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A CLASS OF RENORMINGS OF ℓ_2 WITH THE FIXED POINT PROPERTY

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Abstract. A famous open question in metric fixed point theory is whether every Banach space which is isomorphic to the Hilbert space ℓ_2 has the fixed point property for nonexpansive mappings. In this paper, we give a fixed point theorem for a class of renormings of ℓ_2 which generalizes some previous results. We also show that some spaces of this class lack of the more recent sufficient conditions for the fixed point property given by Prus and Szczpanik, Fetter and Gamboa de Buen, and Dowling et al.

1. Introduction

Suppose that $(X, \|\cdot\|)$ is a Banach space, C is a subset of X and $T: C \to X$ is a mapping. We say that T is *nonexpansive* if $\|T(x) - T(y)\| \leq \|x - y\|$ for all $x, y \in C$, and we say that $(X, \|\cdot\|)$ has the *fixed point property* (FPP for short) if every nonexpansive self-mapping of each nonempty bounded closed convex subset C of X has a fixed point. If the same property holds for every weakly compact convex subset of X we say that $(X, \|\cdot\|)$ has the *weak fixed point property* (WFPP for short). Obviously, FPP equals WFPP for reflexive spaces.

In 1965, W.A. Kirk [20] proved that those Banach spaces which satisfy a geometrical property named *normal structure* have the WFPP. This result includes the case in which the Banach space $(X, \|\cdot\|)$ is a Hilbert space or, more generally, a uniformly convex space. Since then, it has been discovered a forest of geometrical conditions which can replace normal structure in Kirk's theorem. For instance, three recent conditions which we shall be concerned with in this paper are the Prus-Szczepanik condition, the E-convexity and the WORTH property.

On the other hand, a fundamental question of this theory, *Does every reflexive Banach space have the FPP?*, remains unanswered. (See [21] for more about this problem). Two results that may make us inclined to think that the answer is yes were given by D.E. Alspach in 1981 [1] and by P.N. Dowling and C.J. Lennard in 1997 [4], who proved, respectively, that $L^1([0,1])$ fails to have the WFPP and that a subspace of $L^1([0,1])$ has the WFPP if and only if it is reflexive.

A more restrictive, but also unsolved, question in the theory is:

Does every (equivalent) renorming of ℓ_2 have the FPP?

Of course, the answer to this question would be affirmative if the same were true for the more general previous question, or if it were true that the FPP is

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invariant under renormings, but presently little is known about this fact. However, positive results have been given for some specific classes of renormings of ℓ_2 (See, for instance, [6, 19, 29]).

In this note we prove that renormings of ℓ_2 of the form $|x| = \max\{\|x\|_2, p(x)\}$, where $p(x)$ is a certain seminorm, have the FPP, thus generalizing the results obtained in [6, 19, 29]). We also show that some members of this family fail to have any of the three geometrical conditions mentioned above, as well as orthogonal convexity and asymptotic normal structure.

All the results of this paper are established in ℓ_2, the classical real space of all sequences $x = (x_n) = \big(x(n)\big)$ for which $\sum_{i=1}^{\infty} x_i^2 < \infty$. The Euclidean norm $\|x\|_2 := \sqrt{\sum_{i=1}^{\infty} x_i^2}$ is associated to the ordinary inner product $< x, y >= \sum_{n=1}^{\infty} x_n y_n$. Also the "sup" norm $\|x\|_\infty = \sup\{|x(n)| : n = 1, \ldots\}$ will be sometimes considered. The standard Schauder basis of $(\ell_2, \|\cdot\|_2)$ will be denoted by (e_n).

If $\|\cdot\|$ is a norm on ℓ_2 equivalent to $\|\cdot\|_2$, we say that $\|\cdot\|$ is a renorming of ℓ_2. Given such a renorming, we denote the closed unit ball and unit sphere as $B_{\|\cdot\|} := \{x \in \ell_2 : \|x\| \leq 1\}$, $S_{\|\cdot\|} := \{x \in \ell_2 : \|x\| = 1\}$.

Recall that the modulus of convexity of $(X, \|\cdot\|)$ is the function $\delta_X : [0, 2] \to [0, 1]$ given by

$$\delta_X(\varepsilon) \quad := \inf\left\{1 - \left\|\tfrac{1}{2}(x+y)\right\| : x, y \in B_X, \|x-y\| \geq \varepsilon\right\} \ ,$$

and also that the *characteristic of convexity of* $(X, \|\cdot\|)$ is the real number

$$\varepsilon_0(X) := \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}\,.$$

We say that $(X, \|\cdot\|)$ is *uniformly convex* if $\varepsilon_0(X) = 0$, and that $(X, \|\cdot\|)$ is *uniformly nonsquare* if $\varepsilon_0(X) < 2$.

2. A family of renormings of ℓ_2 enjoying the FPP

Suppose that $p : \ell_2 \to [0, \infty)$ is a seminorm on ℓ_2 and consider the norm $|\cdot|$ defined by

$$|x| = \max\{\|x\|_2, p(x)\}\,.$$

This norm is equivalent to $\|\cdot\|_2$ if p satisfies

(H1) There exists $L > 0$ such that $p(x) \leq L\|x\|_2$ for all $x \in \ell_2$.

We shall also need consider the additional assumption

(H2) There exists $k \in \mathbb{N}$ such that for all $x_1, \ldots, x_k$ in ℓ_2 with pairwise disjoint supports we have

$$p(z) \leq \max\{p(z - x_1), \ldots, p(z - x_k)\}$$

for all $z \in \ell_2$ (recall that the support of $x \in \ell_2$ is the set *supp*$(x) = \{n \in \mathbb{N} : x(n) \neq 0\}$).

We are going to proof that the Banach spaces $(\ell_2, |\cdot|)$ enjoy the FPP. To do this we will follow standard arguments that make use of some well known facts in fixed point theory for nonexpansive maps. Next we recall some of them.

Suppose that $T : K \to K$ is a nonexpansive map defined on a nonempty, weakly compact and convex subset of the Banach space $(Y, |\cdot|)$. The first fact is an immediate consequence of Zorn's lemma.

(F1) There exists a nonempty, weakly compact and convex subset C of K which is minimal for T, that is, minimal for the properties of being nonempty, weakly compact, convex and T-invariant.

The next fact is a consequence of the Banach contraction principle.

(F2) There exists a sequence (x_n) in K such that $x_n - T(x_n) \to 0$.
We say that (x_n) is an approximate fixed point sequence (a.f.p.s.) for T in K.

The third fact shows a strange behavior of the a.f.p.s.'s for T in a minimal set, and is known as the Goebel-Karlovitz Lemma (see [22])

(F3) If C is minimal for T and (x_n) is an a.f.p.s. for T in C, then $|x_n - x| \to \operatorname{diam}(C)$ for all $x \in C$.
Observe that if T has no fixed point in C, then $d = \operatorname{diam}(C) > 0$, and (F3) exhibits a behavior of a.f.p.s.'s which is not merely strange, but even pathological. Observe also that those Banach spaces which do not admit this pathology have the FPP.
This observation suggests a strategy to obtain new fixed point theorems since the more the pathology, the larger the class of Banach spaces enjoying the FPP is. A remarkable result in this direction is due to Maurey [24], who proved that for any two a.f.p.s.'s (x_n) and (y_n), there is another a.f.p.s. (z_n), with z_n an almost metric midpoint of x_n and y_n. The following fact is a generalization of Maurey's result, which may be attributed to Elton et al. [6].

(F4) If $\{x_n^1\}, \ldots, \{x_n^k\}$ are k a.f.p.s.'s for T in K, then there exists an a.f.p.s. for T in K, (z_n), such that

$$\limsup |z_n - x_n^j| \le \frac{k-1}{k} d$$

for all $j \in \{1, \ldots, k\}$.
We are now ready to prove our theorem.

Theorem 2.1. *Suppose that p is a seminorm on ℓ_2 and consider the norm defined by*

$$|x| = \max\{\|x\|_2, p(x)\}$$

for all $x \in \ell_2$. If p satisfies (H1) and (H2), then $(\ell_2, |\cdot|)$ has the FPP.

Proof. We argue by contradiction. Suppose that $(\ell_2, |\cdot|)$ lacks of the FPP and let $T : K \to K$ be a fixed point free nonexpansive map defined on the nonempty, closed, bounded and convex set C. Observe that the norm $|\cdot|$ is equivalent to $\|\cdot\|_2$ because p satisfies (H1). Hence, K is weakly compact and, by (F1), we may assume that K is minimal for T. Let (x_n) be an a.f.p.s. for T in K. Since K is weakly compact, (x_n) has a weakly convergent subsequence, which is also an a.f.p.s. for T, and that we keep denoting by (x_n). Since $d = \operatorname{diam}(K) > 0$, we may additionally assume, by translating and dilating K, that $0 \in K$, $d = 1$ and that (x_n) is weakly null. Using that (x_n) is weakly null and that the unitary vectors of ℓ_2 form a Schauder basis of ℓ_2, we may obtain a subsequence of (x_n), still denoted by (x_n), and a sequence (u_n) in ℓ_2 such that
a) $|u_n - x_n| \to 0$.
b) For every positive integer n, the set $\operatorname{supp}(u_n) = \{i : u_n(i) \neq 0\}$ is finite.

c) max supp $(u_n) <$ min supp (u_{n+1}) for all $n \in \mathbb{N}$.

By assumption, p satisfies (H2) and, since (H2) is satisfied for any $k' > k$, we may assume that $k > L$.

Consider the a.f.p.s.'s for T given by $x_n^1 = x_{kn}, \ldots, x_n^k = x_{kn+k-1}$, and use (F4) to obtain an a.f.p.s. for T, (z_n), such that

$$\limsup_{n\to\infty} |z_n - x_n^j| \le \frac{k-1}{k} \tag{2.1}$$

for all $j \in \{1, \ldots, k\}$.

Since $0 \in K$, then $\limsup |z_n| = 1$, by (F3). Next, we shall obtain a contradiction with this. Since $|z_n| = \max\{\|z_n\|_2, p(z_n)\}$, we need prove that $\limsup \|z_n\|_2 < 1$ and that $\limsup p(z_n) < 1$.

Start with $\limsup \|z_n\|_2 < 1$. Denote by $\{u_n^1\}, \ldots, \{u_n^k\}$ the sequences defined by $u_n^1 = u_{kn}, \ldots, u_n^k = u_{kn+k-1}$ and use a) and (2.1) to obtain

$$\limsup_{n\to\infty} |z_n - u_n^j| \le \frac{k-1}{k}$$

for all $j \in \{1, \ldots, k\}$. Next, use b) to check the equality

$$\sum_{j=1}^{k} \|z_n - u_n^j\|_2^2 = \Big\| \sum_{j=1}^{k} u_n^j - z_n \Big\|_2^2 + (k-1)\|z_n\|_2^2 ,$$

and use that $\|x\|_2 \le |x|$ for all $x \in \ell_2$ to obtain

$$\|z_n\|_2^2 \le \frac{1}{k-1} \sum_{j=1}^{k} |z_n - u_n^j|^2$$

for all $n \in \mathbb{N}$. From this and (2.1),

$$\begin{aligned} \limsup \|z_n\|_2^2 &\le \tfrac{1}{k-1} \textstyle\sum_{j=1}^{k} \limsup |z_n - u_n^j|^2 \\ &\le \tfrac{1}{k-1} \textstyle\sum_{j=1}^{k} \tfrac{(k-1)^2}{k^2} \\ &\le \tfrac{k-1}{k} . \end{aligned}$$

To prove that $\limsup p(z_n) < 1$, use (H2), a), b) and (2.1) to obtain

$$\begin{aligned} \limsup p(z_n) &\le \limsup \max\{p(z_n - u_n^1), \ldots, p(z_n - u_n^k)\} \\ &\le \max\{\limsup |z_n - u_n^1|, \ldots, \limsup |z_n - u_n^k|\} \\ &\le \tfrac{k-1}{k} . \end{aligned}$$

Hence, we have arrived to a contradiction and, consequently, $(\ell_2, |\cdot|)$ has the FPP. □

3. A large subfamily

In this section we introduce a family of renormings of ℓ_2, $X = X(a, b, c, d)$, and use the previous theorem to show that each member of the family has the FPP.

For $x \in \ell_2$ define

$$\mathcal{M}(x) := \sup\{|x(2i-1)| + |x(2j)| : i, j \in \mathbb{N}\},$$

and

$$\mathcal{S}(x) := \sup\{|x(1) + x(n) + x(n+1) + x(n+2)| : n \ge 2\}.$$

It is easy to see that, for every $x \in \ell_2$, $\mathcal{M}(x) \leq \sqrt{2}\|x\|_2$ and $\mathcal{S}(x) \leq 2\|x\|_2$.

For $a > 0$ and $b, c, d \geq 0$, let $X = X(a, b, c, d)$ be the Banach space $(\ell_2, \|\cdot\|)$, where the norm $\|\cdot\|$ is defined by

$$\|x\| := \max\{a\|x\|_2, b\,\mathcal{M}(x), c\,\mathcal{S}(x), d\,\|x\|_\infty\}.$$

The norm $\|\cdot\|$ is equivalent to $\|\cdot\|_2$, since for every $x \in \ell_2$ we have

$$a\|x\|_2 \leq \|x\| \leq m\|x\|_2,$$

where $m = \max\{a, b\sqrt{2}, 2c, d\}$.

Corollary 3.1. *X has the FPP.*

Proof. Observe that the norm $\|\cdot\|$ is of the form $\|x\| = a|x|$, where

$$|x| = \max\{\|x\|_2, p(x)\},$$

and p is the seminorm on ℓ_2 defined by

$$p(x) = \max\left\{\frac{b}{a}\mathcal{M}(x), \frac{c}{a}\mathcal{S}(x), \frac{d}{a}\|x\|_\infty\right\}.$$

We shall prove that $(\ell_2, |\cdot|)$ has the FPP as a consequence of the previous theorem, and it is not hard to obtain from this that X also has the FPP. Hence, we only need check that p satisfies (H1) and (H2). That p satisfies (H1), with $L = \frac{1}{a}\max\{b\sqrt{2}, 2c, d\}$, is easy. We proceed to check that p also satisfies (H2), with $K = 5$. Hence, assume that $x_1, \ldots, x_5$ are points in ℓ_2 with pairwise disjoint supports and let $z \in \ell_2$ be arbitrarily chosen. Then

$$p(z) = \max\left\{\frac{b}{a}\mathcal{M}(z), \frac{c}{a}\mathcal{S}(z), \frac{d}{a}\|z\|_\infty\} \leq \max\{p(z - x_1), \ldots, p(z - x_5)\right\}.$$

We shall only check that

$$\begin{aligned}\tfrac{c}{a}\mathcal{S}(z) &= \tfrac{c}{a}\sup\{|z(1) + z(n) + z(n+1) + z(n+2)| : n \geq 2\} \\ &\leq \max\{p(z - x_1), \ldots, p(z - x_5)\},\end{aligned}$$

since the other cases are similar. For this, take an integer $n \geq 2$ at random. Since the supports of $x_1, \ldots, x_5$ are pairwise disjoint, there is one of them, say x_j, such that $supp(x_j) \bigcap \{1, n, n+1, n+2\} = \emptyset$. Then,

$$\begin{aligned}&\frac{c}{a}|z(1) + z(n) + z(n+1) + z(n+2)| = \\ &\quad = \frac{c}{a}|(z - x_j)(1) + (z - x_j)(n) + (z - x_j)(n+1) + (z - x_j)(n+2)| \\ &\quad \leq \frac{c}{a}S(z - x_j) \\ &\quad \leq p(z - x_j) \\ &\quad \leq \max\{p(z - x_1), \ldots, p(z - x_5)\}.\end{aligned}$$

$\square$

4. A quite pathological space. The case $a = \frac{1}{3},\ b = 1,\ c = 1,\ d = 0$

In this section we show that the space $\tilde{X} = X(\frac{1}{3}, 1, 1, 0)$, which have the FPP, fails to have the most recent geometrical properties which have been known that are sufficient conditions for the FPP. To start with, we shall check that $\tilde{X}$ also lacks of the more classical conditions known as asymptotic normal structure and orthogonal convexity.

4.1. **Asymptotic normal structure.** Browder and Göhde, independently, proved in 1965 that uniformly convex Banach spaces have the FPP. In the same year, W.A. Kirk discovered that those reflexive Banach spaces that have a geometrical property called normal structure (NS) also have the FPP. (See [20]). After this seminal work, many authors have obtained a wide range of sufficient conditions for NS in reflexive spaces. Among many others,

(1) Uniform convexity (Belluce, Kirk, 1967).
(2) $\varepsilon_0(X) < 1$ (Goebel, 1970).
(3) Uniform smoothness (Turret, 1982).
(4) Uniform convexity in every direction (Garkavi, 1962).
(5) Opial condition, (Gossez Lami Dozo, 1972).
(6) Near uniform convexity (Van Dulst, 1981).
(7) F-convexity (Saejung, 2008).

Asymptotic normal structure (ANS) is a geometric property of Banach spaces introduced by Baillon and Schöneberg in 1981 [2]. It is a generalization of NS that also implies the FPP, and is defined as follows: we say that a Banach space $(X, \|\cdot\|)$ has asymptotic normal structure if for each nonempty, closed, bounded an convex subset C of X and any sequence (x_n) in X, with $x_n - x_{n+1} \to 0$, there exists a point $x \in C$ such that $\liminf \|x_n - x\| <$ diam (C).

Proposition 4.1. *$\tilde{X}$ does not have asymptotic normal structure.*

Proof. Consider the set C defined as

$$C = \left\{ \big(x(n)\big) : \|x\|_2 \le 1,\ \|x\|_\infty \le \frac{1}{2},\ x(n) \ge 0\ (n \ge 1),\ x(3k+1) = 0\ (k = 0, 1, ...) \right\}.$$

We claim that C is a closed, bounded and convex set, with $\text{diam}_{\|\cdot\|}(C) = 1$, which contains a diametral sequence (x_n) with $x_{n+1} - x_n \to 0$.

That C is closed and convex is easy. To see that $\text{diam}_{\|\cdot\|}(C) = 1$ observe that for all $x, y \in C$,

$$\frac{1}{3}\|x - y\|_2 = \frac{1}{3}[\|x\|_2^2 + \|y\|_2^2 - 2\langle x, y\rangle]^{\frac{1}{2}} \le \frac{1}{3}[\|x\|_2^2 + \|y\|_2^2]^{\frac{1}{2}} \le 1.$$

Moreover for positive integers $i \ne j$ we have that $x(i), y(i), x(j), y(j) \in [0, \frac{1}{2}]$, which implies that

$$|x(i) - y(i)| + |x(j) - y(j)| \le 1.$$

Thus, $\mathcal{M}(x - y) \le 1$.

That $\mathcal{S}(x-y) \leq 1$ follows from $x(1) = y(1) = 0$, and the fact that in the following inequality one of the three summands on the right hand side is zero,

$$\begin{aligned}&|x(n) - y(n) + x(n+1) - y(n+1) + x(n+2) - y(n+2)| \\ &\qquad \leq |x(n) - y(n)| + |x(n+1) - y(n+1)| + |x(n+2) - y(n+2)| \,.\end{aligned}$$

These previous calculations show that diam $_{\|.\|}(C) \leq 1$. To see the equality consider the sequence (v_n) in C defined by $v_n := \frac{1}{2}(e_{3n+2} + e_{3n+3})$, and notice that $\|v_n - v_{n+1}\| = 1$.

Let us consider the sequence (x_n) in C defined by

$$\left.\begin{aligned} x_n &:= \frac{(2k+1)^2 - n}{4k+1} v_k + v_{k+1} && \text{if } (2k)^2 < n \leq (2k+1)^2 \\ x_n &:= v_{k+1} + \frac{n - (2k+1)^2}{4k+3} v_{k+2} && \text{if } (2k+1)^2 < n \leq (2k+2)^2 \end{aligned}\right\}$$

It is straightforward (but tedious) to check that $x_{n+1} - x_n \to 0$.

To see that (x_n) is a diametral sequence observe that v_k, $v_{k+1} = \frac{1}{2}(e_{3k+5} + e_{3k+6})$ and v_{k+2} have disjoint supports, and then we can write

$$x_n - x = \Big(\ldots, \frac{1}{2} - x(3k+5), \frac{1}{2} - x(3k+6), \ldots\Big)$$

where either $(2k)^2 < n \leq (2k+1)^2$ or $(2k+1)^2 < n \leq (2k+2)^2$. Consequently,

$$1 \geq \|x_n - x\| \geq M(x_n - x) \geq \Big|\frac{1}{2} - x(3k+5)\Big| + \Big|\frac{1}{2} - x(3k+6)\Big| = 1 - x(3k+5) - x(3k+6)\,,$$

and since $x \in \ell_2$, this shows that

$$\lim_{n\to\infty} \|x_n - x\| = 1 = \text{ diam } (C).$$

$\square$

4.2. **Orthogonal convexity.** Orthogonal convexity is a geometric property of Banach spaces which is independent of normal structure and implies the WFPP (See [16, 17]). It is defined as follows: we say that a Banach space $(X, \|\cdot\|)$ is orthogonally convex (OC) if for every weakly null sequence (x_n) with $D[(x_n)] := \limsup_n(\limsup_m \|x_n - x_m\|) > 0$ there exists $\lambda > 0$ such that $A_\lambda[(x_n)] < D[(x_n)]$, where

$$A_\lambda[(x_n)] := \limsup_n(\limsup_m |M_\lambda(x_n, x_m)|),$$

$$M_\lambda(x, y) := \{z \in X : \max\{\|z - x\|, \|z - y\|\} \leq \frac{1+\lambda}{2}\|x - y\|\}$$

for any $x, y \in X$, and $|A| := \sup\{\|z\| : z \in A\}$ for any subset A of X.

Proposition 4.2. *The space $\tilde{X}$ is not orthogonally convex.*

Proof. Consider the sequence of unitary vectors $x_n = e_{n+1}$. Then, $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $\|x_n - x_{n+2k+1}\| = 2$ for all $n, k \in \mathbb{N}$, from which $D[(x_n)] = 2$. We also have that $x_n + x_{n+2k+1}$ is a metric midpoint of x_n and x_{n+2k+1}, and since $\|x_n + x_{n+2k+1}\| = 2$, we conclude that $A_\lambda[(x_n)] = D[(x_n)] = 2$ for all $\lambda > 0$. Hence, $\tilde{X}$ is not OC. $\square$

Among the more recent and important results in the fixed point theory of nonexpansive mappings in superreflexive spaces is the one due to Eva Mazcuñán Navarro who showed in 2003 (see [25], [14]) that every Banach space X with characteristic of convexity $\varepsilon_0(X) < 2$ has the FPP.

Needless to say that $\varepsilon_0(\tilde{X}) = 2$, because $\|e_1\| = 1 = \|e_2\|$ and $\|e_1 \pm e_2\| = 2$. Thus, $\tilde{X}$ does no fall into the scope of the Mazcuñán Theorem. Very recently, Mazcuñán Theorem has been generalized by S. Prus and M. Szczepanik [28] and also by P.N. Dowling, B. Randrianantoanina and B. Turett [5]. In the next two sections we show that $\tilde{X}$ also fails to satisfy these two new conditions.

4.3. **Prus-Szczepanik condition (PSz).** This property was introduced by S. Prus and M. Szczepanik in 2005 [28]. Given a Banach space X, put

$$d(\varepsilon, x) = \inf_{(y_m)\in\mathcal{N}_X} \limsup_{m\to\infty} \|x + \varepsilon y_m\| - \|x\|,$$

and

$$b_1(\varepsilon, x) = \sup_{(y_m)\in\mathcal{M}_X} \liminf_{m\to\infty} \|x + \varepsilon y_m\| - \|x\|.$$

where $\mathcal{N}_X := \{(x_n) : x_n \in S_X\ n = 1, 2, \ldots, x_n \rightharpoonup 0_X\}$ and $\mathcal{M}_X := \{(x_n) : x_n \in B_X\ n = 1, 2, \ldots, D[(x_n)] \leq 1, x_n \rightharpoonup 0_X\}$.

Definition 4.3. Let X be a non-Schur Banach space. If there exists $\varepsilon \in (0, 1)$ such that for every $x \in S_X$ it is the case that $b_1(1, x) < 1 - \varepsilon$ or $d(1, x) > \varepsilon$ we say that $(X, \|\cdot\|)$ satisfies the (PSz) condition.

It was shown in [28] that every non-Schur reflexive Banach space with the property (PSz) has the FPP.

Properties which are stronger than (PSz) condition are the following.

(1) Uniform noncreasyness (introduced by S. Prus in 1997) (see [27]) and its generalizations. (See [13, 8, 7]).
(2) Property $M(X) > 1$. (See [3]). In particular this last condition covers all the uniformly nonsquare Banach spaces. (See [25, 14]). Other reflexive Banach spaces X with $M(X) > 1$ are those satisfying $R(X) < 2$. (See [11]).

Next, we check that $\tilde{X}$ also fails (PSz) condition.

Proposition 4.4. *$\tilde{X}$ lacks of the (PSz) condition.*

Proof. We shall prove that, for $x = e_2$, we have $b_1(1, x) \geq 1$ and $d(1, x) = 0$.

That $b_1(1, x) \geq 1$ follows from the fact that for the sequence (x_n) given as $x_n = e_{4n+1}$, $n \geq 1$, we have that $x_n \rightharpoonup 0$, $D(x_n) \leq 1$, $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and also $\|x + x_n\| = 2$ for all $n \in \mathbb{N}$.

To see that $d(1, x) = 0$, consider the sequence $y_n = e_{4n}$. Then, $\|y_n\| = 1$ for all $n \in \mathbb{N}$, $y_n \rightharpoonup 0$ and $\|x + y_n\| = 1$ for all $n \in \mathbb{N}$. □

4.4. **Banach spaces with O-convex dual.** Very recently, in 2008, P.N. Dowling, B. Randrianantoanina and B. Turett [5], showed that (superreflexive) spaces with O-convex dual have the FPP. We will recall this result with a bit more details.

For $\varepsilon \in (0,2)$, a subset A of X is said to be symmetrically ε-separated if the distance between any two distinct points of $A \cup (-A)$ is at least ε and a Banach space X is O-convex if the unit ball B_X contains no symmetrically $(2-\varepsilon)$-separated subset of cardinality n for some $\varepsilon > 0$ and some $n \in \mathbb{N}$. O-convex Banach spaces are superreflexive ([5]).

Theorem 4.5 ([5])**.** *If X^* is O-convex, then the Banach space X has the fixed point property for nonexpansive mappings.*

Since $\varepsilon_0(X) < 2$ if and only if $\varepsilon_0(X^*) < 2$, uniformly nonsquare Banach spaces have O-convex dual, and then this theorem is a generalization of the Mazcuñán's result above reported.

Naidu and Sastry [26] also characterized the dual property to O-convexity. For $\varepsilon > 0$, a convex subset A of B_X is an ε-flat if $A \cap (1-\varepsilon)B_X = \emptyset$. A collection $\mathcal{D}$ of ε-flats is jointly complemented (jcc in short) if, for each distinct ε-flats A and B in $\mathcal{D}$, the sets $A \cap B$ and $A \cap (-B)$ are nonempty. Define

$$E(n,X) = \inf\{\varepsilon > 0 : B_X \text{contains a jcc of } \varepsilon - \text{flats of cardinality } n\}.$$

A Banach space X is said to be E-convex if $E(n,X) > 0$ for some $n \in \mathbb{N}$. Since a Banach space is E-convex if and only if its dual space is O-convex, Theorem 4.5 can be restated by saying that E-convex Banach spaces have the fixed point property for nonexpansive mappings.

Proposition 4.6. *The space $\tilde{X}$ fails to have O-convex dual, and hence it is not E-convex.*

Proof. To see that $\tilde{X}^*$ is not O-convex consider the functionals $f_n \in \tilde{X}^*$ defined on ℓ_2 by $f_n(x) = x_{2n-1} + x_{2n}$, where $x = (x_n)$, and n is any positive integer.

For every $x \in \ell_2$,

$$|f_n(x)| \le |x_{2n-1}| + |x_{2n}| \le \mathcal{M}(x) \le \|x\|.$$

Moreover $\|e_{2n}\| = 1$ and $f(e_{2n}) = 1$. Thus, $\|f_n\| = 1$.

Let $m \neq n$ and take

$$v_{m,n} := \frac{1}{2}(e_{2n-1} + e_{2n} - e_{2m-1} - e_{2m}), \quad w_{m,n} = \frac{1}{2}(e_{2n-1} + e_{2n} + e_{2m-1} + e_{2m}).$$

It is clear that $\frac{1}{3}\|v_{m,n}\|_2 = \frac{1}{3}\|w_{m,n}\| = \frac{1}{3}$ and that $\mathcal{M}(v_{m,n}) = \mathcal{M}(w_{m,n}) = 1$. Moreover, if $\min\{m,n\} \ge 2$ and $|m-n| \ge 4$, we also have $S(v_{m,n}) = S(w_{m,n}) = 1$.

Now consider the set $A = \{f_{4p} : p = 1,2,\ldots\}$. For $p \neq q$ we have that $\min\{4p,4q\} \ge 2$ and $|4p-4q| \ge 4$. Then, $\|v_{4p,4q}\| = \|w_{4p,4q}\| = 1$, and since $(f_{4p}-f_{4q})(v_{4p,4q}) = (f_{4p}+f_{4q})(w_{4p,4q}) = 2$, we have $\|f_4p - f_4q\|_{\tilde{X}^*} = \|f_{4p}+f_{4q}\|_{\tilde{X}^*} = 2$. Thus, A is an infinite 2-symmetrically separated subset of $S_{\tilde{X}^*}$ and hence $\tilde{X}^*$ is not O-convex. ☐

4.5. **Property WORTH.** A Banach space X has the WORTH property (Rosenthal, 1983; Sims 1988) if $\lim_n |\|x_n - x\| - \|x_n + x\|| = 0$ for all $x \in X$ and all weakly null sequence (x_n) in X. The coefficient $\mu(X) := \inf\{r > 0 : \limsup \|x_n + x\| \leq r \limsup \|x_n - x\| : x_n \rightharpoonup 0_X \ , \ x \in X\}$ was introduced in [18], and measures how far a Banach space X is from satisfying WORTH property. Obvious consequences of the definition of $\mu(X)$ are the following.

- $1 \leq \mu(X) \leq 3$.
- $\mu(X) = 1 \Leftrightarrow X$ has WORTH.

B. Sims raised the problem of whether reflexive spaces with WORTH property have FPP. Many partial affirmative answers were obtained (see, for instance [10, 31]) and, very recently, H. Fetter and B. Gamboa [9] solved this problem affirmatively. In fact they showed that reflexive spaces with WORTH property enjoy a sufficient condition for FPP introduced seventeen years before in [12].

Theorem 4.7 (Fetter-Gamboa, 2010)**.** *If X is reflexive and $\mu(X) = 1$, then X enjoys FPP.*

However, the space $\tilde{X}$ does not have property WORTH.

Proposition 4.8. $\mu(\tilde{X}) \geq 2$.

Proof. For $x_n := \big(0, \dots, 0, \overset{(2n+1)}{1/2}, 1/2, 1/2, 0, \dots\big)$, one has $x_n \rightharpoonup 0_{\tilde{X}}$ and $\|x_n\| = 3/2$ $(n = 1, 2, \dots)$. Since $e_1 \in S_{\tilde{X}}$, and for $n > 1$,

$$\|\tfrac{1}{2}e_1 + x_n\| = \big\|\big(\tfrac{1}{2}, 0, \dots, 0, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 0, \dots\big)\big\| = 2.$$

$$\|\tfrac{1}{2}e_1 - x_n\| = \big\|\big(\tfrac{1}{2}, 0, \dots, 0, \tfrac{-1}{2}, \tfrac{-1}{2}, \tfrac{-1}{2}, 0, \dots\big)\big\| = 1 \ ,$$

it follows that $\mu(\tilde{X}) \geq 2$. □

Remark 4.9. Asymptotic normal structure, Orthogonal convexity, Prus-Szczepanik condition, E-convexity and WORTH property are some of the most relevant (likely pairwise independent) sufficient conditions for FPP in reflexive spaces. Each one is maximal in the sense that no other weaker sufficient condition for FPP in such spaces is known. However, this list of sufficient conditions for FPP is not exhaustive. There exist other sufficient conditions for FPP, again of geometric type, which are stated in terms of ultrapowers of the space under consideration, as, for instance, Property AMC in [12] and Property (S) in [32]. For these kind of properties to check that a given Banach space enjoys one of them is not an easy task, and we omit it for our space $\tilde{X}$.

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