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FIXED POINT THEOREMS FOR SINGLE-VALUED AND MULTI-VALUED MIXED MONOTONE OPERATORS OF MEIR-KEELER TYPE

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Abstract. In (2008), Zhang proved the existence of fixed points of mixed monotone operators along with certain convexity and concavity conditions. In this paper, mixed monotone single-valued and multi-valued operators of Meir-Keeler type are defined and two fixed point theorems are proved. Our results, extend the results of Zhang.

1. Introduction

In (1969), Meir and Keeler [4] introduced a new fixed point theorem as a generalization of Banach fixed point theorem. On the other hand, in (1987), mixed monotone operators were introduced by Guo and Lakshmikantham [1]. Then many authors studied them in Banach spaces and obtained lots of interesting results (see [2, 3] and [5]-[9]).

In this paper, weak and strong mixed monotone single-valued and multi-valued operator of Meir-Keeler type are defined. Then two fixed point theorems for this kind of operators are proved. These results extend the results given by Zhang [7].

Let E be a real Banach space. The zero element of E is denoted by θ. A subset P of E is called a cone if and only if:

- P is closed, nonempty and $P \neq \{\theta\}$,
- $a, b \in \mathbb{R}$, $a, b \geq 0$ and $x, y \in P$ imply that $ax + by \in P$,
- $x \in P$ and $-x \in P$ imply that $x = \theta$.

Given a cone $P \subset E$, a partial ordering $\leq$ with respect to P is defined by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there exist a number $K > 0$ such that $\theta \leq x \leq y$ implies $||x|| \leq K||y||$, for every $x, y \in E$. The least positive number satisfying this, is called the normal constant of P.

Assume $u_0, v_0 \in E$ and $u_0 \leq v_0$. The set $\{x \in E : u_0 \leq x \leq v_0\}$ is denoted by $[u_0, v_0]$.

Now, we recall the following definitions from [2, 3].

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Definition 1.1. Let P be a cone of a real Banach space E. Suppose $D \subset P$ and $\alpha \in (-\infty, +\infty)$. An operator $A : D \to D$ is said to be α-convex (α-concave), if it satisfies $A(tx) \leq t^\alpha Ax$ ($A(tx) \geq t^\alpha Ax$) for $(t, x) \in (0, 1) \times D$.

Definition 1.2. Let E be an ordered Banach space and $D \subset E$. An operator is called mixed monotone on $D \times D$, if $A : D \times D \to E$ and $A(x_1, y_1) \leq A(x_2, y_2)$ for any $x_1, x_2, y_1, y_2 \in D$, where $x_1 \leq x_2$ and $y_2 \geq y_1$. Also, $x^* \in D$ is called a fixed point of A, if $A(x^*, x^*) = x^*$.

Let $\mathcal{C}(E)$ be the collection of all closed subsets of E.

Definition 1.3. For two subsets X, Y of E, we write

- $X \leq Y$, if for all $x \in X$, there exists $y \in Y$ such that $x \leq y$.
- $x \ll X$, if there exists $z \in X$, such that $x \ll z$,
- $X \ll x$, if for all $z \in X$, $z \ll x$.

Definition 1.4. Let D be a nonempty subset of E. $T : D \to \mathcal{C}(E)$ is called, increasing (decreasing) upward, if $u, v \in D$, $u \leq v$ and $x \in T(u)$ imply there exists $y \in T(v)$ such that $x \leq y$ ($x \geq y$). Similarly, $T : D \to \mathcal{C}(E)$ is called increasing (decreasing) downward, if $u, v \in D$, $u \leq v$ and $y \in T(v)$ imply there exists $x \in T(u)$ such that $x \leq y$ ($x \geq y$). T is called increasing (decreasing), if T is an increasing (decreasing) upward and downward.

Definition 1.5. Let D be a nonempty subset of E. A multi-valued operator $T : D \times D \to \mathcal{C}(E)$ is said to be mixed monotone upward, if $T(x, y)$ is increasing upward in x and decreasing upward in y, i.e.,

(A_1)**:** for each $y \in D$ and any $x_1, x_2 \in D$ with $x_1 \leq x_2$, if $u_1 \in T(x_1, y)$ then there exists a $u_2 \in T(x_2, y)$ such that $u_1 \leq u_2$;

(A_2)**:** for each $x \in D$ and any $y_1, y_2 \in D$ with $y_1 \leq y_2$, if $v_1 \in T(x, y_1)$ then there exists a $v_2 \in T(x, y_2)$ such that $v_1 \geq v_2$.

Definition 1.6. $x^* \in D$ is called a fixed point of T if $x^* \in T(x^*, x^*)$.

2. Main results

In this section, we introduce two new fixed point theorems in the class of mixed monotone operators. These are new generalizations of the results given by Zhang [7]. Due to this, two definitions are presented as follows:

Definition 2.1. A mixed monotone operator $A : D \times D \to E$ is said to be a **W**eak **M**ixed **M**onotone single-valued operator of **M**eir-**K**eeler type(**WM$_3$K** property, for short), if for each $\epsilon \gg 0$ and $t \in (0, 1)$, there exists $\delta \gg 0$ such that

$$\epsilon \leq A(x, ty) < \epsilon + \delta \quad \text{implies} \quad A(tx, y) < \epsilon, \tag{2.1}$$

for all $(x, y) \in D \times D$.

Definition 2.2. A mixed monotone operator $A : D \times D \to E$ is said to be a **S**trong **M**ixed **M**onotone single-valued operator of **M**eir-**K**eeler type(**SM$_3$K** property, for short), if for each $\epsilon \gg 0$ and $t \in (0, 1)$, there exists $\delta \gg 0$ such that

$$\epsilon \leq A(x, tx) < \epsilon + \delta \quad \text{implies} \quad A(tx, x) < \epsilon, \tag{2.2}$$

for all $x \in D$.

Theorem 2.3. *Let P be a normal cone of E, S be a completely ordered closed subset of E with $S_0 = S\backslash\{\theta\} \subset intP$ and $\lambda S \subset S$, for all $\lambda \in [0,1]$. Let $u_0, v_0 \in S_0$, $A : P \times P \to E$ be a weak mixed monotone operator of Meir-Keeler type with $A(([\theta, v_0] \cap S) \times ([\theta, v_0] \cap S)) \subset S$ and satisfies the following conitions:*

(I) *there exists $r_0 > 0$ such that $u_0 \geq r_0 v_0$,*

(II) *$A(u_0, v_0) \ll u_0 \ll v_0 \ll A(v_0, u_0)$,*

(III) *for $u, v \in [u_0, v_0] \cap S$ with $A(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq A(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll A(v, u)$, there exists $v' \in S$ such hat $u \ll v' \ll A(v', u) \leq v$.*

Then, A has at least one fixed point $x^* \in [u_0, v_0] \cap S$.

Proof. By the above condition (III), there exists $u_1 \in S$ such that $u_0 \leq A(u_1, v_0) \ll u_1 \ll v_0$. Then, there exists $v_1 \in S$ such that $u_1 \ll v_1 \ll A(v_1, u_1) \leq v_0$. Likewise, there exists $u_2 \in S$ such that $u_1 \leq A(u_2, v_1) \ll u_2 \ll v_1$. Then, there exists $v_2 \in S$ such that $u_2 \ll v_2 \ll A(v_2, u_2) \leq v_1$. In general, there exists $u_n \in S$ such that $u_{n-1} \leq A(u_n, v_{n-1}) \ll u_n \ll v_{n-1}$. Then, there exists $v_n \in S$ such that $u_n \ll v_n \ll A(v_n, u_n) \leq v_{n-1}$ $(n = 1, 2, \dots)$.

Take $r_n = \sup\{r \in (0,1) : u_n \geq r v_n\}$, thus $0 < r_0 < r_1 < \cdots < r_n < r_{n+1} < \cdots < 1$, and $\lim_{n\to\infty} r_n = \sup\{r_n : n = 0, 1, 2, \dots\} = r^* \in (0,1]$. Since $r_{n+1} > r_n = \sup\{r \in (0,1) : u_n \geq r v_n\}$, thus $u_n \not\geq r_{n+1} v_n$. In addition, S is completely ordered and $\lambda S \subset S$, for all $\lambda \in [0,1]$, then $u_n < r_{n+1} v_n$. Now, one can prove $r^* = 1$. Otherwise, $r^* \in (0,1)$. Note that

$$u_n \leq A(u_{n+1}, v_{n+1}) \leq A((1/r^*)u_{n+1}, r^* v_{n+1}). \tag{2.3}$$

Since $u_n < r_{n+1} v_n$ and $r_{n+1} < r^*$, hence, $u_n < r^* v_n$. So, if

$$\epsilon = r_n A((1/r^*)u_{n+1}, r^* v_{n+1}), \tag{2.4}$$

there exists $\delta \gg 0$ such that

$$\epsilon \leq A((1/r^*)u_{n+1}, r^* v_{n+1}) \leq r_n A((1/r^*)u_{n+1}, r^* v_{n+1}) + \delta = \epsilon + \delta. \tag{2.5}$$

Therefore, by (2.1), (2.5) and $u_n < r^* v_n$, we have

$$\begin{aligned} u_n &\leq A(u_{n+1}, v_{n+1}) \\ &= A(r^*(1/r^*)u_{n+1}, v_{n+1}) \\ &< \epsilon \\ &= r_n A((1/r^*)u_{n+1}, r^* v_{n+1}) \\ &< r_n A(v_{n+1}, u_{n+1}) \\ &\leq r_n v_n. \end{aligned} \tag{2.6}$$

Hence, $u_n < r_n v_n$, which is a contradiction. Thus, $r^* = 1$ and

$$||v_n - u_n|| \leq K(1 - r_n)||v_n|| \leq K^2(1 - r_n)||v_0||. \tag{2.7}$$

This means that, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$.

For all $n, p \geq 1$,

$$||v_n - v_{n+p}|| \leq K||v_n - u_n|| \to 0 \quad (n, p \to \infty). \tag{2.8}$$

Also,

$$||u_{n+p} - u_n|| \leq K||v_n - u_n|| \to 0 \quad (n, p \to \infty). \tag{2.9}$$

Hence, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in E, then there exist $u^*, v^* \in E$, such that $u_n \to u^*, v_n \to v^*$ $(n \to \infty)$ and $u^* = v^*$. Write $x^* = u^* = v^*$.

It is easy to see, $u_0 \leq u_n \leq u^* \leq v_n \leq v_0$, for all $n = 1, 2, \ldots$. In addition, S is closed, then $u^* \in [u_n, v_n] \cap S \subset [u_0, v_0] \cap S$ $(n = 0, 1, 2, \ldots)$.

Finally, by the mixed monotone property of A,

$$u_{n-1} \leq A(u_n, v_n) \leq A(x^*, x^*) \leq A(u_n, v_n) \leq u_{n-1}. \tag{2.10}$$

By taking limit from (2.10) when $n \to \infty$, we have $A(x^*, x^*) = x^*$. This means x^* is a fixed point of A in $[u_0, v_0] \cap S$. □

Theorem 2.4. *Let P be a normal cone of E, S be a completely ordered closed subset of E with $S_0 = S \backslash \{\theta\} \subset intP$ and $\lambda S \subset S$, for all $\lambda \in [0, 1]$. Let $u_0, v_0 \in S_0$, $A : P \times P \to E$ be a strong mixed monotone operator of Meir-Keeler type with $A(([\theta, v_0] \cap S) \times ([\theta, v_0] \cap S)) \subset S$ and satisfies the following conditions:*

(I) *there exists $r_0 > 0$ such that $u_0 \geq r_0 v_0$,*
(II) *$A(u_0, v_0) \ll u_0 \ll v_0 \ll A(v_0, u_0)$,*
(III) *for $u, v \in [u_0, v_0] \cap S$ with $A(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq A(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll A(v, u)$, there exists $v' \in S$ such hat $u \ll v' \ll A(v', u) \leq v$.*

Then, A has at least one fixed point $x^* \in [u_0, v_0] \cap S$.

Proof. By the above condition (III), there exists $u_1 \in S$ such that $u_0 \leq A(u_1, v_0) \ll u_1 \ll v_0$. Then, there exists $v_1 \in S$ such that $u_1 \ll v_1 \ll A(v_1, u_1) \leq v_0$. Likewise, there exists $u_2 \in S$ such that $u_1 \leq A(u_2, v_1) \ll u_2 \ll v_1$. Then, there exists $v_2 \in S$ such that $u_2 \ll v_2 \ll A(v_2, u_2) \leq v_1$. In general, there exists $u_n \in S$ such that $u_{n-1} \leq A(u_n, v_{n-1}) \ll u_n \ll v_{n-1}$. Then, there exists $v_n \in S$ such that $u_n \ll v_n \ll A(v_n, u_n) \leq v_{n-1}$ $(n = 1, 2, \ldots)$.

Take $r_n = \sup\{r \in (0, 1) : u_n \geq r v_n\}$, then $0 < r_0 < r_1 < \cdots < r_n < r_{n+1} < \cdots < 1$, and $\lim_{n\to\infty} r_n = \sup\{r_n : n = 0, 1, 2, \ldots\} = r^* \in (0, 1]$. Since $r_{n+1} > r_n = \sup\{r \in (0, 1) : u_n \geq r v_n\}$, then $u_n \not\geq r_{n+1} v_n$. In addition, S is completely ordered and $\lambda S \subset S$, for all $\lambda \in [0, 1]$, then $u_n < r_{n+1} v_n$. Now, one can prove $r^* = 1$. Otherwise, $r^* \in (0, 1)$. Note that

$$u_n \leq A(u_{n+1}, v_{n+1}) \leq A(u_{n+1}, (1/r^*) u_{n+1}). \tag{2.11}$$

Since $u_n < r_{n+1} v_n$ and $r_{n+1} < r^*$, hence, $u_n < r^* v_n$. So, if $\epsilon = r_n A((1/r^*) u_{n+1}, u_{n+1})$, there exists $\delta \gg 0$ such that

$$\epsilon \leq A((1/r^*) u_{n+1}, u_{n+1}) \leq r_n A((1/r^*) u_{n+1}, u_{n+1}) + \delta = \epsilon + \delta. \tag{2.12}$$

Therefore, by (2.2), (2.12) and $u_n < r^* v_n$, we have

$$\begin{aligned} u_n &\leq A(u_{n+1}, v_{n+1}) \\ &\leq A(u_{n+1}, (1/r^*) u_{n+1}) \\ &< \epsilon \\ &= r_n A((1/r^*) u_{n+1}, u_{n+1}) \\ &< r_n A(v_{n+1}, u_{n+1}) \\ &\leq r_n v_n. \end{aligned} \tag{2.13}$$

Hence, $u_n < r_n v_n$, which is a contradiction. Thus, $r^* = 1$ and

$$||v_n - u_n|| \leq K(1 - r_n)||v_n|| \leq K^2(1 - r_n)||v_0||. \tag{2.14}$$

This means that, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$.

For all $n, p \geq 1$,

$$||v_n - v_{n+p}|| \leq K||v_n - u_n|| \to 0 \ (n, p \to \infty). \tag{2.15}$$

Also,

$$||u_{n+p} - u_n|| \leq K||v_n - u_n|| \to 0 \ (n, p \to \infty). \tag{2.16}$$

Hence, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in E, then there exist $u^*, v^* \in E$, such that $u_n \to u^*, v_n \to v^*$ $(n \to \infty)$ and $u^* = v^*$. Write $x^* = u^* = v^*$.

It is easy to see, $u_0 \leq u_n \leq u^* \leq v_n \leq v_0$, for all $n = 1, 2, \dots$. In addition S is closed, then $u^* \in [u_n, v_n] \cap S \subset [u_0, v_0] \cap S$ $(n = 0, 1, 2, \dots)$.

Finally, by the mixed monotone property of A,

$$u_{n-1} \leq A(u_n, v_n) \leq A(x^*, x^*) \leq A(u_n, v_n) \leq u_{n-1}. \tag{2.17}$$

By taking limit from (2.17), when $n \to \infty$, we have $A(x^*, x^*) = x^*$. This means x^* is a fixed point of A in $[u_0, v_0] \cap S$. □

Definition 2.5. A function $\Psi : [0, 1) \times P \times P \times E \to E$ is called a $\mathcal{L}'-$function, if $\Psi(t, x, y, 0) = 0$, $\Psi(t, x, y, s) \gg 0$ for $s \gg 0$, and for all $s \gg 0$, there exits $\delta \gg 0$ such that $\Psi(t, x, y, z) \leq s$ for all $s \leq z \leq s + \delta$. Also, $\Psi(t, x, x, z) < tz$, for all $(t, x, x, z) \in [0, 1) \times P \times P \times E$.

Corollary 2.6. *Let P be a normal cone of E, S be a completely ordered closed subset of E with $S_0 = S \backslash \{\theta\} \subset intP$ and $\lambda S \subset S$, for all $\lambda \in [0, 1]$. Let $u_0, v_0 \in S_0$, $A : P \times P \to E$ satisfies*

$$A(tx, y) < \Psi(t, x, y, A(x, ty)), \tag{2.18}$$

for each $x, y \in P$ and $t \in [0, 1]$, where Ψ is a $\mathcal{L}'-$function, $A(([\theta, v_0] \cap S) \times ([\theta, v_0] \cap S)) \subset S$ and satisfies the following conditions:

(I) *there exists $r_0 > 0$ such that $u_0 \geq r_0 v_0$,*
(II) *$A(u_0, v_0) \ll u_0 \ll v_0 \ll A(v_0, u_0)$,*
(III) *for $u, v \in [u_0, v_0] \cap S$ with $A(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq A(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll A(v, u)$, there exists $v' \in S$ such hat $u \ll v' \ll A(v', u) \leq v$.*

Then, A has at least one fixed point $x^* \in [u_0, v_0] \cap S$.

Proof. Let $(x, y) \in P \times P$, $\epsilon \gg 0$ and $t \in (0, 1)$. Choose $\delta \gg 0$ such that $\epsilon \leq A(x, ty) < \epsilon + \delta$. Since Ψ is a $\mathcal{L}'-$function thus

$$\Psi(t, x, y, A(x, ty)) < \epsilon. \tag{2.19}$$

Therefore, by (2.18), $A(tx, y) < \Psi(t, x, y, A(x, ty)) < \epsilon$. Hence, A satisfies (2.1) and by Theorem 2.3, A has a fixed point x^*. □

Corollary 2.7. *Let P be a normal cone of E, S be a completely ordered closed subset of E with $S_0 = S\backslash\{\theta\} \subset intP$ and $\lambda S \subset S$, for all $\lambda \in [0,1]$. Let $u_0, v_0 \in S_0$, $A : P\times P \to E$ be a mixed monotone operator with $A(([\theta, v_0]\cap S)\times([\theta, v_0]\cap S)) \subset S$ and $A(u_0, v_0) \ll u_0 \ll v_0 \ll A(v_0, u_0)$. Assume, there exists a function $\phi : (0,1) \times ([u_0, v_0]\cap S)\times([u_0, v_0]\cap S) \to (0,+\infty)$ such that $A(tx, y) \leq \phi(t, x, y)A(x, ty)$, where $0 < \phi(t, x, x) < t$, for all $(t, x, y) \in (0,1) \times ([u_0, v_0] \cap S) \times ([u_0, v_0] \cap S)$. Suppose*

- (I) *for $u, v \in [u_0, v_0] \cap S$ with $A(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq A(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll A(v, u)$, there exists $v' \in S$ such hat $u \ll v' \ll A(v', u) \leq v$.*
- (II) *there exists an element $w_0 \in [u_0, v_0]\cap S$ such that $\phi(t, x, x) \leq \phi(t, w_0, w_0)$, for all $(t, x) \in (0,1) \times ([u_0, v_0] \cap S)$, and $\lim_{s\to t^-} \phi(s, w_0, w_0) < t$, for all $t \in (0,1)$.*

Then, A has at least one fixed point $x^* \in [u_0, v_0] \cap S$.

Proof. Set $\Psi(t, x, y, z) = z\phi(t, x, y)$. Then, Ψ satisfies all the conditions of Definition 2.5. Thus, by Corollary 2.6, the result is obtained. □

Corollary 2.8. *Let P be a normal cone of E, S be a completely ordered closed subset of E with $S_0 = S\backslash\{\theta\} \subset intP$ and $\lambda S \subset S$, for all $\lambda \in [0,1]$. Let $u_0, v_0 \in S_0$, $A : P\times P \to E$ be a mixed monotone operator with $A(([\theta, v_0]\cap S)\times([\theta, v_0]\cap S)) \subset S$ and $A(u_0, v_0) \ll u_0 \ll v_0 \ll A(v_0, u_0)$. Assume, there exists a function $\eta : (0,1) \times ([u_0, v_0] \cap S) \to (0,+\infty)$ such that $A(tx, x) \leq \eta(t, x)A(x, tx)$, where $0 < \eta(t, x) < t$, for all $(t, x) \in (0,1) \times ([u_0, v_0] \cap S)$. Suppose*

- (I) *for $u, v \in [u_0, v_0] \cap S$ with $A(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq A(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll A(v, u)$, there exists $v' \in S$ such hat $u \ll v' \ll A(v', u) \leq v$.*
- (II) *there exists an element $w_0 \in [u_0, v_0] \cap S$ such that $\eta(t, x) \leq \eta(t, w_0)$, for all $(t, x) \in (0,1) \times ([u_0, v_0] \cap S)$, and $\lim_{s\to t^-} \eta(s, w_0) < t$, for all $t \in (0,1)$.*

Then, A has at least one fixed point $x^* \in [u_0, v_0] \cap S$.

Proof. Set $\Psi(t, x, y, z) = z\eta(t, x)$. Then, Ψ satisfies all conditions of Definition 2.5. Thus, by Corollary 2.6, the result is obtained. □

Remark 2.9. The following notes are considerable:

- Corollary 2.6 is a direct result of Theorem 2.3.
- Corollary 2.7 and Corollary 2.8 are direct results of Corollary 2.6.

3. M_4K property

In this section, we introduce a new fixed point theorem in the class of multi-valued mixed monotone operators. Due to this, the following definition is given.

Definition 3.1. A mixed monotone operator $T : D \times D \to \mathcal{C}(E)$ is said to be a **M**ixed **M**onotone **M**ulti-valued operator of **M**eir-**K**eeler type(**M$_4$K** property, for short), if for each $\epsilon \gg 0$ and $t \in (0,1)$, there exits $\delta \gg 0$ such that $w < \epsilon+\delta$, implies there exists $z \in T(tx, y)$ such that $z < \epsilon$, for all $w \in T(x, ty)$ and $(x, y) \in D \times D$.

Theorem 3.2. *Let P be a normal cone of E, S be a completely ordered closed subset of E with $S_0 = S\backslash\{\theta\} \subset intP$ and $\lambda S \subset S$, for all $\lambda \in [0,1]$. Let $u_0, v_0 \in S_0$,*

$T : P \times P \to \mathcal{C}(E)$ be a mixed monotone multi-valued operator of Meir-Keeler type with $T(([\theta, v_0] \cap S) \times ([\theta, v_0] \cap S)) \subset S$ and satisfies the following conditions:

(I) *there exists $r_0 > 0$ such that $u_0 \geq r_0 v_0$,*

(II) *$T(u_0, v_0) \ll u_0 \ll v_0 \ll T(v_0, u_0)$,*

(III) *for $u, v \in [u_0, v_0] \cap S$ with $T(u, v) \ll u \ll v$, there exists $u' \in S$ such that $u \leq T(u', v) \ll u' \ll v$; similarly, for $u, v \in [u_0, v_0] \cap S$ with $u \ll v \ll T(v, u)$, there exists $v' \in S$ such hat $u \ll v' \ll T(v', u) \leq v$.*

Then, T has at least one fixed point $x^* \in [u_0, v_0] \cap S$.

Proof. By the above condition (III), there exists $u_1 \in S$ such that $u_0 \leq T(u_1, v_0) \ll u_1 \ll v_0$. Then, there exists $v_1 \in S$ such that $u_1 \ll v_1 \ll T(v_1, u_1) \leq v_0$. Likewise, there exists $u_2 \in S$ such that $u_1 \leq T(u_2, v_1) \ll u_2 \ll v_1$. Then, there exists $v_2 \in S$ such that $u_2 \ll v_2 \ll T(v_2, u_2) \leq v_1$. In general, there exists $u_n \in S$ such that $u_{n-1} \leq T(u_n, v_{n-1}) \ll u_n \ll v_{n-1}$. Then, there exists $v_n \in S$ such that $u_n \ll v_n \ll T(v_n, u_n) \leq v_{n-1}$ $(n = 1, 2, \dots)$.

Take $r_n = \sup\{r \in (0, 1) : u_n \geq r v_n\}$, thus $0 < r_0 < r_1 < \cdots < r_n < r_{n+1} < \cdots < 1$, and $\lim_{n\to\infty} r_n = \sup\{r_n : n = 0, 1, 2, \dots\} = r^* \in (0, 1]$. Since $r_{n+1} > r_n = \sup\{r \in (0, 1) : u_n \geq r v_n\}$, thus $u_n \not\geq r_{n+1} v_n$. In addition, S is completely ordered and $\lambda S \subset S$, for all $\lambda \in [0, 1]$, then $u_n < r_{n+1} v_n$. Now, one can prove $r^* = 1$. Otherwise, $r^* \in (0, 1)$. We claim

$$T(u_{n+1}, v_{n+1}) \leq T((1/r^*)u_{n+1}, r^* v_{n+1}). \tag{3.1}$$

Suppose $x \in T(u_{n+1}, v_{n+1})$ be arbitrary. We have $u_{n+1} \leq (1/r^*)u_{n+1}$. If $x_1 = u_{n+1}$, $x_2 = (1/r^*)u_{n+1}$ and $y = v_{n+1}$, then by (A_1) of Definition 1.5, there exists $z \in T((1/r^*)u_{n+1}, v_{n+1})$ such that $x \leq z$. Thus, $T(u_{n+1}, v_{n+1}) \leq T((1/r^*)u_{n+1}, v_{n+1})$.

Also, if $y_1 = r^* v_{n+1}$, $y_2 = v_{n+1}$ and $x = (1/r^*)u_{n+1}$, then for $w \in T((1/r^*)u_{n+1}, r^* v_{n+1})$, there exists $h \in T((1/r^*)u_{n+1}, v_{n+1})$ such that $w \geq h$. It means that,

$$T((1/r^*)u_{n+1}, r^* v_{n+1}) \geq T((1/r^*)u_{n+1}, v_{n+1}). \tag{3.2}$$

Thus,

$$T(u_{n+1}, v_{n+1}) \leq T((1/r^*)u_{n+1}, r^* v_{n+1}). \tag{3.3}$$

Assume $k_n \in T((1/r^*)u_{n+1}, r^* v_{n+1})$ is arbitrary. So, if $\epsilon = r_n k_n$, there exists $\delta > 0$ such that $k_n < r_n k_n + \delta$. By Definition 3.1, for all

$$s \in T(r^*((1/r^*)u_{n+1}), v_{n+1}) = T(u_{n+1}, v_{n+1}), \tag{3.4}$$

$s < r_n k_n$. Since $u_n \leq T(u_{n+1}, v_{n+1})$ by Definition 1.3, there exists $l_n \in T(u_{n+1}, v_{n+1})$ such that $u_n \leq l_n$. Also, $l_n < r_n k_n$, thus $u_n < r_n k_n$.

If we apply the above argument again, we gain $k_n \leq v_n$. Thus, $u_n < r_n v_n$, which is a contradiction by the choice of r_n. Therefore, $r^* = 1$ and

$$||v_n - u_n|| \leq K(1 - r_n)||v_n|| \leq K^2(1 - r_n)||v_0||, \tag{3.5}$$

It means that, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$.

For all $n, p \geq 1$,

$$||v_n - v_{n+p}|| \leq K||v_n - u_n|| \to 0 \ (n, p \to \infty). \tag{3.6}$$

Also,

$$||u_{n+p} - u_n|| \leq K||v_n - u_n|| \to 0 \ (n, p \to \infty). \tag{3.7}$$

Hence, $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in E, then there exists $u^*, v^* \in E$, such that $u_n \to u^*, v_n \to v^*$ $(n \to \infty)$ and $u^* = v^*$. Write $x^* = u^* = v^*$.

It is easy to see that $u_n \leq T(u_{n+1}, v_{n+1}) \leq T(x^*, x^*) \leq T(v_{n+1}, u_{n+1}) \leq v_n$, for all $n = 1, 2, \ldots$. Thus, there exists $z_n \in T(x^*, x^*)$ such that $u_n \leq z_n \leq v_n$. By normality of P and taking limit from both sides of (3.5),

$$||z_n - u_n|| \leq K(1 - r_n)||v_n|| \leq K^2(1 - r_n)||v_0|| \to 0. \tag{3.8}$$

So $z_n \to x^*$. Since T has closed values, then $x^* \in T(x^*, x^*)$ and $x^* \in [u_n, v_n] \cap S \subset [u_0, v_0] \cap S$. □

Remark 3.3. In Theorem 3.2, set $T(x, y) = \{A(x, y)\}$, where A is a mixed monotone single-valued operator, then Theorem 2.3 is concluded.

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