Journal of Nonlinear and Convex Analysis
Volume 14, Number 2, 2013, 301–317

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL


# QUASI INTERIOR-TYPE OPTIMALITY CONDITIONS IN SET-VALUED DUALITY

ANCA GRAD

Abstract. This article addresses a new approach to duality in set-valued optimization, by means of solutions defined with the help of the nonempty quasi interior of a convex cone, employing a set-type criterion. We define and characterize a qi-conjugate function associated with a set-valued function, and a qi-subdifferential, in analogy to the conjugate function and subdifferential from scalar optimization. Weak duality theorems and theorems containing optimality conditions are proved for general unconstrained set-valued optimization problems, in connection to a newly proposed dual problem. In the particular case when the perturbation function extends the Lagrange perturbation from scalar optimization we prove a strong duality theorem for constrained set-valued optimization problems. An application of our strong set-valued Lagrange duality theorem by means of qi-efficiency, stated in $\ell^2(\mathbb{R})$, ends the article.

## 1. Introduction

Let $X$ be a nonempty set, let $Y$ be a topological vector space partially ordered by a pointed, convex cone $K \subset Y$, and let $F : X \to \mathcal{P}(Y)$ be a set-valued function. Recall that $\mathcal{P}(Y) := \{A : A \subseteq Y\}$. Generally speaking, given the set-valued optimization problem

$$(P_0^{sv}) \qquad \min_{x \in X} F(x),$$

the so-far published results concerning $(P_0^{sv})$ can be split into two categories: an approach using the so-called **vector criterion**, and another one using the so-called **set criterion**.

Several authors tackled the set-valued optimization theory from perspectives involving the extension of results concerning vector-valued functions. The vector criterion employs the determination of vector-like efficient points ((Pareto)-efficient, weakly-efficient, strongly-efficient, etc.) within the entire set $F(X) = \cup_{x \in X} F(x)$. This means that $\overline{x} \in X$ is an efficient solution to $(P_0^{sv})$ if there exists an $\overline{y} \in F(\overline{x})$ such that $\overline{y}$ is an efficient element to the set $F(X)$. Such an approach can be found in H. W. Corley [11], D. T. Luc[26], T. Tanino [31], T. Tanino and Y. Sawaragi [32], W. Song [28], [29], [30]. In Chapter 7 of R. I. Boţ, S. M. Grad and G. Wanka [9] is presented a survey on set-valued duality using different vector-type extended notions.

---

2010 *Mathematics Subject Classification.* 90C46, 49N15, 90C48, 49M29.

*Key words and phrases.* Set-valued optimization, duality, set criterion, quasi interior, optimality conditions.

This article was supported by the Exploratory Research Project PN2-IDEI-PCE-2008-2, Code CNCSIS 2261, Contract No. 543.

The set criterion involves a direct comparison of the sets $F(x)$, for all $x \in X$, and it is based on an ordering relation on $\mathcal{P}(Y)$ rather than on an order on $Y$, case encountered when working with a vector criterion. This is the reason why, at times, the set criterion is also called the **natural criterion** in set-valued optimization. Connected to it, T. KUROIWA published a series of articles [18], [19], [20], [21], [22], [23]. We also mention KUROIWA D., TANAKA T. and TRUONG X. D. H. [24]. KUROIWA's approach was expanded in several recent papers written by E. HERNÁNDEZ and L. RODRÍGUEZ-MARIN [15], [16], [17], and M. ALONSO and L. RODRÍGUEZ-MARIN [1].

In this article we propose a new duality perspective for set-valued optimization problems, by employing efficient solutions defined with the help of the quasi interior of a convex cone.

Section 2, entitled Preliminaries, familiarizes the reader with the notions and results taken from the specialized literature. We present definitions and characterization properties for the quasi relative interior and quasi interior of a set. As well, we mention some separation theorems that use these notions.

Section 3 is centered on two new set-relations defined with the help of the quasi interior. We prove some of their properties, and emphasize the connection to other set relations previously introduced in the literature. Moreover, we present four new set-efficiency notions.

Section 4 contains the definition and some characterization properties for a qi-conjugate function associated with a set-valued function, and a qi-subdifferential, in analogy to the conjugate functions and subdifferential from scalar optimization.

Using a general perturbation approach, we construct in Section 5 a new set-valued duality theory, employing qi-efficient solutions. For the particular case when the perturbation function extends the Lagrange perturbation from scalar optimization, we prove a strong duality theorem. Our results are more general than those established by E. HERNÁNDEZ and L. RODRÍGUEZ-MARIN [16] for weak efficieny, since the quasi interior of a set is a more general notion than the interior. It is important to notice that in the particular case of a scalar optimization problem with vector constraints, our set-valued qi-efficiency conditions collapse into the classical ones, for example those in R. I. BOŢ, E. R. CSETNEK and A. MOLDOVAN [7].

The article ends with a section devoted to an application of our set-valued Lagrange duality theorem by means of qi-efficiency, application stated in $\ell^2(\mathbb{R})$.

## 2. PRELIMINARIES

Suppose that $X$ is a topological vector space, and let $X^*$ be the topological dual space of $X$. Given a linear continuous functional $x^* \in X^*$ and a point $x \in X$, we denote by $\langle x^*, x\rangle$ the value of $x^*$ at $x$.

The **normal cone** associated with a set $M \subseteq X$ is defined by

$$N_M(x) := \begin{cases} \{x^* \in X^* : \langle x^*, y-x\rangle \leq 0 \text{ for all } y \in M\} & \text{if } x \in M \\ \emptyset & \text{otherwise.} \end{cases}$$

Given a nonempty cone $C \subseteq X$ (all cones considered in this article are assumed to contain 0), its **dual cone** is the set

$$C^+ := \{x^* \in X^* : \langle x^*, x\rangle \geq 0 \text{ for all } x \in C\}.$$

Consider now a separated topological vector space $X$, and let $M \subseteq X$ be a set. The **quasi-relative interior** of $M$ is

$$\operatorname{qri} M := \big\{x \in M : \operatorname{cl}\operatorname{cone}(M - x) \text{ is a linear subspace of } X\big\}$$

(see J. M. Borwein and A. S. Lewis [3]).

The **quasi interior** of a set $M \subseteq X$ is tightly connected to the quasi-relative interior and appeared in the literature prior to it. It is defined by

$$\operatorname{qi} M := \big\{x \in M : \operatorname{cl}\operatorname{cone}(M - x) = X\big\}.$$

When $M$ is a convex set, then $\operatorname{qi} M \subseteq \operatorname{qri} M$ and $\operatorname{qri}\{x\} = \{x\}$ for all $x \in X$. Moreover, whenever $\operatorname{qi} M \neq \emptyset$, then $\operatorname{qi} M = \operatorname{qri} M$.

When $X$ is a separated locally convex space, and $M \subseteq X$ is a convex set, then the following chain of inclusions hold:

$$\operatorname{int} M \subseteq \operatorname{core} M \subseteq \operatorname{qi} M \subseteq \operatorname{qri} M \subseteq M. \tag{2.1}$$

When $\operatorname{int} M \neq \emptyset$, then all the generalized interior notions in (2.1) collapse in equality to $\operatorname{int} M$, i.e. the topological interior of $M$.

Let us consider the case when $X = \mathbb{R}^n$, with $n \in \mathbb{N}$, and let $M \subseteq \mathbb{R}^n$ be a convex set. Then the following chain of equalities is valid:

$$\operatorname{core} M = \operatorname{qi} M = \operatorname{int} M \tag{2.2}$$

(according to M.A. Limber and R.K. Goodrich [25], R. T. Rockafellar [27]).

For further details and connections among generalized interiors we refer the reader to J. M. Borwein and R. Goebel [2], J. M. Borwein and A. S. Lewis [3], R. I. Boţ, E. R. Csetnek and G. Wanka [8], R. I. Boţ [5], E. R. Csetnek [12], and R. I. Boţ and E. R. Csetnek [6].

We continue by presenting some characterizations of the quasi interior and quasi-relative interior of convex sets in separated locally convex spaces.

**Theorem 2.1** (J. M. Borwein, A. S. Lewis[3])**.** *Let $M$ be a convex subset of a separated locally convex space $X$, and let $x \in M$. Then $x \in \operatorname{qri} M$ if and only if $N_M(x)$ is a linear subspace of $X^*$.*

The following characterization of the quasi interior of a convex set can be found in P. Daniele, S. Giuffré, G. Idone and A. Maugeri [13], where it was stated in reflexive Banach spaces. Nevertheless, it can be extended to separated locally convex spaces, as mentioned and proved by R. I. Boţ, E. R. Csetnek and G. Wanka [8].

**Theorem 2.2.** *Let $M$ be a convex subset of a separated locally convex space $X$, and let $x \in M$. Then $x \in \operatorname{qi} M$ if and only if $N_M(x) = \{0\}$.*

Let $X$ be a separated locally convex space, and let $C \subseteq X$ be a convex cone. Then the equality

$$\operatorname{qi} C + C = \operatorname{qi} C \tag{2.3}$$

holds. This is due to E. R. Csetnek [12, Proposition 2.3 (v)], and F. Cammaroto and B. Di Bella [10, Proposition 1.12].

**Proposition 2.3.** *Let $C$ be a nonempty convex cone of a separated locally convex space $X$. Then, for all $x^* \in C^+\setminus\{0\}$ and for all $x \in \operatorname{qi} C$, the following inequality holds:*

$$\langle x^*, x\rangle > 0. \tag{2.4}$$

*Proof.* The proof can approached similarly to R. I. Boţ, S. M. Grad and G Wanka. [9, Proposition 2.1.1]. □

What follows is a separation theorem.

**Theorem 2.4** (R. I. Boţ, E. R. Csetnek, G. Wanka [8] )**.** *Let $M$ be a nonempty convex subset of a separated locally convex space $X$, and let $\overline{x} \in M$. If $\overline{x} \notin \operatorname{qri} M$, then there exists an $x^* \in X^*\setminus\{0\}$ such that*

$$\langle x^*, \overline{x}\rangle \leq \langle x^*, x\rangle \text{ for all } x \in M. \tag{2.5}$$

*Viceversa, if $0 \in \operatorname{qi}(M - M)$ and there exists an $x^* \in X^*\setminus\{0\}$ satisfying (2.5), then $\overline{x} \notin \operatorname{qri} M$.*

## 3. Two new set relations defined by means of the quasi interior

We consider the following framework throughout the current section:

$$\begin{cases} Y \text{ is a separated locally convex space;} \\ K \subset Y \text{ is a pointed, convex cone with } \operatorname{qi} K \neq \emptyset. \end{cases} \tag{3.1}$$

Let us recall that

$$\mathcal{P}_0(Y) := \{A : A \subseteq Y \text{ and } A \neq \emptyset\}.$$

We start by presenting some set relations defined with the help of a convex cone, introduced by D. Kuroiwa [18].

**Definition 3.1** (D. Kuroiwa [18])**.** *Let $A$ and $B$ belong to $\mathcal{P}_0(Y)$. Then we write:*

(a) *$A \leq^l B$ if $B \subseteq A + K$;*
(b) *$A \leq^u B$ if $A \subseteq B - K$;*
(c) *$A \sim^l B$ if $A \leq^l B$ and $B \leq^l A$.*

D. Kuroiwa [18] proved that $\sim^l$ is an equivalence relation on $\mathcal{P}_0(Y)$. □

**Definition 3.2.** *Let $A$ and $B$ belong to $\mathcal{P}_0(Y)$. Then we write:*

(a) *$A \trianglelefteq^l_{\operatorname{qi} K} B$ if $B \subseteq A + \operatorname{qi} K$;*
(b) *$A \trianglelefteq^u_{\operatorname{qi} K} B$ if $A \subseteq B - \operatorname{qi} K$.*

The definitions above justify the choice made in (3.1) for $K \neq Y$. If $K = Y$ the new two set relations $\trianglelefteq^l_{\operatorname{qi} K}$ and $\trianglelefteq^u_{\operatorname{qi} K}$ would be useless.

The relations $\trianglelefteq^l_{\operatorname{qi} K}$ and $\trianglelefteq^u_{\operatorname{qi} K}$ are transitive.

**Proposition 3.3.** *Let $A$ and $B$ belong to $\mathcal{P}_0(Y)$. Then the following statements are true:*

(a) *$A \sim^l B$ if and only if $A + K = B + K$.*
(b) *If $A \sim^l B$, then $A + \operatorname{qi} K = B + \operatorname{qi} K$.*
(c) *If $A \trianglelefteq^l_{\operatorname{qi} K} B$ and $B \trianglelefteq^l_{\operatorname{qi} K} A$, then $A \sim^l B$.*
(d) *If $A \trianglelefteq^l_{\operatorname{qi} K} B$ and $B \leq^l A$, then $B \trianglelefteq^l_{\operatorname{qi} K} A$.*

(e) *$A \trianglelefteq^l_{\mathrm{qi}\,K} B$ if and only if $-B \trianglelefteq^u_{\mathrm{qi}\,K} -A$.*
(f) *If $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ and $y \in Y$, then $A + y \trianglelefteq^l_{\mathrm{qi}\,K} B + y$.*
(g) *If $A \sim^l B$ and $y \in Y$, then $A + y \sim^l B + y$.*

*Proof.* (a) See Lemma 2.1 (i) in [16].

(b) We make use of (a). Thus, on one hand $A \sim^l B$ implies that $A+K = B+K$, hence $A + K + \mathrm{qi}\,K = B + K + \mathrm{qi}\,K$. On the other hand, we know from (2.3) that $\mathrm{qi}\,K + K = \mathrm{qi}\,K$, therefore $A + \mathrm{qi}\,K = B + \mathrm{qi}\,K$.

(c) From $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ it follows that $B \subseteq A + \mathrm{qi}\,K \subseteq A + K$, i.e $A \leq^l B$. Similarly, from $B \trianglelefteq^l_{\mathrm{qi}\,K} A$ we obtain that $A \subseteq B + \mathrm{qi}\,K \subseteq B + K$, i.e. $B \leq^l A$. Consequently, we have $A \sim^l B$.

(d) From $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ we have $B \subseteq A + \mathrm{qi}\,K$, and from $B \leq^l A$, we have $A \subseteq B + K$. Then, considering (2.3) we deduce the following chain of inclusions:

$$A \subseteq B + K \subseteq A + \mathrm{qi}\,K + K = A + \mathrm{qi}\,K \subseteq B + K + \mathrm{qi}\,K = B + \mathrm{qi}\,K.$$

This means that $B \trianglelefteq^l_{\mathrm{qi}\,K} A$.

(e) Obviously we have

$$B \subseteq A + \mathrm{qi}\,K \text{ if and only if } -B \subseteq -A - \mathrm{qi}\,K.$$

This means that $A \trianglelefteq^l_{\mathrm{qi}\,K} B$ is equivalent to $-B \trianglelefteq^u_{\mathrm{qi}\,K} -A$.

(f) and (g) have straightforward proofs starting from the definitions of the relations $\trianglelefteq^l_{\mathrm{qi}\,K}$ and $\sim^l$. □

**Remark 3.4.** From the statements (b) and (c) in Proposition 3.3 we obtain

$$A \trianglelefteq^l_{\mathrm{qi}\,K} B \text{ and } B \trianglelefteq^l_{\mathrm{qi}\,K} A \Longrightarrow A \sim^l B \Longrightarrow A + \mathrm{qi}\,K = B + \mathrm{qi}\,K.$$

This chain of implications cannot be reversed, as it is explained in the following.

(a) The converse of statement (b) in Proposition 3.3 does not hold, i.e.

$$A + \mathrm{qi}\,K = B + \mathrm{qi}\,K \not\Longrightarrow A \sim^l B. \tag{3.2}$$

Take for instance $Y := \mathbb{R}$, $K := \mathbb{R}_+$, $A := \{-1\}$ and $B := (-1, 1)$. As well, one reaches the same conclusion, by taking $A := K$ and $B := \mathrm{qi}\,K$, if it is considered a cone $K$ such that $\mathrm{qi}\,K \neq K$.

(b) The converse of statement (c) in Proposition 3.3 does not hold, i.e.

$$A \sim^l B \not\Longrightarrow A \trianglelefteq^l_{\mathrm{qi}\,K} B \text{ and } B \trianglelefteq^l_{\mathrm{qi}\,K} A. \tag{3.3}$$

Choose for instance $Y := \mathbb{R}$, $K := \mathbb{R}_+$, $A := \{-1\}$ and $B := [-1, 0)$. As well, one reaches the same conclusion, by taking $A := K$ and $B := \{0\}$, if it is considered a cone $K$ such that $\mathrm{qi}\,K \neq K$. □

With the help of the relations introduced in Definition 3.2 we define four new efficiency notions for sets.

**Definition 3.5.** *Let us consider a set $\mathcal{S} \subseteq \mathcal{P}_0(Y)$. A set $A \in \mathcal{S}$ is said to be:*

(a) *an* l-Min$_{\mathrm{qi}}$**-efficient set** *of $\mathcal{S}$, if for each set $B \in \mathcal{S}$ satisfying*

$$B \trianglelefteq^l_{\mathrm{qi}\,K} A, \text{ the relation } A \trianglelefteq^l_{\mathrm{qi}\,K} B \text{ holds.}$$

(b) *an* l-Max$_{\text{qi}}$-**efficient set** *of* $\mathcal{S}$*, if for each set* $B \in \mathcal{S}$ *satisfying*

$$A \trianglelefteq^{l}_{\text{qi}\,K} B, \text{ the relation } B \trianglelefteq^{l}_{\text{qi}\,K} A \text{ holds.}$$

(c) *an* u-Min$_{\text{qi}}$-**efficient set** *of* $\mathcal{S}$*, if for each set* $B \in \mathcal{S}$ *satisfying*

$$B \trianglelefteq^{u}_{\text{qi}\,K} A, \text{ the relation } A \trianglelefteq^{u}_{\text{qi}\,K} B \text{ holds.}$$

(d) *an* u-Max$_{\text{qi}}$-**efficient set** *of* $\mathcal{S}$*, if for each set* $B \in \mathcal{S}$ *satisfying*

$$A \trianglelefteq^{u}_{\text{qi}\,K} B, \text{ the relation } B \trianglelefteq^{u}_{\text{qi}\,K} A \text{ holds.}$$

The sets of all l-Min$_{\text{qi}}$-efficient, l-Max$_{\text{qi}}$-efficient, u-Min$_{\text{qi}}$-efficient and u-Max$_{\text{qi}}$-efficient sets of $\mathcal{S}$ are denoted by

$$\text{l-Min}_{\text{qi}}\,\mathcal{S}, \text{l-Max}_{\text{qi}}\,\mathcal{S}, \text{u-Min}_{\text{qi}}\,\mathcal{S} \text{ and u-Max}_{\text{qi}}\,\mathcal{S},$$

respectively.

**Remark 3.6.** E. Hernández and L. Rodríguez-Marin [15], [16] considered some weak set-efficiency notions defined with $\leq^l$ when dealing with convex cones with nonempty topological interior. Our notions are more general and can be used even in the case when the topological interior of the cones is empty. □

**Proposition 3.7.** *Let us consider a set* $\mathcal{S} \subseteq \mathcal{P}_0(Y)$*. Then the following equality holds:*

$$\text{l-Min}_{\text{qi}}(-\mathcal{S}) = -\,\text{u-Max}_{\text{qi}}\,\mathcal{S}, \tag{3.4}$$

*where* $-\mathcal{S} = \{-A : A \in \mathcal{S}\}$.

The proof is straightforward.

## 4. qi-conjugate functions and qi-subgradients

We consider the following hypotheses as valid throughout this section:

$$\left\{ \begin{array}{l} X \text{ is a topological vector space;} \\ Y \text{ is a separated locally convex space;} \\ K \subset Y \text{ is a pointed, convex cone with } \operatorname{qi} K \neq \emptyset; \\ F : X \to \mathcal{P}(Y) \text{ is a proper set-valued function.} \end{array} \right. \tag{4.1}$$

Let us recall that under the hypotheses (4.1) the **domain** of the set-valued function $F$ is the set

$$\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\}.$$

The function $F$ is said to be **proper** if $\operatorname{dom} F \neq \emptyset$.

### 4.1. qi-Conjugate Set-Valued Functions.

**Definition 4.1.** *The* qi-**conjugate** *function of* $F$ *is the set-valued function* $F^*_{\text{qi}\,K} : \mathcal{L}(X,Y) \to \mathcal{P}(\mathcal{P}(Y))$ *defined by*

$$F^*_{\text{qi}\,K}(T) := \text{u-Max}_{\text{qi}}\{Tx - F(x) : x \in X\} \textit{ for all } T \in \mathcal{L}(X,Y).$$

Taking into consideration Definition 3.5 (d) and the hypotheses (4.1), it results that the following equality holds:

$$F^*_{\mathrm{qi}\,K}(T) = \text{u-Max}_{\mathrm{qi}}\{Tx - F(x) : x \in \operatorname{dom} F\}.$$

We emphasize at this point that for each set $A \in F^*_{\mathrm{qi}\,K}(T)$ there exists $x_A \in \operatorname{dom} F$ such that $A = Tx_A - F(x_A)$.

In the following we prove a result which may be regarded as an **extension of the Fenchel-Young inequality** to this set-valued setting.

**Theorem 4.2.** *Let $x_0, x_1 \in \operatorname{dom} F$, and let $T \in \mathcal{L}(X, Y)$ be such that*

$$F(x_1) - Tx_1 \in -F^*_{\mathrm{qi}_K}(T). \tag{4.2}$$

*Then the following statements hold:*

(a) *If $F(x_0) - Tx_0 \trianglelefteq^l_{\mathrm{qi}_K} F(x_1) - Tx_1$, then $F(x_1) - Tx_1 \trianglelefteq^l_{\mathrm{qi}_K} F(x_0) - Tx_0$.*
(b) *If $F(x_0) - Tx_0 \trianglelefteq^l_{\mathrm{qi}_K} F(x_1) - Tx_1$, then $F(x_1) - Tx_1 \sim^l F(x_0) - Tx_0$.*

*Proof.* (a) Assume that

$$F(x_0) - Tx_0 \trianglelefteq^l_{\mathrm{qi}\,K} F(x_1) - Tx_1. \tag{4.3}$$

By applying Proposition 3.7 we have $-F^*_{\mathrm{qi}_K}(T) = \text{l-Min}_{\mathrm{qi}}\{F(x) - Tx : x \in X\}$. Then from (4.2) and (4.3) we get that

$$F(x_1) - Tx_1 \trianglelefteq^l_{\mathrm{qi}_K} F(x_0) - Tx_0. \tag{4.4}$$

(b) From (4.3) and (4.4) we obtain, applying Proposition 3.3 (c), that $F(x_0) - Tx_0 \sim^l F(x_1) - Tx_1$. □

4.2. qi**-Subgradients of Set-Valued Functions.** With the help of the qi-conjugate function we extend the notions of the subgradient and subdifferential to set-valued functions.

**Definition 4.3.** *Let $\overline{x} \in \operatorname{dom} F$.*

(a) *An operator $T \in \mathcal{L}(X, Y)$ is said to be a* qi**-subgradient** *of the set-valued function $F$ at $\overline{x}$ if*

$$T\overline{x} - F(\overline{x}) \in F^*_{\mathrm{qi}_K}(T).$$

(b) *The set of all* qi*-subgradients of the set-valued function $F$ at $\overline{x}$ is called the* qi**-subdifferential** *of $F$ at $\overline{x}$ and is denoted by $\partial_{\mathrm{qi}_K} F(\overline{x})$.*

*By convention, if $\overline{x} \notin \operatorname{dom} F$, then we consider that $\partial_{\mathrm{qi}\,K} F(\overline{x}) = \emptyset$.*

In light of Definition 4.3, the condition (4.2) in Theorem 4.2 can be equivalently rewritten as $T \in \partial_{\mathrm{qi}\,K} F(x_1)$.

Similar to the scalar and vector case, we prove the following property.

**Proposition 4.4.** *Let $\overline{x} \in \operatorname{dom} F$. Then*

$$F(\overline{x}) \in \text{l-Min}_{\mathrm{qi}}\{F(x) : x \in X\} \textit{ if and only if } 0 \in \partial_{\mathrm{qi}_K} F(\overline{x}).$$

*Proof.* By making use of Proposition 3.7 we obtain

$$\begin{aligned} F(\overline{x}) \in \text{l-Min}_{\text{qi}}\{F(x) : x \in X\} &\Longleftrightarrow 0\overline{x} - F(\overline{x}) \in \text{u-Max}_{\text{qi}}\{0x - F(x) : x \in X\} \\ &\Longleftrightarrow 0\overline{x} - F(\overline{x}) \in F^*_{\text{qi}_K}(0) \\ &\Longleftrightarrow 0 \in \partial_{\text{qi}_K} F(\overline{x}). \end{aligned}$$

□

## 5. A quasi interior perturbation approach in set-valued optimization

5.1. **Unconstrained Set-Valued Optimization.** In this subsection we consider the unconstrained set-valued optimization problem

$$(P^{sv}_{\text{qi}}) \qquad \underset{x \in X}{\text{l-Min}_{\text{qi}}}\, F(x),$$

which will be studied under the hypotheses (4.1) and the additional assumption that

$$(5.1) \qquad W \text{ is a topological vector space.}$$

**Definition 5.1.** *An element* $\overline{x} \in \operatorname{dom} F$ *is a* qi**-efficient solution** *to* $(P^{sv}_{\text{qi}})$ *if*

$$F(\overline{x}) \in \text{l-Min}_{\text{qi}}\{F(x) : x \in X\} = \text{l-Min}_{\text{qi}}\{F(x) : x \in \operatorname{dom} F\}.$$

We develop a general duality theory based on a quasi interior perturbation approach. A set-valued function $\Phi : X \times W \to \mathcal{P}(Y)$, which satisfies the equality

$$\Phi(x, 0) = F(x) \text{ for all } x \in X,$$

is called a **perturbation function** associated with $F$.

Consider an arbitrary perturbation function $\Phi : X \times W \to \mathcal{P}(Y)$ associated with $F$. The qi-conjugate function of $\Phi$ is the set-valued function $\Phi^*_{\text{qi}_K} : \mathcal{L}(X,Y) \times \mathcal{L}(W,Y) \to \mathcal{P}(\mathcal{P}(Y))$ defined by

$$\Phi^*_{\text{qi}_K}(H,T) := \text{u-Max}_{\text{qi}}\{Hx + Tw - \Phi(x,w) : (x,w) \in X \times W\}$$

for all $(H,T) \in \mathcal{L}(X,Y) \times \mathcal{L}(W,Y)$.

We introduce the following new set-valued dual problem associated with $(P^{sv}_{\text{qi}})$:

$$(D^{sv}_{\text{qi}}) \qquad \underset{T \in \mathcal{L}(W,Y)}{\text{l-Max}_{\text{qi}}} \left[-\Phi^*_{\text{qi}\,K}(0,T)\right].$$

For the sake of simplicity we consider the notation

$$\begin{aligned} \mathcal{A}_{D^{sv}_{\text{qi}}} :&= \left\{(T,x,w) \in \mathcal{L}(W,Y) \times \operatorname{dom}\Phi : -Tw + \Phi(x,w) \in -\Phi^*_{\text{qi}_K}(0,T)\right\} \\ &= \left\{(T,x,w) \in \mathcal{L}(W,Y) \times X \times W : (0,T) \in \partial_{\text{qi}_K}\Phi(x,w)\right\}. \end{aligned}$$

**Definition 5.2.** *An operator* $\widetilde{T} \in \mathcal{L}(W,Y)$ *is said to be a* qi**-efficient solution** *to the dual problem* $(D^{sv}_{\text{qi}})$ *if there exists an* $(\widetilde{x}, \widetilde{w}) \in \operatorname{dom}\Phi$ *such that*

$$(5.2) \qquad (\widetilde{T}, \widetilde{x}, \widetilde{w}) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$$

*and*

$$(5.3) \qquad -\widetilde{T}\widetilde{w} + \Phi(\widetilde{x}, \widetilde{w}) \in \text{l-Max}_{\text{qi}}\{-Tw + \Phi(x,w) : (T,x,w) \in \mathcal{A}_{D^{sv}_{\text{qi}}}\}.$$

With the help of the following **set-valued weak duality** theorem we certify that $(D^{sv}_{\text{qi}})$ is actually a dual problem to $(P^{sv}_{\text{qi}})$.

**Theorem 5.3.** *Let $x_0 \in \operatorname{dom} F$, and let $(T, x, w) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$. Then the following statements are true:*

(a) *If $F(x_0) \trianglelefteq^l_{\text{qi}_K} -Tw + \Phi(x, w)$, then $-Tw + \Phi(x, w) \trianglelefteq^l_{\text{qi}_K} F(x_0)$.*
(b) *If $F(x_0) \trianglelefteq^l_{\text{qi}_K} -Tw + \Phi(x, w)$, then $-Tw + \Phi(x, w) \sim^l F(x_0)$.*

*Proof.* The proof relies on Theorem 4.2 considered for the function $\Phi$ and the linear continuous operator $(0, T) \in \mathcal{L}(X, Y) \times \mathcal{L}(W, Y)$. □

The forthcoming result contains some **optimality conditions** for the primal-dual pair $(P^{sv}_{\text{qi}}, D^{sv}_{\text{qi}})$ of set-valued optimization problems.

**Theorem 5.4.** *Let $\overline{x} \in \operatorname{dom} F$, and let $(\widetilde{T}, \widetilde{x}, \widetilde{w}) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$ be such that*

$$F(\overline{x}) \trianglelefteq^l_{\text{qi}\,K} -\widetilde{T}\widetilde{w} + \Phi(\widetilde{x}, \widetilde{w}). \tag{5.4}$$

*Then the following statements are true:*

(a) *$\overline{x}$ is a* qi-*efficient solution to $(P^{sv}_{\text{qi}})$.*
(b) *$\widetilde{T}$ is a* qi-*efficient solution to $(D^{sv}_{\text{qi}})$.*

*Proof.* (a) Let $x \in X$ be such that $F(x) \trianglelefteq^l_{\text{qi}\,K} F(\overline{x})$. Taking into account that the relation $\trianglelefteq^l_{\text{qi}\,K}$ is transitive, we get $F(x) \trianglelefteq^l_{\text{qi}\,K} -\widetilde{T}\widetilde{w} + \Phi(\widetilde{x}, \widetilde{w})$. From Theorem 5.3 (a) it follows that $-\widetilde{T}\widetilde{w} + \Phi(\widetilde{x}, \widetilde{w}) \trianglelefteq^l_{\text{qi}\,K} F(x)$. Using again the transitivity of the relation $\trianglelefteq^l_{\text{qi}\,K}$ and (5.4) we get that $F(\overline{x}) \trianglelefteq^l_{\text{qi}\,K} F(x)$, which means that $\overline{x}$ is a qi-efficient solution to $(P^{sv}_{\text{qi}})$.

(b) Let us consider $(T, x, w) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$ such that

$$-\widetilde{T}\widetilde{w} + \Phi(\widetilde{x}, \widetilde{w}) \trianglelefteq^l_{\text{qi}\,K} -Tw + \Phi(x, w).$$

Using the transitivity of the relation $\trianglelefteq^l_{\text{qi}\,K}$ and (5.4), we get that $F(\overline{x}) \trianglelefteq^l_{\text{qi}\,K} -Tw + \Phi(x, w)$, which, from Theorem 5.3 (a) implies $-Tw + \Phi(x, w) \trianglelefteq^l_{\text{qi}\,K} F(\overline{x})$. Applying again (5.4), we reach the conclusion

$$-Tw + \Phi(x, w) \trianglelefteq^l_{\text{qi}\,K} -\widetilde{T}\widetilde{w} + \Phi(\widetilde{x}, \widetilde{w}),$$

which means that $(\widetilde{T}, \widetilde{x}, \widetilde{w}) \in \text{l-Max}_{\text{qi}}\{-Tw + \Phi(x, w) : (T, x, w) \in \mathcal{A}_{D^{sv}_{\text{qi}}}\}$. Thus the conditions (5.3) holds. Moreover, from the hypotheses we know that (5.2) is also satisfied, implying that $\widetilde{T}$ is a qi-efficient solution to the dual problem $(D^{sv}_{\text{qi}})$. □

The next theorem contains further **optimality conditions** for the dual problem $(D^{sv}_{\text{qi}})$.

**Theorem 5.5.** *Let $\overline{x} \in \operatorname{dom} F$. If there exists an operator $\overline{T} \in \mathcal{L}(W, Y)$ such that $(\overline{T}, \overline{x}, 0) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$, then $\overline{T}$ is a* qi-*efficient solution to $(D^{sv}_{\text{qi}})$.*

*Proof.* Let us start by noticing that, by assuming that $(\overline{T}, \overline{x}, 0) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$, the condition (5.2) in Definition 5.2 is satisfied. Let now $(T, x, w) \in \mathcal{A}_{D^{sv}_{\text{qi}}}$ be such that

$$-\overline{T}0 + \Phi(\overline{x}, 0) \trianglelefteq^{l}_{\text{qi}\, K} -Tw + \Phi(x, w). \tag{5.5}$$

By Theorem 5.3 (a) we get that

$$-Tw + \Phi(x, w) \trianglelefteq^{l}_{\text{qi}\, K} F(\overline{x}) = -\overline{T}0 + \Phi(\overline{x}, 0).$$

Therefore it holds $-\overline{T}0 + \Phi(\overline{x}, 0) \in \text{l-Max}_{\text{qi}}\{-Tw + \Phi(x, w) : (T, x, w) \in \mathcal{A}_{D^{sv}_{\text{qi}}}\}$, i.e. the condition (5.3) is satisfied. Thus $\overline{T}$ is a qi-efficient solution to $(D^{sv}_{qi})$. □

5.2. **A Set-Valued Lagrange Multipliers Rule.** In this subsection we associate with a general constrained set-valued optimization problem with cone constraints a dual problem, obtained by particularizing the perturbation function in a manner similar to the classical Lagrange approach from the scalar case. For the new prima-dual pair of set-valued optimization problems we are able to provide a Lagrange multipliers rule.

We consider the general set-valued optimization problem with cone constraints

$$(CP^{sv}_{\text{qi}}) \qquad \underset{G(x)\cap -C\neq\emptyset}{\text{l-Min}_{\text{qi}}} \; F(x)$$

stated under the following hypotheses:

$$\left\{\begin{array}{l} X \text{ and } W \text{ are topological vector spaces;} \\ Y \text{ and } Z \text{ are separated locally convex spaces;} \\ K \subset Y \text{ is a pointed, convex cone with } \operatorname{qi} K \neq \emptyset; \\ C \subset Z \text{ is a nonempty, pointed and convex cone;} \\ F : X \to \mathcal{P}(Y) \text{ and } G : X \to \mathcal{P}(Z) \text{ are proper set-valued functions;} \\ \{x \in (\operatorname{dom} F) \cap (\operatorname{dom} G) : G(x) \cap (-C) \neq \emptyset\} \neq \emptyset. \end{array}\right. \tag{5.6}$$

We associate with problem $(CP^{sv}_{\text{qi}})$ the Lagrange-type perturbation function $\Phi^{sv}_{L} : X \times Z \to \mathcal{P}(Y)$, defined by

$$\Phi^{sv}_{L}(x, z) := \left\{\begin{array}{ll} F(x) & \text{if } x \in X \text{ and } (G(x) - z) \cap (-C) \neq \emptyset \\ \emptyset & \text{otherwise.} \end{array}\right.$$

Let us notice that

$$\Phi^{sv}_{L}(x, z) = \left\{\begin{array}{ll} F(x) & \text{if } x \in (\operatorname{dom} F) \cap (\operatorname{dom} G) \text{ and } z \in G(x) + C \\ \emptyset & \text{otherwise.} \end{array}\right.$$

Given an operator $T \in \mathcal{L}(Z, Y)$, the set-valued qi-conjugate function associated with the Lagrange-type perturbation function $\Phi^{sv}_{L}$ at $(0, T)$ is

$$(\Phi^{sv}_{L})^{*}_{\text{qi}\, K}(0, T) = \text{u-Max}_{\text{qi}}\{Tz - F(x) : x \in X, z \in G(x) + C\}.$$

We attach to $(CP^{sv}_{\text{qi}})$ the following set-valued Lagrange-type dual problem:

$$(LCD^{sv}_{\text{qi}}) \qquad \underset{T\in\mathcal{L}(Z,Y)}{\text{l-Max}_{\text{qi}}} \left[ -(\Phi^{sv}_{L})^{*}_{\text{qi}\, K}(0, T) \right].$$

In the following we use the notation

$$\mathcal{A}_{LCD^{sv}_{\text{qi}}} := \left\{ \begin{array}{ll} (T, x, z) : & T \in \mathcal{L}(Z, Y), x \in X, z \in G(x) + C, \\ & -Tz + F(x) \in -(\Phi^{sv}_{L})^{*}_{\text{qi}\, K}(0, T) \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} (T,x,z): & T \in \mathcal{L}(Z,Y), x \in X, z \in G(x)+C, \\ & (0,T) \in \partial \Phi^{sv}_{L\ \mathrm{qi}\, K}(x,z) \end{array} \right\}.$$

**Definition 5.6.** *An operator* $\widetilde{T} \in \mathcal{L}(Z,Y)$ *is said to be a* qi-***efficient solution*** *to the dual problem* $(LCD^{sv}_{\mathrm{qi}})$ *if there exists an* $(\widetilde{x},\widetilde{z}) \in \operatorname{dom}\Phi^{sv}_L$ *such that*

$$(\widetilde{T},\widetilde{x},\widetilde{z}) \in \mathcal{A}_{LCD^{sv}_{\mathrm{qi}}} \tag{5.7}$$

*and*

$$-\widetilde{T}\widetilde{z} + F(\widetilde{x}) \in \text{l-Max}_{\mathrm{qi}} \left\{ -Tz + F(x) : (T,x,z) \in \mathcal{A}_{LCD^{sv}_{\mathrm{qi}}} \right\}. \tag{5.8}$$

The results from Subsection 5.1 are applicable to the primal-dual pair $(CP^{sv}_{\mathrm{qi}}, LCD^{sv}_{\mathrm{qi}})$ of optimization problems. Particularizing the weak duality theorem, as well as the theorems which state optimality conditions in Subsection 5.1, one obtains similar results in this constrained set-valued setting.

We next present a **strong duality theorem**.

**Theorem 5.7.** *Let* $(F,G)$ *be a* $K \times C$*-convex function, and let* $\overline{x} \in \mathcal{A}_{CP^{sv}_{\mathrm{qi}}}$ *be such that there exists* $\overline{y} \in F(\overline{x})$ *with the property*

$$(\overline{y},0) \notin \operatorname{qri}\left((F,G)(X) + K \times C\right). \tag{5.9}$$

*Moreover, assume that*

$$0 \in \operatorname{qi}(G(X)+C). \tag{5.10}$$

*Then there exists an operator* $\overline{T} \in \mathcal{L}(Z,Y)$ *such that* $\overline{T}$ *is a* qi-*efficient solution to the dual problem* $(LCD^{sv}_{\mathrm{qi}})$.

*Proof.* Since $\overline{x} \in \mathcal{A}_{CP^{sv}_{\mathrm{qi}}}$, there exists at least one $\overline{z} \in G(\overline{x}) \cap (-C)$. As $(F,G)$ is a proper $K \times C$-convex function, it follows that the set $(F,G)(X)+K\times C$ is nonempty and convex. Moreover, since $\overline{y} \in F(\overline{x})$, we have $(\overline{y},0) \in (F,G)(X) + K \times C$. Using (5.9), we may apply Theorem 2.4 from which we get that there exists an

$$(y^*,z^*) \in Y^* \times Z^* \setminus \{(0,0)\} \tag{5.11}$$

such that

$$\langle y^*, \overline{y} \rangle \leq \langle y^*, y \rangle + \langle z^*, z \rangle \text{ for all } (y,z) \in (F,G)(X) + K \times C. \tag{5.12}$$

We proceed by proving that

$$y^* \in K^+. \tag{5.13}$$

Since $\overline{z} \in G(\overline{x}) \cap (-C)$, it is clear that $0 \in G(\overline{x}) + C$. Putting $x := \overline{x}$ and $z := 0$ in (5.12), we obtain

$$\langle y^*, \overline{y} \rangle \leq \langle y^*, y \rangle \text{ for all } y \in F(\overline{x}) + K.$$

Moreover, since $\overline{y} \in F(\overline{x})$, hence we can further particularize the inequality above to

$$\langle y^*, \overline{y} \rangle \leq \langle y^*, \overline{y} \rangle + \langle y^*, k \rangle \text{ for all } k \in K,$$

which means that $0 \leq \langle y^*, k \rangle$ for all $k \in K$, i.e. $y^* \in K^+$.

Next we prove that

$$z^* \in C^+. \tag{5.14}$$

Put $x := \overline{x}$ and $y = \overline{y}$ in (5.12). Then we get

$$0 \leq \langle z^*, z\rangle \text{ for all } z \in G(\overline{x}) + C. \tag{5.15}$$

Let us consider an arbitrary $c \in C$. From $\overline{z} \in G(\overline{x}) \cap (-C)$ it follows that

$$c = c + \overline{z} - \overline{z} \in C + G(\overline{x}) - (-C) = G(\overline{x}) + C + C \subseteq G(\overline{x}) + C.$$

This means that from (5.15) we obtain

$$0 \leq \langle z^*, c\rangle \text{ for all } c \in C,$$

whence $z^* \in C^+$.

We now prove that $y^* \neq 0$. Assume that $y^* = 0$. Then (5.12) turns into

$$0 \leq \langle z^*, z\rangle \text{ for all } z \in G(X) + C.$$

Hence, $-z^* \in N_{G(X)+C}(0)$. According to Theorem 2.2, (5.10) implies

$$N_{G(X)+C}(0) = \{0\},$$

therefore $-z^* = 0$. But in this case we have $(y^*, z^*) = (0, 0)$, which contradicts (5.11). Thus $y^* \neq 0$, and from (5.13) we obtain

$$y^* \in K^+ \backslash \{0\}. \tag{5.16}$$

This means that we can choose $\overline{k} \in K$ such that $\langle y^*, \overline{k}\rangle = 1$. We define now the operator $\overline{T} : Z \to Y$ by

$$\overline{T}(z) := \langle z^*, z\rangle(-\overline{k}) \text{ for all } z \in Z$$

and we prove that

$$(\overline{T}, \overline{x}, 0) \in \mathcal{A}_{LCD^{sc}_{\text{qi}}}. \tag{5.17}$$

Let us proceed by contradiction. Assume that there exist

$$x \in (\operatorname{dom} F) \cap (\operatorname{dom} G) \text{ and } z \in G(x) + C$$

such that

$$-\overline{T}z + F(x) \trianglelefteq^l_{\text{qi}\, K} -\overline{T}0 + F(\overline{x}).$$

This means that $F(\overline{x}) \subseteq F(x) - \overline{T}z + \text{qi}\, K$. Thus, for $\overline{y} \in F(\overline{x})$ there exist $y \in F(x)$ and $k \in \text{qi}\, K$ such that

$$\overline{y} = y - \overline{T}z + k \in Y.$$

Since $y^*$ is a linear operator and $\langle y^*, \overline{k}\rangle = 1$, we get

$$\begin{aligned} \langle y^*, \overline{y}\rangle &= \langle y^*, y\rangle - \langle y^*, \langle z^*, z\rangle(-\overline{k})\rangle + \langle y^*, k\rangle \\ &= \langle y^*, y\rangle + \langle z^*, z\rangle\langle y^*, \overline{k}\rangle + \langle y^*, k\rangle \\ &= \langle y^*, y\rangle + \langle z^*, z\rangle + \langle y^*, k\rangle. \end{aligned}$$

Using now the facts that $y^* \in K^+ \backslash \{0\}$, $k \in \text{qi}\, K$, and applying Proposition 2.3, it follows that $\langle y^*, k\rangle > 0$. Therefore we come to the conclusion that

$$\langle y^*, \overline{y}\rangle > \langle y^*, y\rangle + \langle z^*, z\rangle. \tag{5.18}$$

Considering the facts that $x \in (\operatorname{dom} F) \cap (\operatorname{dom} G)$, $y \in F(x) \subseteq F(x) + K$ and $z \in G(x) + C$, it follows that (5.18) is in contradiction to (5.12). This leads to $(\overline{T}, \overline{x}, 0) \in \mathcal{A}_{LCD^{sc}_{\text{qi}}}$.

Applying a result similar to Theorem 5.5, this time for the primal-dual pair $(CP^{sv}_{\mathrm{qi}}, LCD^{sv}_{\mathrm{qi}})$, we get from (5.17) that $\overline{T}$ is a qi-efficient solution to $(LCD^{sv}_{\mathrm{qi}})$. $\square$

**Remark 5.8.** E. Hernández and L. Rodríguez-Marin approached in [16] a Lagrange dual problem in set-valued optimization using the set criterion. We underline at this point the differences between their approach, and the one suggested in the present paper. First of all, our approach is more general, since it uses the quasi-interior, a generalization of the classical topological interior. Then, when looking for dual solutions, we seek linear continuous operators, whilst in [16] the operators considered are the sum of continuous linear affine map, and an element in $-C$. In case $\operatorname{int} K \neq \emptyset$, then $\operatorname{int} K = \operatorname{qi} K$ and the solutions for the primal problems coincide for both [16] and the present paper, however, this does not change the differences between the dual problems. The strong duality theorem, i.e. Theorem 4.1 in [16], has many assumption in contrast to our strong duality result, i.e Theorem 5.7. More precisely, in [16] the authors need except for a generalized Slater condition, an optimal solution $x_0 \in X$ of the primal problem, satisfying properties which involve the existence of weak efficient points for the set $F(x_0)$ along with condition iii), which guarantees a separation, in order to get an optimal solution for the dual.

**Remark 5.9.** Theorem 5.5 extends results from scalar optimization, such as Theorem 4.1 in [7] or Theorem 14 in [14].

## 6. An application to a set-valued optimization problem in $\ell^2(\mathbb{R})$

In this section we present an example of a set-valued optimization problem for which we can apply the strong-duality statements of Theorem 5.7. Our application was inspired by Example 2.10 in E. R. Csetnek [12], and is formulated under the following particular instance of the framework (5.6):

$$(6.1)\quad \begin{cases} X := \ell^2(\mathbb{R}), Y := \mathbb{R}, Z := \ell^2(\mathbb{R}), K := \mathbb{R}_+, C := \ell^2_+(\mathbb{R}); \\ F : \ell^2(\mathbb{R}) \to \mathcal{P}(\mathbb{R}) \text{ is defined by } F(\mu) := \begin{cases} \{\|\mu\|_{\ell^2(\mathbb{R})}\} & \text{if } \mu \in \ell^2_+(\mathbb{R}) \\ \emptyset & \text{otherwise} \end{cases}; \\ G : \ell^2(\mathbb{R}) \to \mathcal{P}(\ell^2(\mathbb{R})) \text{ is defined by } G(\mu) := \begin{cases} \{-\mu\} & \text{if } \mu \in \ell^2_+(\mathbb{R}) \\ \emptyset & \text{otherwise} \end{cases}. \end{cases}$$

We notice that $(\operatorname{dom} F) \cap (\operatorname{dom} G) = \ell^2_+(\mathbb{R})$ and $\operatorname{qi} K = \operatorname{qi} \mathbb{R}_+$. As $\mathbb{R}$ is a finite dimensional space, we have $\operatorname{qi} \mathbb{R}_+ = \operatorname{int} \mathbb{R}_+ = (0, +\infty)$.

Let us recall some important properties concerning the set

$$\ell^2(\mathbb{R}) := \{\mu : \mathbb{R} \to \mathbb{R} : \sum_{x \in \mathbb{R}} |\mu(x)|^2 < +\infty\}.$$

The function $\|\cdot\|_{\ell^2(\mathbb{R})} : \ell^2(\mathbb{R}) \to \mathbb{R}$ defined by

$$\|\mu\|_{\ell^2(\mathbb{R})} := \left(\sum_{x \in \mathbb{R}} |\mu(x)|^2\right)^{\frac{1}{2}} = \left(\sup_{F \in \mathcal{P}_0(\mathbb{R}), F \text{ finite}} \sum_{x \in F} |\mu(x)|^2\right)^{\frac{1}{2}} \text{ for all } \mu \in \ell^2(\mathbb{R})$$

is a norm on $\ell^2(\mathbb{R})$, and the vector space $\ell^2(\mathbb{R})$, equipped with this norm is a Banach space. Moreover, the norm $\|\cdot\|_{\ell^2(\mathbb{R})}$ is generated by the scalar product

$\langle\cdot,\cdot\rangle_{\ell^2(\mathbb{R})}:\ell^2(R)\times\ell^2(R)\to\mathbb{R}$ defined by

$$\langle\mu,\mu'\rangle_{\ell^2(\mathbb{R})}:=\sup_{F\in\mathcal{P}_0(\mathbb{R}),F\text{ finite}}\sum_{x\in F}\mu(x)\mu'(x)\text{ for all }\mu,\mu'\in\ell^2(\mathbb{R}).$$

Thus, the vector space $\ell^2(\mathbb{R})$ equipped with the above defined scalar product is a Hilbert space. The dual space $(\ell^2(\mathbb{R}))^*$ is identified with $\ell^2(\mathbb{R})$. Moreover, the set

$$\ell^2_+(\mathbb{R}):=\{\mu\in\ell^2(\mathbb{R}):\mu(x)\geq 0\text{ for all }x\in\mathbb{R}\}$$

is a pointed convex cone, and from J. M. Borwein, Y. Lucet and B. Mordukhovich [4, Remark 2.20] we know that $\operatorname{qri}(\ell^2_+(\mathbb{R}))=\emptyset$. Furthermore, it holds

$$\ell^2_+(\mathbb{R})-\ell^2_+(\mathbb{R})=\ell^2(\mathbb{R}).\tag{6.2}$$

It is easy to see that $\overline{\mu}:=0\in\ell^2(\mathbb{R})$ is a qi-efficient solution to the set-valued optimization problem

$$(P^{sv}_{\ell^2(\mathbb{R})})\qquad \underset{\substack{\mu\in\ell^2_+(\mathbb{R})\\ G(\mu)\cap-\ell^2_+(\mathbb{R})\neq\emptyset}}{\text{l-Min}_{\text{qi}}}\{\|\mu\|_{\ell^2(\mathbb{R})}\}.$$

We associate with $(P^{sv}_{\ell^2(\mathbb{R})})$ a Lagrange-type set-valued dual problem, with the help of the perturbation function $\Phi_{\ell^2(\mathbb{R})}:\ell^2(\mathbb{R})\times\ell^2(\mathbb{R})\to\mathbb{R}$ defined by

$$\begin{aligned}\Phi_{\ell^2(\mathbb{R})}(\mu,\zeta):&=\left\{F(\mu):\mu\in\ell^2(\mathbb{R}),G(\mu)-\zeta\in-\ell^2_+(\mathbb{R})\right\}\\&=\left\{\|\mu\|_{\ell^2(\mathbb{R})}:\mu\in\ell^2_+(\mathbb{R}),\zeta\in-\mu+\ell^2_+(\mathbb{R})\right\},\end{aligned}$$

for all $(\mu,\zeta)\in\ell^2(\mathbb{R})\times\ell^2(\mathbb{R})$.

The qi-conjugate set-valued function associated with $\Phi_{\ell^2(\mathbb{R})}$ is

$$(\Phi_{\ell^2(\mathbb{R})})^*_{\text{qi}\,\mathbb{R}_+}:\mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R})\times\mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R})\to\mathcal{P}(\mathcal{P}(\mathbb{R})),$$

defined by

$$\begin{aligned}(\Phi_{\ell^2(\mathbb{R})})^*_{\text{qi}\,\mathbb{R}_+}(0,T):&=\text{u-Max}_{\text{qi}}\{T\zeta-\Phi(\mu,\zeta):(\mu,\zeta)\in\ell^2(\mathbb{R})\times\ell^2(\mathbb{R})\}\\&=\text{u-Max}_{\text{qi}}\{T\zeta-\|\mu\|_{\ell^2(\mathbb{R})}:\mu\in\ell^2_+(\mathbb{R}),\zeta\in-\mu+\ell^2_+(\mathbb{R})\}\end{aligned}$$

for all $T\in\mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R})$.

A Lagrange-type set-valued dual problem associated with $(P^{sv}_{\ell^2(\mathbb{R})})$ is

$$(LD^{sv}_{\ell^2(\mathbb{R})})\qquad \underset{T\in\mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R})}{\text{l-Max}_{\text{qi}}}\left[-(\Phi_{\ell^2(\mathbb{R})})^*_{\text{qi}\,\mathbb{R}_+}(0,T)\right].$$

In the following we use the notation

$$\mathcal{A}_{LD^{sv}_{\ell^2(\mathbb{R})}}:=\left\{\begin{array}{ll}(T,\mu,\zeta): & T\in\mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R}),\mu\in\ell^2(\mathbb{R}),\zeta\in\ell^2(\mathbb{R}),\\ & T\zeta-\Phi(\mu,\zeta)\in-(\Phi_{\ell^2(\mathbb{R})})^*_{\text{qi}\,\mathbb{R}_+}(0,T)\end{array}\right\}.$$

Let us further notice that

$$\begin{aligned}\mathcal{A}_{LD^{sv}_{\ell^2(\mathbb{R})}}&=\left\{\begin{array}{ll}(T,\mu,\zeta): & T\in\mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R}),\mu\in\ell^2_+(\mathbb{R}),\zeta\in-\mu+\ell^2_+(\mathbb{R})\\ & T\zeta-\|\mu\|_{\ell^2(\mathbb{R})}\in-(\Phi_{\ell^2(\mathbb{R})})^*_{\text{qi}\,\mathbb{R}_+}(0,T)\end{array}\right\}\\&=\left\{\begin{array}{ll}(T,\mu,\zeta): & T\in\mathcal{L}(\ell^2(\mathbb{R}),\mathbb{R}),\mu\in\ell^2_+(\mathbb{R}),\zeta\in-\mu+\ell^2_+(\mathbb{R}),\\ & (0,T)\in\partial_{\text{qi}\,\mathbb{R}_+}(\Phi_{\ell^2(\mathbb{R})})(\mu,\zeta)\end{array}\right\}.\end{aligned}$$

As we have previously mentioned, $\overline{\mu} = 0$ is a qi-efficient solution to $(P^{sv}_{\ell^2(\mathbb{R})})$. We check the hypotheses of Theorem 5.7. First of all we notice that $(F, G)$ is a $\mathbb{R}_+ \times \ell^2_+(\mathbb{R})$-convex function. Next, from (6.2) it follows that

$$0 \in \operatorname{qi}\left[-\ell^2_+(\mathbb{R}) + \ell^2_+(\mathbb{R})\right] = \operatorname{qi}\ell^2(\mathbb{R}) = \ell^2(\mathbb{R}),$$

which implies that (5.10) is automatically satisfied. The relation (5.9) turns into

$$(0,0) \notin \operatorname{qri}\left[(F,G)(\ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell^2_+(\mathbb{R})\right]. \tag{6.3}$$

By contradiction we prove that (6.3) is satisfied. Assume that

$$(0,0) \in \operatorname{qri}\left[(F,G)(\ell^2(\mathbb{R})) + \mathbb{R}_+ \times \ell^2_+(\mathbb{R})\right]$$

and consider an arbitrary $(r, \mu^*) \in N_{(F,G)(\ell^2(\mathbb{R}))+\mathbb{R}_+\times\ell^2_+(\mathbb{R})}\{(0,0)\}$. Then, for all $\mu \in \ell^2_+(\mathbb{R})$, all $\lambda \geq 0$, and all $\chi \in \ell^2_+(\mathbb{R})$ it holds

$$r(\|\mu\|_{\ell^2(\mathbb{R})} + \lambda) + \langle \mu^*, -\mu + \chi\rangle_{\ell^2(\mathbb{R})} \leq 0. \tag{6.4}$$

We notice that

$$(-1,0) \in N_{(F,G)(\ell^2(\mathbb{R}))+\mathbb{R}_+\times\ell^2_+(\mathbb{R})}\{(0,0)\}.$$

By applying Theorem 2.1 we get

$$(1,0) \in N_{(F,G)(\ell^2(\mathbb{R}))+\mathbb{R}_+\times\ell^2_+(\mathbb{R})}\{(0,0)\}.$$

This means that (6.4) turns into

$$\|u\|_{\ell^2(\mathbb{R})} + \lambda \leq 0 \text{ for all } \mu \in \ell^2_+(\mathbb{R}) \text{ and all } \lambda \geq 0,$$

which is obviously a contradiction. Hence (6.3) is satisfied.

Summing up the facts listed above, we conclude that the hypotheses of Theorem 5.7 are satisfied. This means that there exists an operator $\overline{T} \in \mathcal{L}(\ell^2(\mathbb{R}), \mathbb{R})$ such that $\overline{T}$ is a qi-efficient solution to $(LD^{sv}_{\ell^2(\mathbb{R})})$.

## Acknowledgment

The author is thankful to an anonymous reviewer for suggestions that significantly improved the quality of the paper. As well, the author is grateful to P.D. Dr. rer. nat. habil. Radu Ioan Boţ and to Dr. rer. nat. Ernö Robert Csetnek from Chemnitz University of Technology for reading and correcting initial versions of the results presented here.

## References

[1] M. Alonso and L. Rodríguez-Marin, *Set-relations and optimality conditions in set-valued maps*, Nonlinear Anal. **63** (2005), 1167–1179.

[2] J. M. Borwein and R. Goebel, *Notions of relative interior in Banach spaces*, J. Math. Sci. (N.Y.) **115** (2003), 2542–2553.

[3] J. M. Borwein and A. S. Lewis, *Partially finite convex programming, Part I: Quasi relative interiors and duality theory*, Math. Programming Ser. B **57** (1992), 15–48.

[4] J. M. Borwein Y. Lucet and B. Mordukhovich, *Compactly epi-Lipschitzian convex sets and functions in normed spaces*, J. Convex Anal. **7** (2000), 375–393.

[5] R. I. Boţ, *Conjugate Duality in Convex Optimization*, Lecture Notes in Economics and Mathematical Systems **637**, Springer, Berlin-Heidelberg, 2010.

[6] R. I. Boţ and E. R. Csetnek, *Regularity conditions via generalized interiority notions in convex optimization: new achievements and their relation to some classical statements*, to apear in Optimization, arXiv:0906.0453, posted 2 June (2009).
[7] R. I. Boţ, E. R. Csetnek and A. Moldovan, *Revisiting some duality theorems via the quasirelative interior in convex optimization*, J. Optim. Theory Appl. **139** (2008), 67–84.
[8] R. I. Boţ, E. R. Csetnek and G. Wanka, *Regularity conditions via quasi-relative interior in convex programming*, SIAM J. Optim. **19** (2008), 217–233.
[9] R. I. Boţ, S. M. Grad and G. Wanka, *Duality in Vector Optimization*, Springer, Berlin-Heidelberg, 2009.
[10] F. Cammaroto and B. Di Bella, *On a separation theorem involving the quasi-relative interior*, Proc. Edinb. Math. Soc.(2) **50** (2007), 605–610.
[11] H. W. Corley, *Optimality conditions for maximizations of set-valued functions*, J. Optim. Theory Appl. **58** (1988), 1–10.
[12] E. R. Csetnek, *Overcoming the Failure of the Classical Generalized Interior-point Regularity Conditions in Convex Optimization. Applications of the Duality Theory to Enlargements of Maximal Monotone Operators*, Logos Verlag, Berlin, 2010.
[13] P. Daniele, S. Giuffré, G. Idone and A. Maugeri, *Infinite dimensional duality and applications*, Math. Ann. **339** (2007), 221–239.
[14] A. Grad, *Quasi-relative interior-type constraint qualifications ensuring strong Lagrange duality for optimization problems with cone and affine constraints*, J. Math. Anal. Appl. **361** (2010), 86–95.
[15] E. Hernández and L. Rodríguez-Marin, *Nonconvex scalarization in set optimization with set-valued maps*, J. Math. Anal. Appl. **325** (2007), 1–18.
[16] E. Hernández and L. Rodríguez-Marin, *Lagrangian duality in set-valued optimization*, J. Optim. Theory Appl. **134** (2007), 119–134.
[17] E. Hernández and L. Rodríguez-Marin, *Existence theorems for set optimization problems*, Nonlinear Anal. **67** (2007), 1726–1736.
[18] D. Kuroiwa, *The natural criteria in set-valued optimization*, RIMS Kokyuroku **1031** (1998), 85–90.
[19] D. Kuroiwa, *On natural criteria in set-valued optimization*, RIMS Kokyuroku **1048** (1998), 86–92.
[20] D. Kuroiwa, *Lagrange duality of set-valued optimization with natural criteria*, RIMS Kokyuroku **1068** (1998), 164–170.
[21] D. Kuroiwa, *On set-valued optimization*, Nonlinea Anal. **47** (2001), 1395–1400.
[22] D. Kuroiwa, *Existence of efficient points of set optimization with weighted criteria*, J. Nonlinear Convex Anal. **4** (2003), 117–123.
[23] D. Kuroiwa, *Existence theorems of set optimization with set-valued maps*, J. Inf. Optim. Sci. **24** (2003), 73–84.
[24] D. Kuroiwa, T. Tanaka and X. D. H. Truong, *On cone convexity of set-valued maps*, Nonlinear Anal. **30** (1997), 1487–1496.
[25] M. A. Limber and R. K. Goodrich, *Quasi interiors, Lagrange multipliers and $L^p$ spectral estimation with lattice bounds*, J. Optim. Theory Appl. **78** (1993), 143–161.
[26] D. T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems **319**, Springer, Berlin-Heidelberg-New York, 1989.
[27] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.
[28] W. Song, *Conjugate duality in set-valued vector optimization*, J. Math. Anal. Appl. **216** (1997), 265–283.
[29] W. Song, *A generalization of Fenchel duality in set-valued vector optimization*, Math. Methods Oper. Res. **8** (1998), 259–272.
[30] W. Song, *Duality in set-valued optimization*, Dissertationes Math. (Rozprawy Mat.) **375** 1998.
[31] T. Tanino, *Conjugate duality in vector optimization*, J. Math. Anal. Appl. **167** (1992), 84–97.
[32] T. Tanino and Y. Sawaragi, *Conjugate maps and duality in multiobjective optimization*, J. Optim. Theory Appl. **31** (1980), 473–499.

*Manuscript received August 25, 2009*
*revised July 6, 2010*

Anca Grad
Faculty of Mathematics and Computer Science, Babeş-Bolyai University Cluj-Napoca, Romania
*E-mail address*: dumitruanca@math.ubbcluj.ro