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FIXED POINT THEOREMS AND CONVERGENCE THEOREMS FOR GENERALIZED HYBRID NON-SELF MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first prove a fixed point theorem for normal generalized hybrid non-self mappings in a Hilbert space. In the proof, we show that widely more generalized hybrid mappings are deduced from normal generalized hybrid non-self mappings. Next, we prove a weak convergence theorem of Mann's type [20] for widely more generalized hybrid non-self mappings in a Hilbert space. For the proof, we use the demi-closedness property for widely more generalized hybrid non-self mappings. Finally, using an idea of mean convergence by Shimizu and Takahashi [21] and [22], we prove a mean strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space. This theorem generalizes Hojo and Takahashi's mean strong convergence theorem [9] for generalized hybrid mappings.

1. Introduction

Let H be a real Hilbert space and let C be a non-empty subset of H. Kocourek, Takahashi and Yao [16] introduced a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings, nonspreading mappings [18] and hybrid mappings [27]. A mapping $T : C \to H$ is said to be generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. We call such a mapping an (α, β)-generalized hybrid mapping. An (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, i.e.,

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. It is nonspreading for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2$$

for all $x, y \in C$. Furthermore, it is hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|y - x\|^2$$

for all $x, y \in C$. They proved fixed point theorems and nonlinear ergodic theorems of Baillon's type [2] for generalized hybrid mappings; see also Kohsaka and Takahashi [17] and Iemoto and Takahashi [12]. Very recently, Kawasaki and Takahashi

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[15] introduced a more broad class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}(1.1)\qquad \alpha\|Tx-Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for all $x, y \in C$. Such a mapping T is called an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. In particular, an $(\alpha, \beta, \gamma, \delta, 0, 0, 0)$-widely more generalized hybrid mapping is called $(\alpha, \beta, \gamma, \delta)$-normal generalized hybrid; see Takahashi, Wong and Yao [29]. An $(\alpha, \beta, \gamma, \delta)$-normal generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [16] if $\alpha + \beta = -\gamma - \delta = 1$. and $\varepsilon = \zeta = \eta = 0$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a widely more generalized hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. In [15], Kawasaki and Takahashi proved fixed point theorems and nonlinear ergodic theorems of Baillon's type for such new mappings in a Hilbert space. In particular, by using their fixed point theorems, they proved directly Browder and Petryshyn's fixed point theorem [5] for strict pseudo-contractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [16] for super generalized hybrid mappings.

In this paper, motivated by these mappings and results, we first prove a fixed point theorem for normal generalized hybrid non-self mappings in a Hilbert space. In the proof, we show that widely more generalized hybrid mappings are deduced from normal generalized hybrid non-self mappings and then we prove a fixed point theorem for the mappings by using Kawasaki and Takahashi's fixed point theorem for widely more generalized hybrid mappings. Next, we prove a weak convergence theorem of Mann's type [20] for widely more generalized hybrid non-self mappings in a Hilbert space. For the proof, we use the demi-closedness property for widely more generalized hybrid non-self mappings. Finally, using an idea of mean convergence by Shimizu and Takahashi [21] and [22], we prove a mean strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space. This theorem generalizes Hojo and Takahashi's mean strong convergence theorem [9] for generalized hybrid mappings.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [26], we know the following basic equality: For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$(2.1)\qquad \|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.$$

Furthermore, we know that for $x, y, u, v \in H$

$$(2.2)\qquad 2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.$$

Let C be a nonempty subset of H and let T be a mapping from C into itself. Then, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$

is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Ty\| \leq \|x - y\|$ for all $x \in F(T)$ and $y \in C$. Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for all $x \in H$ and $u \in C$. Furthermore, we know that

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y \rangle \tag{2.3}$$

for all $x, y \in H$; see [26] for more details. For proving main results in this paper, we also need the following lemmas proved in [28] and [1].

Lemma 2.1. *Let D be a nonempty closed convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Lemma 2.2 (Aoyama-Kimura-Takahashi-Toyoda [1])**.** *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [24] for the proof of existence of a Banach limit and its other elementary properties.

3. Fixed point theorem for non-self mappings

In this section, we prove a fixed point theorem for normal generalized hybrid non-self mappings in a Hilbert space. For proving the result, we need the following fixed point theorem obtained by Kawasaki and Takahashi [15].

Theorem 3.1 ([15])**.** *Let H be a Hilbert space, let C be a non-empty, closed and convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself, i.e., there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2$$

$$+ \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$*.*

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2)*.*

In particular, we have the following theorem from Theorem 3.1.

Theorem 3.2. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \geq 0$*.*

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2)*.*

Using Theorem 3.2, we obtain the following fixed point theorem for normal generalized hybrid non-self mappings in a Hilbert space.

Theorem 3.3. *Let C be a non-empty, bounded, closed and convex subset of a Hilbert space H. Let $T : C \to H$ be an $(\alpha, \beta, \gamma, \delta)$-normal generalized hybrid mapping, i.e., there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that*

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. Suppose that it satisfies the following condition (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\alpha + \beta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\alpha + \gamma \geq 0$*.*

Assume that there exists $m > 1$ such that for any $x \in C$,

$$Tx = x + t(y - x)$$

for some $y \in C$ and t with $0 < t \leq m$. Then T has a fixed point in C. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2)*.*

Proof. We give the proof for the case of (1). By the assumption, we have that for any $x \in C$, there exist $y \in C$ and t with $0 < t \leq m$ such that $Tx = x + t(y - x)$. From this, we have $Tx = ty + (1 - t)x$ and hence

$$y = \frac{1}{t}Tx + \frac{t-1}{t}x.$$

Define $Ux \in C$ as follows:

$$Ux = \left(1 - \frac{t}{m}\right)x + \frac{t}{m}y = \left(1 - \frac{t}{m}\right)x + \frac{t}{m}\left(\frac{1}{t}Tx + \frac{t-1}{t}x\right).$$

We obtain that $Ux = \frac{1}{m}Tx + \frac{m-1}{m}x$. Taking $\lambda > 0$ with $m = 1 + \lambda$, we have that

$$Ux = \frac{1}{1+\lambda}Tx + \frac{\lambda}{1+\lambda}x$$

and hence

$$T = (1+\lambda)U - \lambda I. \tag{3.1}$$

Since $T : C \to H$ is an $(\alpha, \beta, \gamma, \delta)$-normal generalized hybrid mapping, we have from (3.1) and (2.1) that for any $x, y \in C$,

$$\begin{aligned}
&\alpha\|(1+\lambda)Ux - \lambda x - ((1+\lambda)Uy - \lambda y)\|^2 \\
&\quad + \beta\|x - ((1+\lambda)Uy - \lambda y)\|^2 + \gamma\|(1+\lambda)Ux - \lambda x - y\|^2 + \delta\|x-y\|^2 \\
&= \alpha\|(1+\lambda)(Ux - Uy) - \lambda(x-y)\|^2 \\
&\quad + \beta\|(1+\lambda)(x - Uy) - \lambda(x-y)\|^2 + \gamma\|(1+\lambda)(Ux - y) - \lambda(x-y)\|^2 \\
&\quad + \delta\|x-y\|^2 \\
&= \alpha(1+\lambda)\|Ux - Uy\|^2 - \alpha\lambda\|x-y\|^2 + \alpha\lambda(1+\lambda)\|x - y - (Ux - Uy)\|^2 \\
&\quad + \beta(1+\lambda)\|x - Uy\|^2 - \beta\lambda\|x-y\|^2 + \beta\lambda(1+\lambda)\|y - Uy\|^2 \\
&\quad + \gamma(1+\lambda)\|Ux - y\|^2 - \gamma\lambda\|x-y\|^2 + \gamma\lambda(1+\lambda)\|x - Ux\|^2 + \delta\|x-y\|^2 \\
&= \alpha(1+\lambda)\|Ux - Uy\|^2 + \beta(1+\lambda)\|x - Uy\|^2 + \gamma(1+\lambda)\|Ux - y\|^2 \\
&\quad + (-\alpha\lambda - \beta\lambda - \gamma\lambda + \delta)\|x-y\|^2 \\
&\quad + \gamma\lambda(1+\lambda)\|x - Ux\|^2 + \beta\lambda(1+\lambda)\|y - Uy\|^2 \\
&\quad + \alpha\lambda(1+\lambda)\|x - y - (Ux - Uy)\|^2 \le 0.
\end{aligned}$$

This implies that U is widely more generalized hybrid. Since $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma > 0$ and $\alpha + \beta \ge 0$, we obtain that

$$\alpha(1+\lambda) + \beta(1+\lambda) + \gamma(1+\lambda) - \alpha\lambda - \beta\lambda - \gamma\lambda + \delta = \alpha + \beta + \gamma + \delta \ge 0,$$
$$\alpha(1+\lambda) + \gamma(1+\lambda) + \gamma\lambda(1+\lambda) + \alpha\lambda(1+\lambda) = (\alpha + \gamma)(1+\lambda)^2 > 0,$$
$$\beta\lambda(1+\lambda) + \alpha\lambda(1+\lambda) = (\alpha + \beta)\lambda(1+\lambda) \ge 0.$$

By Theorem 3.2, we obtain that $F(U) \neq \emptyset$ and hence $F(T) \neq \emptyset$ from $F(U) = F(T)$. Suppose that $\alpha + \beta + \gamma + \delta > 0$. Let p_1 and p_2 be fixed points of T. We have that

$$\begin{aligned}
&\alpha\|Tp_1 - Tp_2\|^2 + \beta\|p_1 - Tp_2\|^2 + \gamma\|Tp_1 - p_2\|^2 + \delta\|p_1 - p_2\|^2 \\
&= (\alpha + \beta + \gamma + \delta)\|p_1 - p_2\|^2 \le 0
\end{aligned}$$

and hence $p_1 = p_2$. Therefore a fixed point of T is unique.

Similarly, we can obtain the desired result for the case when $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta > 0$ and $\alpha + \gamma \ge 0$. This completes the proof. □

Let us give an example of mappings $T : C \to H$ such that for any $x \in C$, there are $y \in C$ and t with $0 < t \le m$ such that $Tx = x + t(y - x)$. In the case of $H = \mathbb{R}$, consider a mapping $T : [0,1] \to \mathbb{R}$:

$$Tx = (1+2x)\cos x - 2x^2, \quad \forall x \in [0,1].$$

Then, we have

$$Tx = (1+2x)(\cos x - x) + x, \quad \forall x \in [0,1].$$

Take $m = 3$. For any $x \in [0,1]$, take $t = 1 + 2x$ and $y = \cos x$. Then, we have that $Tx = t(y - x) + x$, $y = \cos x \in [0,1]$ and $0 < t = 1 + 2x \le 3$.

4. Weak convergence theorems of Mann's type

In this section, we prove a weak convergence theorem of Mann's type [20] for widely more generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following two lemmas. As in the proof in [15], we can first prove the following lemma for widely more generalized hybrid non-self mappings.

Lemma 4.1. *Let C be a non-empty, closed and convex subset of a Hilbert space H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into H with $F(T) \neq \emptyset$ which satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$*.*

Then T is quasi-nonexpansive.

Proof. Suppose that the condition (2) holds. We have from (1.1) that for any $x \in C$ and for any $y \in F(T)$,

$$\begin{aligned}\alpha\|Tx - Ty\|^2 &+ \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &+\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \\ &= (\alpha + \gamma)\|Tx - y\|^2 + (\beta + \delta)\|x - y\|^2 + (\varepsilon + \eta)\|x - Tx\|^2 \leq 0.\end{aligned}$$

We obtain from $\alpha + \gamma > 0$ that

$$\|Tx - y\|^2 \leq -\frac{\beta + \delta}{\alpha + \gamma}\|x - y\|^2 - \frac{\varepsilon + \eta}{\alpha + \gamma}\|x - Tx\|^2.$$

Since $-\frac{\beta+\delta}{\alpha+\gamma} \leq 1$ from $\alpha+\beta+\gamma+\delta \geq 0$ and $-\frac{\varepsilon+\eta}{\alpha+\gamma} \leq 0$ from $\varepsilon+\eta \geq 0$, we obtain that $\|Tx-y\|^2 \leq \|x-y\|^2$ and hence $\|Tx-y\| \leq \|x-y\|$. Thus T is quasi-nonexpansive. Similarly, we can obtain the desired result for the case of the condition (1). □

If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [13]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0, 1]$, put $z = \alpha x + (1 - \alpha)y$. Then we have from (2.1) that

$$\begin{aligned}\|z - Tz\|^2 &= \|\alpha x + (1 - \alpha)y - Tz\|^2 \\ &= \alpha\|x - Tz\|^2 + (1 - \alpha)\|y - Tz\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &\leq \alpha\|x - z\|^2 + (1 - \alpha)\|y - z\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)^2\|x - y\|^2 + (1 - \alpha)\alpha^2\|x - y\|^2 - \alpha(1 - \alpha)\|x - y\|^2 \\ &= \alpha(1 - \alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 \\ &= 0\end{aligned}$$

and hence $Tz = z$. This implies that $F(T)$ is convex.

Lemma 4.2. *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to H$ be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping. Suppose that it satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon + \eta > 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta + \eta > 0$*.*

If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in F(T)$.

Proof. We give the proof for the case of (2). Let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping and suppose that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Replacing x by x_n in (1.1), we have that

$$\begin{aligned} \alpha\|Tx_n - Ty\|^2 + \beta\|x_n - Ty\|^2 + \gamma\|Tx_n - y\|^2 + \delta\|x_n - y\|^2 \\ + \varepsilon\|x_n - Tx_n\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x_n - Tx_n) - (y - Ty)\|^2 \leq 0. \end{aligned} \tag{4.1}$$

From this inequality, we have that

$$\begin{aligned} &\alpha(\|Tx_n - x_n\|^2 + \|x_n - Ty\|^2 + 2\langle Tx_n - x_n, x_n - Ty\rangle) + \beta\|x_n - Ty\|^2 \\ &\quad + \gamma(\|Tx_n - x_n\|^2 + \|x_n - y\|^2 + 2\langle Tx_n - x_n, x_n - y\rangle) + \delta\|x_n - y\|^2 \\ &\quad + \varepsilon\|x_n - Tx_n\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x_n - Tx_n) - (y - Ty)\|^2 \leq 0. \end{aligned}$$

We apply a Banach limit μ to both sides of this inequality. We have that

$$\begin{aligned} &\alpha\mu_n(\|Tx_n - x_n\|^2 + \|x_n - Ty\|^2 + 2\langle Tx_n - x_n, x_n - Ty\rangle) + \beta\mu_n\|x_n - Ty\|^2 \\ &\quad + \gamma\mu_n(\|Tx_n - x_n\|^2 + \|x_n - y\|^2 + 2\langle Tx_n - x_n, x_n - y\rangle) + \delta\mu_n\|x_n - y\|^2 \\ &\quad + \varepsilon\mu_n\|x_n - Tx_n\|^2 + \zeta\mu_n\|y - Ty\|^2 + \eta\mu_n\|(x_n - Tx_n) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

and hence

$$\begin{aligned} \alpha\mu_n\|x_n - Ty\|^2 + \beta\mu_n\|x_n - Ty\|^2 + \gamma\mu_n\|x_n - y\|^2 + \delta\mu_n\|x_n - y\|^2 \\ + \zeta\mu_n\|y - Ty\|^2 + \eta\mu_n\|y - Ty\|^2 \leq 0. \end{aligned}$$

Thus we have

$$(\alpha + \beta)\mu_n\|x_n - Ty\|^2 + (\gamma + \delta)\mu_n\|x_n - y\|^2 + (\zeta + \eta)\|y - Ty\|^2 \leq 0.$$

From $\|x_n - Ty\|^2 = \|x_n - y\|^2 + \|y - Ty\|^2 + 2\langle x_n - y, y - Ty\rangle$, we also have

$$\begin{aligned} &(\alpha + \beta)(\mu_n\|x_n - y\|^2 + \|y - Ty\|^2 + 2\mu_n\langle x_n - y, y - Ty\rangle) \\ &\qquad + (\gamma + \delta)\mu_n\|x_n - y\|^2 + (\zeta + \eta)\|y - Ty\|^2 \leq 0. \end{aligned}$$

From $\alpha + \beta + \gamma + \delta \geq 0$ we obtain that

$$\begin{aligned} (\alpha + \beta)\|y - Ty\|^2 + 2(\alpha + \beta)\mu_n\langle x_n - y, y - Ty\rangle \\ + (\zeta + \eta)\|y - Ty\|^2 \leq 0 \end{aligned}$$

and hence

$$(\alpha + \beta + \zeta + \eta)\|y - Ty\|^2 + 2(\alpha + \beta)\mu_n\langle x_n - y, y - Ty\rangle \leq 0.$$

Since $x_n \rightharpoonup z$, we have that

$$(\alpha + \beta + \zeta + \eta)\|y - Ty\|^2 + 2(\alpha + \beta)\langle z - y, y - Ty\rangle \leq 0.$$

Putting $y = z$, we have that

$$(\alpha + \beta + \zeta + \eta)\|z - Tz\|^2 \leq 0.$$

Since $\alpha + \beta + \zeta + \eta > 0$, we have that $z \in F(T)$.

Similarly, by replacing the variables x and y in (1.1), we can obtain the desired result for the case when $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma + \varepsilon + \eta > 0$. This completes the proof. □

Using Lemmas 4.1, 4.2 and the technique developed by Ibaraki and Takahashi [10, 11], we can prove the following weak convergence theorem.

Theorem 4.3. *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to H$ be a widely more generalized hybrid mapping with $F(T) \neq \emptyset$ which satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma > 0$ *and* $\varepsilon + \eta \geq 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$.

Let P be the mertic projection of H onto $F(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C\left(\alpha_n x_n + (1 - \alpha_n)Tx_n\right), \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ converges weakly to $v \in F(T)$, where $v = \lim_{n\to\infty} Px_n$.

Proof. Since $T : C \to H$ is quasi-nonexpansive, we have from Lemma 4.1 that $F(T)$ is closed and convex. Furthermore, we have that for any $z \in F(T)$,

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 \\ &= \|x_n - z\|^2\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \|x_n - z\|^2$ exists. Then $\{x_n\}$ is bounded. We also have from (2.1) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2 \\ &= \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2.\end{aligned}$$

Thus we have

$$\alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty} \|x_n - z\|^2$ exists and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, we have that

$$\|Tx_n - x_n\| \to 0. \tag{4.2}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. By Lemma 4.2 and (4.2), we obtain that $v \in F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. To complete the proof, we show $v_1 = v_2$. We know that $v_1, v_2 \in F(T)$ and hence $\lim_{n\to\infty} \|x_n - v_1\|^2$ and $\lim_{n\to\infty} \|x_n - v_2\|^2$ exist. Put

$$a = \lim_{n\to\infty}(\|x_n - v_1\|^2 - \|x_n - v_2\|^2).$$

Note that for $n = 1, 2, \dots,$

$$\|x_n - v_1\|^2 - \|x_n - v_2\|^2 = 2\langle x_n, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2.$$

From $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$, we have

$$a = 2\langle v_1, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2 \tag{4.3}$$

and

$$a = 2\langle v_2, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2. \tag{4.4}$$

Combining (4.3) and (4.4), we obtain $0 = 2\langle v_2 - v_1, v_2 - v_1\rangle$. Thus we obtain $v_2 = v_1$. This implies that $\{x_n\}$ converges weakly to an element $v \in F(T)$. Since $\|x_{n+1} - z\| \le \|x_n - z\|$ for all $z \in F(T)$ and $n \in \mathbb{N}$, we obtain from Lemma 2.1 that $\{Px_n\}$ converges strongly to an element $p \in F(T)$. On the other hand, we have from the property of P that

$$\langle x_n - Px_n, Px_n - u\rangle \ge 0$$

for all $u \in F(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $Px_n \to p$, we obtain

$$\langle v - p, p - u\rangle \ge 0$$

for all $u \in F(T)$. Putting $u = v$, we obtain $p = v$. This means $v = \lim_{n\to\infty} Px_n$. This completes the proof. □

Using Theorem 4.3, we can show the following weak convergence theorem of Mann's type for generalized hybrid mappings in a Hilbert space.

Theorem 4.4 (Kocourek, Takahashi and Yao [16])**.** *Let H be a Hilbert space and let C be a non-empty, closed and convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Suppose that $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $v \in F(T)$.

Proof. Since $T : C \to C$ is a generalized hybrid mapping, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - Ty\|^2 + (1 - \beta)\|x - Ty\|^2$$

for all $x, y \in C$. We have that an (α, β)-generalized hybrid mapping is an $(\alpha, 1 - \alpha, -\beta, -(1 - \beta), 0, 0, 0)$-widely more generalized hybrid mapping which satisfies the condition (2) in Theorem 4.3. Therefore, we have the desired result from Theorem 4.3. □

5. Strong convergence theorem

In this section, using an idea of mean convergence by Shimizu and Takahashi [21] and [22], we prove the following strong convergence theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 5.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let T be a widely more generalized hybrid mapping of C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma > 0$, $\varepsilon+\eta \geq 0$ *and* $\zeta+\eta \geq 0$*;*
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$, $\zeta+\eta \geq 0$ *and* $\varepsilon+\eta \geq 0$.

Let $u \in C$ and define sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \dots,$ where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T) \neq \emptyset$, then $\{x_n\}$ and $\{z_n\}$ converge strongly to Pu, where P is the metric projection of H onto $F(T)$.

Proof. Since $T : C \to C$ be a widely more generalized hybrid mapping, there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned} \tag{5.1}$$

for any $x, y \in C$. Since $F(T) \neq \emptyset$, we have that for all $q \in F(T)$ and $n = 1, 2, 3, \dots,$

$$\begin{aligned} \|z_n - q\| &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n - q \right\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x_n - q\| \\ &\leq \frac{1}{n} \sum_{k=0}^{n-1} \|x_n - q\| = \|x_n - q\|. \end{aligned} \tag{5.2}$$

Thus we have that

$$\begin{aligned} \|x_{n+1} - q\| &= \|\alpha_n u + (1-\alpha_n) z_n - q\| \\ &\leq \alpha_n \|u - q\| + (1-\alpha_n)\|z_n - q\| \\ &\leq \alpha_n \|u - q\| + (1-\alpha_n)\|x_n - q\|. \end{aligned}$$

Hence, by induction, we obtain

$$\|x_n - q\| \leq \max\{\|u - q\|, \|x - q\|\}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ and $\{z_n\}$ are bounded. Since $\|T^n x_n - q\| \leq \|x_n - q\|$, we have also that $\{T^n x_n\}$ is bounded.

We also obtain from (5.1) that for any $x, z \in C$ and $n \in \mathbb{N}$,

$$\begin{aligned} &\alpha\|Tx - T^{n+1}z\|^2 + \beta\|x - T^{n+1}z\|^2 + \gamma\|Tx - T^n z\|^2 + \delta\|x - T^n z\|^2 \\ &\quad +\varepsilon\|x - Tx\|^2 + \zeta\|T^n z - T^{n+1}z\|^2 + \eta\|(x - Tx) - (T^n z - T^{n+1}z)\|^2 \leq 0 \end{aligned}$$

for any $n \in \mathbb{N} \cup \{0\}$ and $x \in C$. By (2.2) we obtain that

$$\begin{aligned}
&\|(x - Tx) - (T^n z - T^{n+1} z)\|^2 \\
&\quad = \|x - Tx\|^2 + \|T^n z - T^{n+1} z\|^2 - 2\langle x - Tx, T^n z - T^{n+1} z\rangle \\
&\quad = \|x - Tx\|^2 + \|T^n z - T^{n+1} z\|^2 + \|x - T^n z\|^2 + \|Tx - T^{n+1} z\|^2 \\
&\qquad -\|x - T^{n+1} z\|^2 - \|Tx - T^n z\|^2.
\end{aligned}$$

Thus we have that

$$\begin{aligned}
&(\alpha + \eta)\|Tx - T^{n+1} z\|^2 + (\beta - \eta)\|x - T^{n+1} z\|^2 + (\gamma - \eta)\|Tx - T^n z\|^2 \\
&\quad +(\delta + \eta)\|x - T^n z\|^2 + (\varepsilon + \eta)\|x - Tx\|^2 + (\zeta + \eta)\|T^n z - T^{n+1} z\|^2 \leq 0.
\end{aligned}$$

From

$$\begin{aligned}
&(\gamma - \eta)\|Tx - T^n z\|^2 \\
&\quad = (\alpha + \gamma)(\|x - Tx\|^2 + \|x - T^n z\|^2 - 2\langle x - Tx, x - T^n z\rangle) \\
&\qquad -(\alpha + \eta)\|Tx - T^n z\|^2,
\end{aligned}$$

we have that

$$\begin{aligned}
&(\alpha + \eta)\|Tx - T^{n+1} z\|^2 + (\beta - \eta)\|x - T^{n+1} z\|^2 \\
&\quad +(\alpha + \gamma)(\|x - Tx\|^2 + \|x - T^n z\|^2 - 2\langle x - Tx, x - T^n z\rangle) \\
&\quad -(\alpha + \eta)\|Tx - T^n z\|^2 + (\delta + \eta)\|x - T^n z\|^2 \\
&\quad +(\varepsilon + \eta)\|x - Tx\|^2 + (\zeta + \eta)\|T^n z - T^{n+1} z\|^2 \leq 0
\end{aligned}$$

and hence

$$\begin{aligned}
&(\alpha + \eta)(\|Tx - T^{n+1} z\|^2 - \|Tx - T^n z\|^2) + (\beta - \eta)\|x - T^{n+1} z\|^2 \\
&\quad -2(\alpha + \gamma)\langle x - Tx, x - T^n z\rangle + (\alpha + \gamma + \delta + \eta)\|x - T^n z\|^2 \\
&\quad +(\alpha + \gamma + \varepsilon + \eta)\|x - Tx\|^2 + (\zeta + \eta)\|T^n z - T^{n+1} z\|^2 \leq 0.
\end{aligned}$$

By $\alpha + \beta + \gamma + \delta \geq 0$, we have that

$$-(\beta - \eta) = -(\beta + \delta) + \delta + \eta \leq \alpha + \gamma + \delta + \eta.$$

From this inequality and $\zeta + \eta \geq 0$ we obtain that

$$\begin{aligned}
&(\alpha + \eta)(\|Tx - T^{n+1} z\|^2 - \|Tx - T^n z\|^2) \\
&\quad +(\beta - \eta)(\|x - T^{n+1} z\|^2 - \|x - T^n z\|^2) \\
&\quad -2(\alpha + \gamma)\langle x - Tx, x - T^n z\rangle + (\alpha + \gamma + \varepsilon + \eta)\|x - Tx\|^2 \leq 0.
\end{aligned} \tag{5.3}$$

From (5.3), we have that

$$\begin{aligned}
&(\alpha + \eta)(\|Tz - T^{k+1} x_n\|^2 - \|Tz - T^k x_n\|^2) \\
&\quad +(\beta - \eta)(\|z - T^{k+1} x_n\|^2 - \|z - T^k x_n\|^2) \\
&\quad -2(\alpha + \gamma)\langle z - Tz, z - T^k x_n\rangle + (\alpha + \gamma + \varepsilon + \eta)\|z - Tz\|^2 \leq 0
\end{aligned}$$

for any $k \in \mathbb{N} \cup \{0\}$ and $z \in C$. Summing up these inequalities with respect to $k = 0, 1, \ldots, n-1$ and dividing by n, we obtain that

$$\begin{aligned}
&\frac{\alpha + \eta}{n}(\|Tz - T^n x_n\|^2 - \|Tz - x_n\|^2) + \frac{\beta - \eta}{n}(\|z - T^n x_n\|^2 - \|z - x_n\|^2) \\
&\quad -2(\alpha + \gamma)\langle z - Tz, z - z_n\rangle + (\alpha + \gamma + \varepsilon + \eta)\|z - Tz\|^2 \leq 0.
\end{aligned}$$

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup w \in H$. Replacing n by n_i, we have that

$$\frac{\alpha+\eta}{n_i}(\|Tz - T^{n_i}x_{n_i}\|^2 - \|Tz - x_{n_i}\|^2) + \frac{\beta-\eta}{n_i}(\|z - T^{n_i}x_{n_i}\|^2 - \|z - x_{n_i}\|^2)$$
$$-2(\alpha+\gamma)\langle z - Tz, z - z_{n_i}\rangle + (\alpha+\gamma+\varepsilon+\eta)\|z - Tz\|^2 \leq 0.$$

Since $\{x_n\}$ and $\{T^n x_n\}$ are bounded, we have that

$$-2(\alpha+\gamma)\langle z - Tz, z - w\rangle + (\alpha+\gamma+\varepsilon+\eta)\|z - Tz\|^2 \leq 0$$

as $i \to \infty$. Putting $z = w$, we have that

$$(\alpha+\gamma+\varepsilon+\eta)\|w - Tw\|^2 \leq 0.$$

Since $\alpha+\gamma+\varepsilon+\eta > 0$, we have that $w \in F(T)$.

On the other hand, since $x_{n+1} - z_n = \alpha_n(u - z_n)$, $\{z_n\}$ is bounded and $\alpha_n \to 0$, we have $\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0$. Let us show

$$\limsup_{n\to\infty}\langle u - Pu, x_{n+1} - Pu\rangle \leq 0.$$

We may assume without loss of generality that there exists a subsequence $\{x_{n_i+1}\}$ of $\{x_{n+1}\}$ such that

$$\limsup_{n\to\infty}\langle u - Pu, x_{n+1} - Pu\rangle = \lim_{i\to\infty}\langle u - Pu, x_{n_i+1} - Pu\rangle$$

and $x_{n_i+1} \rightharpoonup v$. From $\|x_{n+1} - z_n\| \to 0$, we have $z_{n_i} \rightharpoonup v$. From the above argument, we have $v \in F(T)$. Since P is the metric projection of H onto $F(T)$, we have

$$\lim_{i\to\infty}\langle u - Pu, x_{n_i+1} - Pu\rangle = \langle u - Pu, v - Pu\rangle \leq 0.$$

This implies

$$\limsup_{n\to\infty}\langle u - Pu, x_{n+1} - Pu\rangle \leq 0. \tag{5.4}$$

Since $x_{n+1} - Pu = (1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)$, from (5.2) we have

$$\begin{aligned}\|x_{n+1} - Pu\|^2 &= \|(1-\alpha_n)(z_n - Pu) + \alpha_n(u - Pu)\|^2\\ &\leq (1-\alpha_n)^2\|z_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle\\ &\leq (1-\alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu\rangle.\end{aligned}$$

Putting $s_n = \|x_n - Pu\|^2$, $\beta_n = 0$ and $\gamma_n = 2\langle u - Pu, x_{n+1} - Pu\rangle$ in Lemma 2.2, we have from $\sum_{n=1}^{\infty}\alpha_n = \infty$ and (5.4) that

$$\lim_{n\to\infty}\|x_n - Pu\| = 0.$$

By $\lim_{n\to\infty}\|x_{n+1} - z_n\| = 0$, we also obtain $z_n \to Pu$ as $n \to \infty$.

Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$, $\zeta+\eta \geq 0$ and $\varepsilon+\eta \geq 0$. □

Using Theorem 5.1, we can show the following result obtained by Hojo and Takahashi [9].

Theorem 5.2 (Hojo and Takahashi [9]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let T be a generalized hybrid mapping of C into itself. Let $u \in C$ and define two sequences $\{x_n\}$ and $\{z_n\}$ in C as follows: $x_1 = x \in C$ and*

$$\begin{cases} x_{n+1} = \alpha_n u + (1-\alpha_n) z_n, \\ z_n = \dfrac{1}{n} \displaystyle\sum_{k=0}^{n-1} T^k x_n \end{cases}$$

for all $n = 1, 2, \dots$, where $0 \leq \alpha_n \leq 1$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $F(T)$ is nonempty, then $\{x_n\}$ and $\{z_n\}$ converge strongly to $Pu \in F(T)$, where P is the metric projection of H onto $F(T)$.

Proof. As in the proof of Theorem 4.4, a generalized hybrid mapping is a widely more generalized hybrid mapping. Therefore, we have the desired result from Theorem 5.1. □

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