_{s(j)} = \bar{t}$. By (4.6), for all $j \in J$, one has

$$\langle w^*_{s(j)} + \sum_{k\in K_{s(j)}} \lambda_{k,s(j)} x^*_{k,s(j)}, u\rangle \le (f_{t_{s(j)}})'_{\eta_{s(j)}}(\bar{x}; u). \tag{4.7}$$

Observe through the last member of (4.5) and through the assumption (i) that the function $(\eta, t) \mapsto (f_t)'_\eta(\bar{x}; u)$ is upper semicontinuous on $[0, +\infty[\times T(\bar{x})$. So, using (e) of Theorem 3.2 along with the equality $\lim_{j\in J} t_{s(j)} = \bar{t}$ and taking the limit superior in both members of the inequality in (4.7) we get

$$\langle x^*\,,\, u\rangle \le (f_{\bar{t}})'(\bar{x}; u). \tag{4.8}$$

• **Step 4.** We prove that $x^* \in \overline{\text{co}}^*(\bigcup_{t\in T(\bar{x})} \partial f_t(\bar{x}))$.

We note that $\bar{t} \in T(\bar{x})$ and $(f_{\bar{t}})'(\bar{x}; u) = \max_{u^*\in\partial f_{\bar{t}}(\bar{x})} \langle u^*\,,\, u\rangle$ according to the continuity of function $f_{\bar{t}}$ at $\bar{x}$. Then, from the inequality in (4.8) we deduce that, for each $u \in X$,

$$\langle x^*\,,\, u\rangle \le \max_{u^*\in\overline{\text{co}}^*(\bigcup_{t\in T(\bar{x})} \partial f_t(\bar{x}))} \langle u^*\,,\, u\rangle,$$

and this ensures the desired inclusion and finishes the proof of the theorem. □

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