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# DIMENSION OF THE SOLUTION SET FOR FRACTIONAL DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we investigate the following question in a general setting: For a multi-valued mapping $F$ on a Banach space $X$, when does $\dim F(x) \geq n$ imply $\dim Fix(F) \geq n$, where $\dim F(x)$ and $\dim Fix(F)$ denote the topological (covering) dimension of $F(x)$ and the fixed points set $Fix(F)$ of $F$ respectively? We apply our results to the solution set of a Cauchy problem for the fractional differential inclusion with nonlocal condition.

## 1. Introduction

Fractional calculus (differentiation and integration of arbitrary order) is proved to be an important mathematical tool in the modelling of dynamical systems associated with phenomena such as fractal and chaos. In fact, this branch of calculus has found its applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, biology, economics, control theory, signal and image processing, polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, aerodynamics, electro-dynamics of complex medium, viscoelasticity and damping, control theory, wave propagation, percolation, identification, fitting of experimental data, etc. ([16, 18, 22, 25, 28]).

The following problem has been discussed during the last twenty years: If $X$ is a Banach space, $F$ is a multi-valued mapping on $X$, when does $\dim F(x) \geq n$ imply $\dim Fix(F) \geq n$, where $\dim F(x)$ denotes the topological (covering) dimension of $F(x)$ and $\dim Fix(F)$ represents the topological dimension of the fixed points set $Fix(F)$ of $F$? Some answers to the above question were given by some authors; for instance, see the papers ([6, 10, 20, 21, 23, 24, 26, 27]). In this paper, we first investigate this question in a general setting. We also illustrate the usefulness of our results by applying to the solution set of the following Cauchy problem for the fractional differential inclusion with nonlocal condition

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \ t \in [0,T] \ (T > 0), \ 0 < q \leq 1 \\ x(0) + g(x) = x_0, \ \ x_0 \in \mathbb{R}^n \end{cases} \tag{1.1}$$

where ${}^cD^q$ denote the Caputo fractional derivative of order $q$, $g : \mathfrak{C}([0,T], \mathbb{R}^n) \to \mathbb{R}^n$ is continuous, and $F : [0,T] \times \mathbb{R}^n \to P(\mathbb{R}^n)$, where $P(\mathbb{R}^n)$ is the family of all

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nonempty subsets of $\mathbb{R}^n$. For more details on fractional differential equations and inclusions, we refer the reader to ([1, 2, 3, 4, 15]).

## 2. Preliminaries

Let $\mathfrak{C}([0,T],\mathbb{R}^n)$ denote the Banach space of continuous functions from $[0,T]$ into $\mathbb{R}^n$ with the norm $\|x\|_\infty = \sup_{t\in[0,T]} ||x(t)||$. Let $L^1([0,T],\mathbb{R}^n)$ be the Banach space of measureable functions $x : [0,T] \to \mathbb{R}^n$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T ||x(t)||dt$.

Now we recall some basic definitions on multi-valued maps (see [10],[14]).

For a nonempty subset $C$ of a Banach space $X := (X, \|.\|)$, let $P(C) = \{Y \subseteq C : Y \neq \emptyset\}$, $P_{cl}(C) = \{Y \in P(C) : Y \text{ is closed}\}$, $P_b(C) = \{Y \in P(C) : Y \text{ is bounded}\}$, $P_{b,cl}(C) = \{Y \in P(C) : Y \text{ is bounded and closed}\}$, $P_{c,cl}(C) = \{Y \in P(C) : Y \text{is closed and convex}\}$, $P_{b,c,cl}(C) = \{Y \in P(C) : Y \text{ is bounded, closed and convex}\}$, $P_{cp}(C) = \{Y \in P(C) : Y \text{ is compact}\}$, and $P_{c,cp}(C) = \{Y \in P(C) : Y \text{ is compact and convex}\}$. A multi-valued map $F : C \to P(X)$ is convex (resp. closed) valued if $F(x)$ is convex (resp. closed) for all $x \in C$. The map $F$ is bounded on bounded sets if $F(\mathbb{B}) = \cup_{x\in\mathbb{B}} F(x)$ is bounded in $X$ for all $\mathbb{B} \in P_b(C)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{||y|| : y \in F(x)\}\} < \infty$). The map $F$ is called upper semi-continuous (u.s.c.) if $\{x \in C : Fx \subset V\}$ is open in $C$ whenever $V \subset X$ is open. $F$ is called lower semi-continuous (l.s.c.) if the set $\{y \in C : F(y) \cap V \neq \emptyset\}$ is open for any open set $V \subset X$. $F$ is called continuous if it is both l.s.c. and u.s.c. $F$ is said to be completely continuous if $F(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in P_b(C)$. A mapping $f : C \to X$ is called a selection of $F : C \to X$ if $f(x) \in F(x)$ for every $x \in C$. We say that the mapping $F$ has a fixed point if there is $x \in X$ such that $x \in F(x)$. The fixed points set of the multivalued operator $F$ will be denoted by $Fix(F)$. A multivalued map $F : [0,T] \to P_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $y \in \mathbb{R}^n$, the function

$$t \longmapsto d(y, F(t)) = \inf\{||y - z|| : z \in F(t)\}$$

is measurable.

**Definition 2.1.** Let $(X, d)$ be a metric space. Consider $H : P(X) \times P(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

where $d(a, B) = \inf_{b\in B} d(a, b)$. $H$ is the (generalized) Pompeiu-Hausdorff functional. It is known that $(P_{b,cl}(X), H)$ is a metric space and $(P_{cl}(X), H)$ is a generalized metric space (see [7, 14]).

**Definition 2.2.** A multivalued operator $F : X \to P_{cl}(X)$ is called (**a**) $k-$Lipschitz if there exists $k > 0$ such that

$$H(F(x), F(y)) \leq kd(x, y) \text{ for each } x, y \in X;$$

(**b**) a $k-$contraction if there exists $0 < k < 1$ such that

$$H(F(x), F(y)) \leq kd(x, y) \text{ for each } x, y \in X.$$

It is known that $F : X \to P_{cp}(X)$ is continuous on $X$ if and only if $F$ is continuous on $X$ with respect to Hausdorff metric. Also, if $F : X \to P_{b,cl}(X)$ is $k$-Lipschitz, then $F$ is continuous with respect to Hausdorff metric.

**Lemma 2.3** ([8]). *Let $(X, d)$ be a complete metric space. If $F : X \to P_{cl}(X)$ is a $k$-contraction, then $Fix(F) \neq \emptyset$.*

**Definition 2.4** ([17]). Let $X$ be a normed space and $C$ a nonempty subset of $X$. The closure, the convex hull and the closed convex hull of $C$ in $X$ are denoted by $\overline{C}$, $co\, C$ and $\overline{co}C$. Let $\Psi$ be a collection of subsets of $\overline{co}C$ with the property that for any $M \in \Psi$, the sets $\overline{M}$, $coM$, $M \cup \{u\}$ $(u \in C)$ and every subset of $M$ belong to $\Psi$.

Let $A$ be a partially ordered set with the partial ordering $\leq$, and $\varphi : A \to A$ a function. A function $\gamma : \Psi \to A$ is said to be a $\varphi$-measure of noncompactness on $C$ if the following conditions are satisfied for any $M \in \Psi$:

(i) $\gamma(\overline{M}) = \gamma(M)$;
(ii) if $u \in C$, then $\gamma(M \cup \{u\}) = \gamma(M)$;
(iii) if $N \subset M$, then $\gamma(N) \leq \gamma(M)$;
(iv) $\gamma(coM) \leq \varphi(\gamma(M))$.
$\gamma$ is called a measure of noncompactness if instead of (iv), we have the following
(iv)* $\gamma(coM) \leq \gamma(M)$.

Let $\gamma$ be a $\varphi$-measure of noncompactness on $C$. A map $F : C \to P_{c,cl}(X)$ is said to be $(\gamma, \varphi)$-condensing if for every $S \subset C$, the inequality $\gamma(S) \leq \varphi(\gamma(F(S)))$ implies that $F(S)$ is relatively compact. In particular, if $\varphi$ is the identity map, then $F$ is called $\gamma$-condensing. For details, we refer the reader to [12, 13, 17].

Let us recall some definitions on fractional calculus [16, 22, 28].

**Definition 2.5.** For a $n-$times continuously differentiable function $w : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order $q$ is defined as

$$^{c}D^{q}w(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} w^{(n)}(s) ds, \ n-1 < q < n, n = [q]+1, \ q > 0,$$

where $[q]$ denotes the integer part of the real number $q$ and $\Gamma$ denotes the gamma function.

**Definition 2.6.** The Riemann-Liouville fractional integral of order $q$ for a continuous function $w$ is defined as

$$I^{q}w(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{w(s)}{(t-s)^{1-q}} ds, \ q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

**Definition 2.7.** The Riemann-Liouville fractional derivative of order $q$ for a continuous function $w$ is defined by

$$D^{q}w(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^{n} \int_0^t \frac{w(s)}{(t-s)^{q-n+1}} ds, \ n = [q]+1, \ q > 0,$$

provided the right hand side is pointwise defined on $(0, \infty)$.

Let $g : \mathfrak{C}([0,T],\mathbb{R}^n) \to \mathbb{R}^n$ satisfy $||g(x) - g(y)|| \leq \kappa_2||x - y||_\infty$ for all $x, y \in \mathfrak{C}([0,T],\mathbb{R}^n)$ and some $\kappa_2 > 0$. To define the solution of (1.1), we consider the following lemma.

**Lemma 2.8.** *For a given $\sigma \in \mathfrak{C}([0,T],\mathbb{R}^n)$, the unique solution of the problem*

$$\begin{cases} {}^cD^qx(t) = \sigma(t),\ 0 < t < T,\ q \in (0,1), \\ \qquad\qquad x(0) + g(x) = x_0 \end{cases} \tag{2.2}$$

*is given by*

$$x(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds. \tag{2.3}$$

**Definition 2.9.** A function $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,T],\mathbb{R}^n)$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,T]$ and

$$x(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds.$$

Let $S_{x_0}([0,\alpha])$ denote the set of all solutions of (1.1) on the interval $[0,\alpha]$, where $0 < \alpha \leq T$.

## 3. Fixed point theorems

We need the following result in the sequel.

**Lemma 3.1** ([17])**.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Suppose that $\gamma$ is a $\varphi$-measure of noncompactness on $C$ and $F : C \to P_{c,cl}(C)$ is an upper semicontinuous $(\gamma,\varphi)$-condensing map. Then $Fix(F)$ is nonempty.*

The following is the well-known Schauder fixed point theorem [11].

**Theorem 3.2** (Schauder's theorem)**.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Suppose that $f$ be a continuous mapping of $C$ into a compact subset of $C$. Then $Fix(f)$ is nonempty.*

**Theorem 3.3.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Suppose that $\gamma$ is a $\varphi$-measure of noncompactness on $C$ and $F : C \to P_{c,cl}(C)$ is an upper semicontinuous $(\gamma,\varphi)$-condensing map. If $f : C \to C$ is a continuous selection of $F$, then $Fix(f)$ is nonempty.*

*Proof.* By Lemma 3.1, $Fix(F)$ is nonempty. Let $u \in C$ and $\Sigma = \{S \subset C : S = \overline{co}S, u \in S, F(S) \subset S\}$. Then clearly $\Sigma$ is nonempty. Let $W = \cap_{S\in\Sigma}S$. Then $W \in \Sigma$ and so $V := \overline{co}(F(W) \cup \{u\}) \subset W$. Since $F(V) \subset F(W) \subset W$, it follows that $V \in \Sigma$ and so $W \subset V$. As a result, we have $W = \overline{co}(F(W) \cup \{u\})$. Now since $\gamma$ is a $\varphi$-measure of noncompactness on $C$,

$$\gamma(W) \leq \varphi(\gamma(F(W) \cup \{u\})) = \varphi(\gamma(F(W))).$$

This implies that $F(W)$ is relatively compact. But $f(W) \subset F(W)$, thus $f(W)$ is relatively compact. Now Schauder's theorem (Theorem 3.2) guarantees that $Fix(f)$ is nonempty. □

It is well-known that if $X$ is a Banach space, $C \subset X$ is closed and $F : C \to P_{cp}(C)$ is a $k$-contraction, then $F$ is condensing with respect to the Hausdorff measure of noncompactness (see [5]). Recall that the Hausdorff measure of noncompactness $\chi$ of a bounded set $A$ is defined by

$$\chi(A) = \inf\{d > 0 : \ A \text{ can be covered by finitely many balls of radii } < d\}.$$

**Lemma 3.4** ([14], pp.101)**.** *Let $C$ be a metric space. Suppose that $F : C \to P_{cp}(\mathbb{R}^n)$ is a Lipschitz map. Then $F$ admits a Lipschitz selection.*

**Corollary 3.5.** *Suppose that $F : \mathbb{R}^n \to P_{cp}(\mathbb{R}^n)$ is a $k$-contraction. Then there exists a Lipschitz selection $f$ of $F$ with $Fix(f)$ is nonempty.*

*Proof.* Since $F$ is Lipschitz, by Lemma 3.4 $F$ admits a Lipschitz selection $f : \mathbb{R}^n \to \mathbb{R}^n$. Since a condensing map with respect to the Hausdorff measure of noncompactness is $(\gamma, \varphi)$-condensing, the result follows from Theorem 3.3. □

We have the following result of Dzedzej and Gelman [10] as a corollary.

**Corollary 3.6.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Suppose that $F : C \to P_{cp}(C)$ is a $k$-contraction. If $f : C \to C$ is a continuous selection of $F$, then $Fix(f)$ is nonempty.*

*Proof.* Since a condensing map with respect to the Hausdorff measure of noncompactness is $(\gamma, \varphi)$-condensing, the result follows from Theorem 3.3. □

We shall need the following result in the sequel.

**Theorem 3.7** (Michael's selection theorem)**.** [19] *Let $C$ be a metric space, $X$ a Banach space and $F : C \to P_{c,cl}(C)$ a lower semicontinuous map. Then there exists a continuous selection $f : C \to X$ of $F$.*

In the next result, we shall make use of the following lemma due to Saint Raymond [26].

**Lemma 3.8.** *Let $K$ be a compact metric space with $\dim K < n$, $X$ a Banach space and $F : K \to P_{b,c,cl}(X)$ a lower semicontinuous map such that $0 \in F(x)$ and $dimF(x) \geq n$ for every $x \in K$. Then there exists a continuous selection $f$ of $F$ such that $f(x) \neq 0$ for each $x \in K$.*

**Theorem 3.9.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Suppose that $\gamma$ is a $\varphi$-measure of noncompactness on $C$ and $F : C \to P_{b,c,cl}(C)$ is a continuous $(\gamma, \varphi)$-condensing map. If $dimF(x) \geq n$ for each $x \in C$, then $dimFix(F) \geq n$.*

*Proof.* By Lemma 3.1, $Fix(F)$ is nonempty. Since $Fix(F) \subset coF(Fix(F))$ and $\gamma$ is a $\varphi$-measure of noncompactness on $C$, we have

$$\gamma(Fix(F)) \leq \gamma(coF(Fix(F))) \leq \varphi(\gamma(F(Fix(F)))).$$

This implies that $F(Fix(F))$ is relatively compact. As a result, $Fix(F)$ is compact. Consider a map $I - F : Fix(F) \to P_{b,c,cl}(C)$, where $I$ is the identity operator. Assume that $\dim Fix(F) < n$. Then, by Lemma 3.8, there is a continuous selection $g$ of $I - F$ such that $g(x) \neq 0$ for each $x \in Fix(F)$. This implies that there exists a continuous selection $h$ of $F : Fix(F) \to P_{b,c,cl}(C)$ without fixed points. Michael's selection theorem (Theorem 3.7) guarantees that $h$ admits an extension to continuous selection $f : C \to C$ of $F$ with no fixed points, which contradicts Theorem 3.3. □

As a corollary, we immediately obtain the following result of Dzedzej and Gelman [10].

**Corollary 3.10.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$. Suppose that $F : C \to P_{c,cp}(C)$ is a $k$-contraction. If $dimF(x) \geq n$ for each $x \in C$, then $dimFix(F) \geq n$.*

## 4. A Cauchy problem for some fractional differential inclusion

**Lemma 4.1.** *Assume that*

- $(\mathbf{K_1})$ *$F : [0,T] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ is such that $F(.,x) : [0,T] \to P_{c,cp}(\mathbb{R}^n)$ is measurable for each $x \in \mathbb{R}^n$;*
- $(\mathbf{K_2})$ *$H(F(t,x),F(t,\bar{x})) \leq \kappa_1(t)\|x-\bar{x}\|$ for almost all $t \in [0,T]$ and $x, \bar{x} \in \mathbb{R}^n$ with $\kappa_1 \in L^1([0,T],\mathbb{R}_+)$ and $||F(t,x)|| = \sup\{||v|| : v \in F(t,x)\} \leq \kappa_1(t)$ for almost all $t \in [0,T]$ and $x \in \mathbb{R}^n$;*
- $(\mathbf{K_3})$ *$g : \mathfrak{C}([0,T],\mathbb{R}^n) \to \mathbb{R}^n$ is continuous and $||g(x)-g(y)|| \leq \kappa_2||x-y||_\infty$ for all $x, y \in \mathfrak{C}([0,T],\mathbb{R}^n)$ and some $\kappa_2 > 0$.*

*Then the Cauchy problem (1.1) has at least one solution on $[0,T]$ if $\kappa_2 + \frac{T^{q-1}}{\Gamma(q)}\|\kappa_1\|_{L^1} < 1$.*

*Proof.* For each $y \in \mathfrak{C}([0,T],\mathbb{R}^n)$, define the set of selections of $F$ by

$$S_{F,y} := \{v \in L^1([0,T],\mathbb{R}^n) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [0,T]\}.$$

Observe that by the assumptions $(K_1)$ and $(K_2)$, $F(.,x(.))$ is measurable and has a measureable selection $v(.)$ (see Theorem III.6 [7]). Also $\kappa_1 \in L^1([0,T],\mathbb{R}_+)$ and

$$||v(t)|| \quad \leq \quad ||F(t,x(t))|| \leq \kappa_1(t).$$

Thus the set $S_{F,x}$ is nonempty for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$. Now we show that the operator $\Omega$ defined by

$$\Omega(x) = \{h \in \mathfrak{C}([0,T],\mathbb{R}^n) : h(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, f \in S_{F,x}\}$$

satisfies the assumptions of Lemma 2.3. To show that $\Omega(x) \in P_{cl}(\mathfrak{C}([0,T],\mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$, let $\{u_n\}_{n\geq 0} \in \Omega(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Then $u \in \mathfrak{C}([0,T],\mathbb{R}^n)$ and there exists $v_n \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$u_n(t) \quad = \quad x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_n(s)ds.$$

As $F$ has compact values, we pass to a subsequence to obtain that $v_n$ converges to $v$ in $L^1([0,T],\mathbb{R}^n)$. Thus, $v \in S_{F,x}$ and for each $t \in [0,T]$,

$$u_n(t) \to u(t) \quad = \quad x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v(s)ds.$$

Hence $u \in \Omega(x)$.
Next we show that

$$H(\Omega(x), \Omega(\bar{x})) \leq k\|x - \bar{x}\|_\infty \text{ for each } x, \bar{x} \in \mathfrak{C}([0,T], \mathbb{R}^n).$$

Let $x, \bar{x} \in \mathfrak{C}([0,T], \mathbb{R}^n)$ and $h_1 \in \Omega(x)$. Then there exists $v_1(t) \in S_{F,x}$ such that, for each $t \in [0,T]$,

$$h_1(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_1(s)ds.$$

By $(K_2)$, we have

$$H(F(t,x), F(t,\bar{x})) \leq \kappa_1(t)||x(t) - \bar{x}(t)||.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$||v_1(t) - w|| \leq \kappa_1(t)\|x(t) - \bar{x}(t)\|, \ t \in [0,T].$$

Define $V : [0,T] \to P(\mathbb{R}^n)$ by

$$V(t) = \{w \in \mathbb{R}^n : \|v_1(t) - w\| \leq \kappa_1(t)\|x(t) - \bar{x}(t)\|\}.$$

Since the nonempty closed valued operator $V(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [7]), there exists a function $v_2(t)$ which is a measurable selection for $V(t) \cap F(t, \bar{x}(t))$. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [0,T]$, we have $\|v_1(t) - v_2(t)\| \leq \kappa_1(t)\|x(t) - \bar{x}(t)\|$.
For each $t \in [0,T]$, let us define

$$h_2(t) = x_0 - g(\bar{x}) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} v_2(s)ds.$$

Thus

$$\|h_1(t) - h_2(t)\| \leq ||g(x) - g(\bar{x})|| + \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)} ||v_1(s) - v_2(s)||ds.$$

Hence

$$\begin{aligned} \|h_1 - h_2\|_\infty &\leq \kappa_2||x - \bar{x}||_\infty + \big(\frac{T^{q-1}}{\Gamma(q)}\big)||\kappa_1||_{L^1}\|x - \overline{x}\|_\infty \\ &= \big(\kappa_2 + \frac{T^{q-1}}{\Gamma(q)}||\kappa_1||_{L^1}\big)\|x - \overline{x}\|_\infty. \end{aligned}$$

Analogously, interchanging the roles of $x$ and $\overline{x}$, we obtain

$$H(\Omega(x), \Omega(\bar{x})) \leq k\|x - \bar{x}\|_\infty \text{ for each } x, \bar{x} \in \mathfrak{C}([0,T], \mathbb{R}^n),$$

where $k = \big(\kappa_2 + \frac{T^{q-1}}{\Gamma(q)}||\kappa_1||_{L^1}\big) < 1$. Since $\Omega$ is a contraction, it follows by Lemma 2.3 that $\Omega$ has a fixed point $x$ which is a solution of (1.1). This completes the proof. □

**Lemma 4.2.** *Let $F : [0,T] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfy $(K_1)$, $(K_2)$ and $(K_3)$ and suppose that $\Omega : \mathfrak{C}([0,T], \mathbb{R}^n) \to P(\mathfrak{C}([0,T], \mathbb{R}^n))$ is defined by*

$$\Omega(x) = \{h \in \mathfrak{C}([0,T], \mathbb{R}^n) : h(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, f \in S_{F,x}\}.$$

*Then $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0,T], \mathbb{R}^n))$ for each $x \in \mathfrak{C}(([0,T], \mathbb{R}^n))$.*

*Proof.* First we show that $\Omega(x)$ is convex for each $x \in \mathfrak{C}([0,T],\mathbb{R}^n)$. For that, let $h_1, h_2 \in \Omega(x)$. Then there exist $f_1, f_2 \in S_{F,x}$ such that for each $t \in [0,T]$, we have

$$h_i(t) \;=\; x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f_i(s)ds, \; i = 1,2.$$

Let $0 \leq \lambda \leq 1$. Then, for each $t \in [0,T]$, we have

$$\begin{aligned}&[\lambda h_1 + (1-\lambda)h_2](t)\\ &= x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}[\lambda f_1(s) + (1-\lambda)f_2(s)]ds.\end{aligned}$$

Since $S_{F,x}$ is convex ($F$ has convex values), therefore it follows that $\lambda h_1 + (1-\lambda)h_2 \in \Omega(x)$.

Next, we show that $\Omega$ maps bounded sets into bounded sets in $\mathfrak{C}([0,T],\mathbb{R}^n)$. For a positive number $r$, let $B_r = \{x \in \mathfrak{C}([0,T],\mathbb{R}^n) : \|x\|_\infty \leq r\}$ be a bounded set in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Then, for each $h \in \Omega(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) \;=\; x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds$$

and in view of $(H_1)$, we have

$$\begin{aligned}||h(t)|| &\leq ||x_0|| + \sup_{x\in B_r}||g(x)|| + \int_0^t \frac{|t-s|^{q-1}}{\Gamma(q)}||f(s)||ds\\ &\leq ||x_0|| + \sup_{x\in B_r}||g(x)|| + \frac{T^{q-1}}{\Gamma(q)}\int_0^T \kappa_1(s)ds.\end{aligned}$$

Thus,

$$\|h\|_\infty \;\leq\; ||x_0|| + \sup_{x\in B_r}||g(x)|| + \frac{T^{q-1}}{\Gamma(q)}||\kappa_1||_{L^1}.$$

Now we show that $\Omega$ maps bounded sets into equicontinuous sets in $\mathfrak{C}([0,T],\mathbb{R}^n)$. Let $t', t'' \in [0,T]$ with $t' < t''$ and $x \in B_r$, where $B_r$ is a bounded set in $\mathfrak{C}([0,T],\mathbb{R}^n)$. For each $h \in \Omega(x)$, we obtain

$$\begin{aligned}||h(t'') - h(t')|| &= \Big|\Big|\int_0^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds - \int_0^{t'}\frac{(t'-s)^{q-1}}{\Gamma(q)}f(s)ds\Big|\Big|\\ &\leq \Big|\Big|\int_0^{t'}\frac{[(t''-s)^{q-1} - (t'-s)^{q-1}]}{\Gamma(q)}f(s)ds\Big|\Big|\\ &\quad + \Big|\Big|\int_{t'}^{t''}\frac{(t''-s)^{q-1}}{\Gamma(q)}f(s)ds\Big|\Big|.\end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_{r'}$ as $t'' - t' \to 0$.. By the Arzela-Ascoli Theorem, $\Omega : \mathfrak{C}([0,T],\mathbb{R}^n) \to P(\mathfrak{C}([0,T],\mathbb{R}^n))$ is completely continuous. As in Lemma 4.1, $\Omega$ is closed-valued. Consequently, $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0,T],\mathbb{R}^n))$ for each $x \in \mathfrak{C}(([0,T],\mathbb{R}^n))$. □

For $0 < \alpha \leq T$, let us consider the operator

$$\Omega(x) = \{h \in \mathfrak{C}([0,\alpha],\mathbb{R}^n) : h(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, f \in S_{F,x}\}.$$

It is well-known that $Fix(\Omega) = S_{x_0}([0,\alpha])$ and, in view of Lemma 4.1, it is nonempty for each $0 < \alpha \leq T$.

The following lemma due to Dzedzej and Gelman [10] is useful in the sequel.

**Lemma 4.3.** *Let $F : [0,\alpha] \to P_{c,cp}(\mathbb{R}^n)$ be a measurable map such that the Lebesgue measure $\mu$ of the set $\{t : dim\ F(t) < 1\}$ is zero. Then there are arbitrarily many linearly independent measurable selections $x_1(.), x_2(.), ..., x_m(.)$ of $F$.*

**Theorem 4.4.** *Let $F : [0,\alpha] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfy $(K_1)$, $(K_2)$ and $(K_3)$ and suppose that the Lebesgue measure $\mu$ of the set$\{t : dim\ F(t,x) < 1$ for some $x \in \mathbb{R}^n\}$ is zero. Then for each $\alpha$, $0 < \alpha < \min\{(\frac{(1-\kappa_2)\Gamma(q)}{\|\kappa_1\|_{L^1}})^{\frac{1}{q-1}}, T\}$, the set $S_{x_0}([0,\alpha])$ of solutions of (1.1) has an infinite dimension for any $x_0$.*

*Proof.* Define the operator $\Omega$ by

$$\Omega(x) = \{h \in \mathfrak{C}([0,\alpha],\mathbb{R}^n) : h(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds, f \in S_{F,x}\}.$$

Then by Lemma 4.2, $\Omega(x) \in P_{c,cp}(\mathfrak{C}([0,\alpha],\mathbb{R}^n))$ for each $x \in \mathfrak{C}([0,\alpha],\mathbb{R}^n)$ and as in the proof of Lemma 4.1, it is a contraction if $\kappa_2 + \frac{\alpha^{q-1}}{\Gamma(q)}||\kappa_1||_{L^1} < 1$ or $\alpha < (\frac{(1-\kappa_2)\Gamma(q)}{\|\kappa_1\|_{L^1}})^{\frac{1}{q-1}}$. We shall show that $\dim\Omega(x) \geq m$ for any $x \in \mathfrak{C}([0,\alpha],\mathbb{R}^n)$ and arbitrary $m \in \mathbb{N}$. Consider $G(t) = F(t,x(t))$. By Lemma 4.3, there exist linearly independent measurable selections $x_1(.), x_2(.), ..., x_m(.)$ of $G$. Set $y_i(t) = x_0 - g(x) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} x_i(s)ds \in \Omega(x)$. Assume that $\sum_{i=1}^m a_i y_i(t) = 0$ a.e. in $[0,\alpha]$. Taking Caputo derivatives a.e. in $[0,\alpha]$, we have $\sum_{i=1}^m a_i x_i(t) = 0$ a.e. in $[0,\alpha]$ and hence $a_i = 0$ for all $i$. As a result, $y_i(.)$ are linearly independent. Thus $\Omega(x)$ contains an $m$-dimensional simplex. So $\dim\Omega(x) \geq m$. By Theorem 3.9, $Fix(\Omega) = S_{x_0}([0,\alpha])$ is infinite dimensional. □

A metric space $X$ is an AR-space if, whenever it is nonempty closed subset of another metric space $Y$, then there exists a continuous retraction $r : Y \to X$, $r(x) = x$ for $x \in X$. In particular, it is contractible (and hence connected).

**Theorem 4.5** ([23])**.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and $F : C \to P_{c,cp}(C)$ a contraction. Then $Fix(F)$ is a nonempty AR-space.*

**Remark 4.6.** Under the assumption of Theorem 4.5, $Fix(F)$ is also compact. This follows as in the proof of Theorem 3.9 since $\Omega$ is condensing with respect to Hausdorff measure of noncompactness.

The following result follows from Theorem 4.4 and Theorem 4.5.

**Corollary 4.7.** *Let $F : [0,\alpha] \times \mathbb{R}^n \to P_{c,cp}(\mathbb{R}^n)$ satisfy $(K_1)$, $(K_2)$ and $(K_3)$ and suppose that the Lebesgue measure $\mu$ of the set$\{t : dim\ F(t,x) < 1$ for some $x \in \mathbb{R}^n\}$ is zero. Then for each $\alpha$, $0 < \alpha < \min\{(\frac{(1-\kappa_2)\Gamma(q)}{\|\kappa_1\|_{L^1}})^{\frac{1}{q-1}}, T\}$, the set $S_{x_0}([0,\alpha])$ of solutions of (1.1) is a compact and infinite dimensional AR-space.*

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