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STRONG CONVERGENCE THEOREMS FOR GENERALIZED EQUILIBRIUM PROBLEMS AND RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. The purpose of this paper is to prove strong convergence theorems for finding a common element of the set of solutions of a generalized equilibrium problem and the set of common fixed points of infinite relatively nonexpansive mappings in Banach spaces.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}$ the set of all real numbers. Let E be a real Banach space with a norm $\|\cdot\|$, E^* the dual space of E, $\langle\cdot,\cdot\rangle$ the pairing between E and E^* and C a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction and A a nonlinear operator of C into E^*. The *generalized equilibrium problem* is finding $u \in C$ such that

$$f(u, y) + \langle Au, y - u\rangle \geq 0 \tag{1.1}$$

for all $y \in C$. The set of solutions of (1.1) is denoted by EP, that is,

$$EP = \{u \in C : f(u, y) + \langle Au, y - u\rangle \geq 0,\ \forall y \in C\}.$$

If $A = 0$, then the problem (1.1) is equivalent to that of finding a point $u \in C$ such that

$$f(u, y) \geq 0 \tag{1.2}$$

for all $y \in C$ which is called the *equilibrium problem.* The set of solutions of (1.2) is denoted by $EP(f)$. If $f = 0$, then the problem (1.1) is equivalent to that of finding a point $u \in C$ such that

$$\langle Au, y - u\rangle \geq 0 \tag{1.3}$$

for all $y \in C$ which is called the *variational inequality.* The set of solutions of (1.3) is denoted by $VI(C, A)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, numerous problems in physics, economics and others. Some methods have been proposed for solving the generalized equilibrium problem, the equilibrium problem and the variational inequality in Hilbert spaces (see [14, 15]) and in Banach spaces ([11, 19]).

Let C be a nonempty closed convex subset of a real Banach space E. A mapping T of C into E is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

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A point $p \in C$ is called a *fixed point* of T if $Tp = p$. The set of fixed points of a mapping T is denoted by $F(T)$. A point $p \in C$ is called an *asymptotic fixed point* of T if there exists a sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\|x_n - Tx_n\| \to 0$. We denote by $\hat{F}(T)$ the set of all asymptotic fixed points of T. A mapping T is said to be *relatively nonexpansive* if $\hat{F}(T) = F(T) \neq \emptyset$ and $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$. Let $\alpha > 0$. A operator A of C into E^* is said to be *α-inverse strongly monotone* if

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. It is known that if A is an α-inverse strongly monotone operator, then A is $1/\alpha$-Lipschitzian.

Example ([3])**.** Let E be a Banach space, f a continuously Fréchet differentiable, convex functional on E and ∇f the gradient of f. If ∇f is $1/\alpha$-Lipschitz continuous, then ∇f is α-inverse strongly monotone.

In 2008, Takahashi and Takahashi [15] proved a strong convergense theorem for finding an element of $F(S) \cap EP$ in a Hilbert space H, where S is a nonexpansive mapping of a nonempty closed convex subset $C \subset H$ into itself and A is an inverse strongly monotone operator of C into H. Recently, Chang, Lee and Chan [4] considered iterative methods for finding an element of $F(S) \cap F(T) \cap EP$ in a certain Banach space E, where S and T are two relatively nonexpansive mappings of a nonempty closed convex subset $C \subset E$ into itself and A is an inverse strongly monotone operator of C into E^*. On the other hand, Matsushita, Nakajo and Takahashi [10] introduced iterative methods for finding an element of $\bigcap_{i=0}^{\infty} F(S_i)$, where S_i is a relatively nonexpansive mapping of C into itself for all $i \geq 0$.

In this paper, motivated by Chang *et al.* [4] and Matsushita *et al.* [10], we introduce new iterative methods for finding an element of $\bigcap_{i=0}^{\infty} F(S_i) \cap EP$, where S_i is a relatively nonexpansive mapping of C into itself for all $i \geq 0$ and A is an inverse-strongly monotone operator of C into E^*. In the next section, we recall some basic notions and give the definition of W-mappings and convex combinations of mappings. We present and prove our main results which are strong convergence theorems of W-mappings and convex combinations in Section 3 and Section 4, respectively.

2. Preliminaries

Throughout this paper, we assume that E is a real Banach space with a norm $\|\cdot\|$, E^* is the dual space of E and $\langle \cdot, \cdot \rangle$ is the pairing between E and E^*. We denote strong convergence of a sequence $\{x_n\}$ to x by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *reflexive* if the natural mapping $E \to E^{**}$ is surjective and we write $E = E^{**}$. A Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in U$ with $x \neq y$. A Banach space E is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\| \frac{x + y}{2} \right\| \leq 1 - \delta.$$

It is well known that a uniformly convex Banach space is reflexive and strictly convex.

A Banach space is said to have the *Kadec-Klee property* if, for every sequence $\{x_n\} \subset E$, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$. It is known that a uniformly convex Banach space has the Kadec-Klee property. Let $G = \{g : [0,\infty) \to [0,\infty) : g(0) = 0,\ g$ is continuous, strictly increasing and convex$\}$. We have the following theorem for a uniformly convex Banach space.

Proposition 2.1 ([20]). *A Banach space E is uniformly convex if and only if, for every bounded subset B of E, there exists $g_B \in G$ such that*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g_B(\|x-y\|)$$

for all $x, y \in B$ and $0 \leq \lambda \leq 1$.

A Banach space E is said to be *smooth* if there exists

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

for all $x, y \in U$. In this case, the norm of E is said to be *Gâteaux differentiable.* A Banach space E is said to be *uniformly Gâteaux differentiable* if for each $y \in U$, the limit defined by (2.1) exists uniformly for $x \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for all $x, y \in U$. It is well known that every uniformly smooth Banach space is reflexive and with uniformly Gâteaux differentiable norm. It is also known that E^* is uniformly convex if E is uniformly smooth.

The mapping J of E into 2^{E^*} defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$ is called the *normalized duality mapping.* By the Hahn-Banach theorem, $J(x) \neq \emptyset$ for each $x \in E$. The normalized duality mapping J has the following properties:

(i) if E is smooth, then J is single-valued;
(ii) if E is strictly convex, then J is one-to-one and $\langle x - y, x^* - y^* \rangle > 0$ holds for all $(x, x^*), (y, y^*) \in J$ with $x \neq y$;
(iii) if E is reflexive, then J is surjective;
(iv) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty closed convex subset of E. Throughout this paper, the *Lyapunov functional* $\phi : E \times E \to \mathbb{R}^+$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$; see [1, 7, 12]. It is obvious that

(i) $\phi(x, y) = 0$ if and only if $x = y$;
(ii) $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$ for all $x, y \in E$.

Proposition 2.2 ([7]). *Let E be a smooth and uniformly convex Banach space and $\{x_n\}, \{y_n\} \subset E$ two sequences. If $\phi(x_n, y_n) \to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a smooth Banach space. It is obvious from the definition of ϕ that $\phi(x_n, y_n) \to 0$ whenever $\|x_n - y_n\| \to 0$. By this fact and Proposition 2.2, if $\{x_n\}$ and $\{y_n\}$ are two bounded sequences in a uniformly smooth and uniformly convex Banach space, then

$$\|x_n - y_n\| \to 0 \Leftrightarrow \|Jx_n - Jy_n\| \to 0 \Leftrightarrow \phi(x_n, y_n) \to 0.$$

Proposition 2.3 ([7]). *Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty closed convex subset of E and $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min_{y \in C} \phi(y, x)$.*

Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty closed convex subset of E. Following Alber [1], the *generalized projection* Π_C of E onto C is defined by

$$\Pi_C x = \arg\min_{y \in C} \phi(y, x)$$

for all $x \in E$. We have the following results for generalized projections.

Proposition 2.4 ([1, 7]). *Let E be a smooth Banach space, C a nonempty convex subset of E, $x \in E$ and $x_0 \in C$. Then $x_0 = \Pi_C x$ if and only if $\langle y - x_0, Jx_0 - Jx \rangle \geq 0$ for all $y \in C$.*

Proposition 2.5 ([1, 7]). *Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty closed convex subset of E and $y \in E$. Then*

$$\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$$

for all $x \in C$.

We denoted by $F(T)$ the set of all fixed points of a mapping T.

Proposition 2.6 ([11]). *Let E be a smooth and strictly convex Banach space, C a nonempty closed convex subset of E and T a relatively nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty closed convex subset of E, $\{S_i\}_{i=0}^{\infty}$ a family of mappings of C into inself and $\{\beta_{n,i} : 0 \leq i \leq n\}_{n=0}^{\infty} \subset [0, 1]$ a sequence of real numbers. For any $n \geq 0$, let us define a mapping W_n of C into itself as follows:

$$\begin{aligned}
&U_{n,n+1} = I, \\
&U_{n,n} = \Pi_C J^{-1}\big(\beta_{n,n} J(S_n U_{n,n+1}) + (1 - \beta_{n,n})J\big), \\
&U_{n,n-1} = \Pi_C J^{-1}\big(\beta_{n,n-1} J(S_{n-1} U_{n,n}) + (1 - \beta_{n,n-1})J\big), \\
&\vdots \\
&U_{n,i} = \Pi_C J^{-1}\big(\beta_{n,i} J(S_i U_{n,i+1}) + (1 - \beta_{n,i})J\big), \qquad (2.2) \\
&\vdots \\
&U_{n,1} = \Pi_C J^{-1}\big(\beta_{n,1} J(S_1 U_{n,2}) + (1 - \beta_{n,1})J\big), \\
&W_n = U_{n,0} = J^{-1}\big(\beta_{n,0} J(S_0 U_{n,1}) + (1 - \beta_{n,0})J\big),
\end{aligned}$$

where I is the identity mapping on C. Such a mapping W_n is called the *W-mapping* generated by $\{S_i\}_{i=0}^n$ and $\{\beta_{n,i}\}_{i=0}^n$. We have the following result for the W-mappings; see [9, 10, 16, 18].

Proposition 2.7 ([10])**.** *Let E be a uniformly smooth and strictly convex Banach space, C a nonempty closed convex subset of E and $\{S_i\}_{i=0}^n$ a family of relatively nonexpansive mappings of C into itself such that $\bigcap_{i=0}^n F(S_i) \neq \emptyset$. Let $\{\beta_{n,i}\}_{i=0}^n$ be a sequence of real numbers such that $0 < \beta_{n,0} \leq 1$ and $0 < \beta_{n,i} < 1$ for every $1 \leq i \leq n$. Let $\{U_{n,i}\}_{i=0}^{n+1}$ be a sequence defined by* (2.2) *and W_n the W-mapping generated by $\{S_i\}_{i=0}^n$ and $\{\beta_{n,i}\}_{i=0}^n$. Then the following hold:*
(i) *$F(W_n) = \bigcap_{i=0}^n F(S_i)$;*
(ii) *for every $0 \leq i \leq n$, $x \in C$ and $z \in F(W_n)$, $\phi(z, U_{n,i}x) \leq \phi(z, x)$ and $\phi(z, S_iU_{n,i+1}x) \leq \phi(z, x)$.*

Let E be a smooth and uniformly convex, C a nonempty closed convex subset of E, $\{S_i\}_{i=0}^\infty$ a family of relatively nonexpansive mappings of C into itself and $\{\lambda_{n,i} : 0 \leq i \leq n\}_{n=0}^\infty \subset [0, 1]$ a sequence of real numbers. For any $n \geq 0$, let V_n be a mapping of C into itself defined by

$$V_n = J^{-1} \sum_{i=0}^n \lambda_{n,i} JS_i. \tag{2.3}$$

We have the following result for convex combinations of relatively nonexpansive mappings.

Proposition 2.8 ([10])**.** *Let E be a smooth and uniformly convex Banach space, C a nonempty closed convex subset of E and $\{S_i\}_{i=0}^\infty$ a family of relatively nonexpansive mappings of C into itself such that $\bigcap_{i=0}^\infty F(S_i) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=0}^n \subset [0, 1]$ such that $\sum_{i=0}^n \lambda_{n,i} = 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \lambda_{n,i} > 0$ for each $i \geq 0$. Let V_n be a mapping of C into itself defined by* (2.3)*. Then the following hold:*
(i) *$\bigcap_{n=0}^\infty F(V_n) = \bigcap_{i=0}^\infty F(S_i)$;*
(ii) *for every $n \geq 0$, $x \in C$ and $z \in \bigcap_{i=0}^\infty F(S_i)$, $\phi(z, V_nx) \leq \phi(z, x)$.*

We denoted by $\hat{F}(T)$ the set of all asymptotic fixed points of a mapping T. For solving the equilibrium problem, let us assume that a bifunction $f : C \times C \to \mathbb{R}$ satisfies the following conditions:
(A_1) $f(x, x) = 0$ for all $x \in C$;
(A_2) f is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A_3) f is upper-hemicontinuous, that is, $\limsup_{t\downarrow 0} f(x + t(z - x), y) \leq f(x, y)$ for all $x, y, z \in C$;
(A_4) the function $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Proposition 2.9 ([19])**.** *Let E be a uniformly smooth and strictly convex Banach space, C a nonempty closed convex subset of E and $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A_1)–(A_4). For $r > 0$ and $x \in E$, define a mapping T_r of E into C as follows:*

$$T_r(x) = \{u \in C : f(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \ \forall y \in C\} \tag{2.4}$$

for all $x \in E$. Then the following hold:

(i) T_r *is single-valued;*

(ii) T_r *is a firmly nonexpansive-type mapping, that is,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$$

for all $x, y \in E$;

(iii) $F(T_r) = \hat{F}(T_r) = EP(f)$;

(iv) $EP(f)$ *is a closed convex set of C.*

Remark. It follows from Proposition 2.9 that the mapping T_r defined by (2.4) is relatively nonexpansive. Indeed, by Proposition 2.9 (ii), we have

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle$$

for all $x, y \in C$. Moreover, we obtain

$$\begin{aligned}
&\phi(T_r x, T_r y) + \phi(T_r y, T_r x)\\
&= 2\|T_r x\|^2 - 2\langle T_r x, JT_r y\rangle - 2\langle T_r y, JT_r x\rangle + 2\|T_r y\|^2\\
&= 2\langle T_r x, JT_r x - JT_r y\rangle + 2\langle T_r y, JT_r y - JT_r x\rangle\\
&= 2\langle T_r x - T_r y, JT_r x - JT_r y\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y)\\
&= \|T_r x\|^2 - 2\langle T_r x, Jy\rangle + \|y\|^2 + \|T_r y\|^2 - 2\langle T_r y, Jx\rangle + \|x\|^2\\
&\quad - \|T_r x\|^2 + 2\langle T_r x, Jx\rangle - \|x\|^2 - \|T_r y\|^2 + 2\langle T_r y, Jy\rangle - \|y\|^2\\
&= 2\langle T_r x, Jx - Jy\rangle - 2\langle T_r y, Jx - Jy\rangle\\
&= 2\langle T_r x - T_r y, Jx - Jy\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
\phi(T_r x, T_r y) + \phi(T_r y, T_r x) &\leq \phi(T_r x, y) + \phi(T_r y, x) - \phi(T_r x, x) - \phi(T_r y, y)\\
&\leq \phi(T_r x, y) + \phi(T_r y, x)
\end{aligned}$$

for all $x, y \in C$. Taking $y = p \in F(T_r)$, we obtain

$$\phi(p, T_r x) \leq \phi(p, x).$$

Thus, by Proposition 2.9 (iii), this implies that T_r is relatively nonexpansive.

Proposition 2.10 ([19]). *Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty closed convex subset of E, $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A_1)–(A_4) and $r > 0$. Let T_r be the mapping defined by* (2.4). *Then*

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x)$$

for all $p \in F(T_r)$ and $x \in E$.

For solving the generalized equilibrium problem, let us assume that a nonlinear operator A of C into E^* is an α-inverse strongly monotone and a bifunction $f : C \times C \to \mathbb{R}$ satisfies the conditions (A_1)–(A_4).

Proposition 2.11 ([4]). *Let E be a smooth, strictly convex and reflexive Banach space, C a nonempty closed convex subset of E and A an α-inverse strongly monotone operator of C into E^*. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1)–(A_4) and $g : C \times C \to \mathbb{R}$ a bifunction defined by*

$$g(x, y) = f(x, y) + \langle Ax, y - x\rangle$$

for all $x, y \in C$. Let $r > 0$ and $x \in E$. Then g satisfies (A_1)–(A_4) and there exists $u \in C$ such that

$$g(u, y) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0$$

for all $y \in C$.

Propositions 2.9 and 2.10 can obtain the following proposition.

Proposition 2.12 ([4]). *Let E be a uniformly smooth and strictly convex Banach space, C a nonempty closed convex subset of E, A an α-inverse strongly monotone operator of C into E^* and $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A_1)–(A_4). For any $r > 0$ and $x \in E$, define a mapping K_r of E into C as follows:*

$$K_r(x) = \{u \in C : f(u, y) + \langle Au, y - u\rangle + \frac{1}{r}\langle y - u, Ju - Jx\rangle \geq 0, \ \forall y \in C\}$$

for all $x \in E$. Then the following hold:
(i) *K_r is single-valued;*
(ii) *K_r is a firmly nonexpansive-type mapping, that is,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$$

for all $x, y \in E$;
(iii) *$F(K_r) = \hat{F}(K_r) = EP$;*
(iv) *EP is a closed convex set of C;*
(v) *$\phi(p, K_r x) + \phi(K_r x, x) \leq \phi(p, x)$ for all $p \in F(K_r)$.*
Moreover, the mapping K_r is relatively nonexpansive.

3. Strong convergence theorems of W-mappings

In this section, we prove a strong convergence theorem of W-mappings for finding a common element of the set of solutions for a generalized equilibrium problem and the set of common fixed points of infinite relatively nonexpansive mappings in a Banach space.

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, C a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A_1)–(A_4) and $\{S_i\}_{i=0}^{\infty}$ an infinite family of relatively nonexpansive mappings of C into itself such that $F := \bigcap_{i=0}^{\infty} F(S_i) \cap EP(f) \neq \emptyset$. Let $\{\beta_{n,i}\}_{i=0}^{n} \subset (0, 1)$ be a sequence real numbers such that $\liminf_{n\to\infty} \beta_{n,i}(1 - \beta_{n,i}) > 0$, W_n the W-mapping* *generated by* $\{S_i\}_{i=0}^n$ *and* $\{\beta_{n,i}\}_{i=0}^n$*. Let* $\{x_n\}$ *be the sequence generated by*

$$(3.1)\quad \begin{cases} x_0 \in C, \\ y_n = W_n x_n, \\ u_n \in T_{\gamma_n} y_n, \text{ that is, } f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \text{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases}$$

for $n \geq 0$*, where* $\Pi_{C_n \cap Q_n}$ *is the generalized projection of* E *onto* $C_n \cap Q_n$ *and* $\{\gamma_n\} \subset [r, \infty)$ *for some* $r > 0$*. Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$*, where* Π_F *is the generalized projection of* E *onto* F*.*

Proof. First we prove that $C_n \cap Q_n \subset C$ is closed convex subset for all $n \geq 0$. In fact, it is obvious that C_n is closed, and Q_n is closed and convex for all $n \geq 0$. It follows that C_n is convex for all $n \geq 0$ because $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2.$$

Thus $C_n \cap Q_n$ is closed and convex for all $n \geq 0$.

Next we prove that $F \subset C_n \cap Q_n$ for all $n \geq 0$. Let $u_n = T_{\gamma_n} y_n$ for all $n \geq 0$ and $u \in F$. It follows from Proposition 2.7 (i) that $u \in F(W_n)$ for all $n \geq 0$. We obtain T_{γ_n} is relatively nonexpansive by Proposition 2.9. Since S_i is also relatively nonexpansive for all $n \geq 0$, by Proposition 2.7 (ii), we have

$$\begin{aligned} \phi(u, u_n) &= \phi(u, T_{\gamma_n} y_n) \leq \phi(u, y_n) = \phi(u, W_n x_n) \\ &= \phi\Big(u, J^{-1}\big(\beta_{n,0} J(S_0 U_{n,1} x_n) + (1 - \beta_{n,0}) Jx_n\big)\Big) \\ &= \|u\|^2 - 2\langle u, \beta_{n,0} J(S_0 U_{n,1} x_n) + (1 - \beta_{n,0}) Jx_n\rangle \\ &\quad + \|\beta_{n,0} J(S_0 U_{n,1} x_n) + (1 - \beta_{n,0}) Jx_n\|^2 \\ &\leq \|u\|^2 - 2\beta_{n,0}\langle u, J(S_0 U_{n,1} x_n)\rangle - 2(1 - \beta_{n,0})\langle u, Jx_n\rangle \\ &\quad + \beta_{n,0}\|S_0 U_{n,1} x_n\|^2 + (1 - \beta_{n,0})\|x_n\|^2 \\ &= \beta_{n,0}\phi(u, S_0 U_{n,1} x_n) + (1 - \beta_{n,0})\phi(u, x_n) \\ &\leq \beta_{n,0}\phi(u, x_n) + (1 - \beta_{n,0})\phi(u, x_n) = \phi(u, x_n). \end{aligned} \tag{3.2}$$

This implies that $u \in C_n$ and so $F \subset C_n$ for all $n \geq 0$. By induction, now we prove that $F \subset C_n \cap Q_n$ for all $n \geq 0$. In fact, since $Q_0 = C$, we have $F \subset C_0 \cap Q_0$. Suppose that $F \subset C_k \cap Q_k$ for some $k \geq 0$. Then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \Pi_{C_k \cap Q_k} x_0$. By the definition of x_{k+1}, we have

$$\langle x_{k+1} - z, Jx_0 - Jx_{k+1}\rangle \geq 0 \tag{3.3}$$

for all $z \in C_k \cap Q_k$. Since $F \subset C_k \cap Q_k$, we obtain (3.3) for all $z \in F$. This shows that $z \in Q_{k+1}$, and so $F \subset Q_{k+1}$. Therefore $F \subset C_n \cap Q_n$ for all $n \geq 0$.

We prove that $\{x_n\}$ is bounded. By the definition of Q_n and Proposition 2.4, we have $x_n = \Pi_{Q_n} x_0$ for all $n \geq 0$. Hence, by Proposition 2.5,

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \leq \phi(u, x_0)$$

for all $u \in F \subset Q_n$ and $n \geq 0$. This implies that $\{\phi(x_n, x_0)\}$ is bounded, and so $\{x_n\}$ and $\{u_n\}$ are bounded in C.

Next we prove that $\|x_n - u_n\| \to 0$ and $\|Jx_n - Ju_n\| \to 0$. Since $x_n = \Pi_{Q_n} x_0$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$, we have $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$ for all $n \geq 0$. This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing, and so there exists the limit $\lim_{n\to\infty} \phi(x_n, x_0)$. By Proposition 2.5, we have

$$\begin{aligned}\phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{Q_n} x_0)\\ &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0)\end{aligned}$$

for all $n \geq 0$. This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.4}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, by the definition of C_n, we obtain

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \tag{3.5}$$

Since E is smooth and uniformly convex, from (3.4), (3.5) and Proposition 2.2 we have

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.6}$$

Since J is uniformly continuous on any bounded subset of E, we obtain

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.7}$$

Next we prove that $\omega(\{x_n\}) \subset F$, where $\omega(\{x_n\})$ is the set consisting all of the weak limits points of $\{x_n\}$. In fact, for any $p \in \omega(\{x_n\})$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup p$. We shall prove that $p \in \bigcap_{i=0}^{\infty} F(S_i)$. We have

$$\begin{aligned}\phi(u, x_n) - \phi(u, u_n) &= \|x_n\|^2 - \|u_n\|^2 + 2\langle u, Ju_n - Jx_n\rangle\\ &\leq \big|\|x_n\| - \|u_n\|\big|(\|x_n\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_n\|\\ &\leq \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|u\|\|Ju_n - Jx_n\|\end{aligned} \tag{3.8}$$

for all $n \geq 0$. From (3.6) and (3.7) we obtain

$$\lim_{n\to\infty} \big(\phi(u, x_n) - \phi(u, u_n)\big) = 0. \tag{3.9}$$

By Proposition 2.7 (ii), we have

$$\phi(u, U_{n,i} x_n) \leq \phi(u, x_n) \quad \text{and} \quad \phi(u, S_i U_{n,i+1} x_n) \leq \phi(u, U_{n,i+1} x_n) \leq \phi(u, x_n)$$

for each $0 \leq i \leq n$. Thus $\{S_i U_{n,i+1} x_n\}_{n \geq i}$ and $\{U_{n,i} x_n\}_{n \geq i}$ are bounded sequences in C for all $i \geq 0$. By Propositions 2.1, 2.5 and 2.7 (ii), we have

$$\begin{aligned}\phi(u, U_{n,i} x_n) &\leq \phi\Big(u, J^{-1}\big(\beta_{n,i} J(S_i U_{n,i+1} x_n) + (1 - \beta_{n,i}) J x_n\big)\Big)\\ &\quad - \phi\Big(U_{n,i} x_n, J^{-1}\big(\beta_{n,i} J(S_i U_{n,i+1} x_n) + (1 - \beta_{n,i}) J x_n\big)\Big)\\ &= \|u\|^2 - 2\langle u, \beta_{n,i} J(S_i U_{n,i+1} x_n) + (1 - \beta_{n,i}) J x_n\rangle\end{aligned}$$

$$
\begin{aligned}
&+ \|\beta_{n,i} J(S_i U_{n,i+1} x_n) + (1-\beta_{n,i}) J x_n\|^2 \\
&- \phi\Big(U_{n,i} x_n, J^{-1}\big(\beta_{n,i} J(S_i U_{n,i+1} x_n) + (1-\beta_{n,i}) J x_n\big)\Big) \\
= {}& \|u\|^2 - 2\langle u, \beta_{n,i} J(S_i U_{n,i+1} x_n) + (1-\beta_{n,i}) J x_n\rangle \\
&+ \beta_{n,i}\|S_i U_{n,i+1} x_n\|^2 + (1-\beta_{n,i})\|x_n\|^2 \\
&- \beta_{n,i}(1-\beta_{n,i}) g(\|J(S_i U_{n,i+1} x_n) - J x_n\|) \\
&- \phi\Big(U_{n,i} x_n, J^{-1}\big(\beta_{n,i} J(S_i U_{n,i+1} x_n) + (1-\beta_{n,i}) J x_n\big)\Big) \\
= {}& \beta_{n,i}\phi(u, S_i U_{n,i+1} x_n) + (1-\beta_{n,i})\phi(u, x_n) \\
&- \beta_{n,i}(1-\beta_{n,i}) g(\|J(S_i U_{n,i+1} x_n) - J x_n\|) \\
&- \phi\Big(U_{n,i} x_n, J^{-1}\big(\beta_{n,i} J(S_i U_{n,i+1} x_n) + (1-\beta_{n,i}) J x_n\big)\Big) \\
\le {}& \beta_{n,i}\phi(u, U_{n,i+1} x_n) + (1-\beta_{n,i})\phi(u, x_n) \\
&- \beta_{n,i}(1-\beta_{n,i}) g(\|J(S_i U_{n,i+1} x_n) - J x_n\|) \\
&- \phi\Big(U_{n,i} x_n, J^{-1}\big(\beta_{n,i} J(S_i U_{n,i+1} x_n) + (1-\beta_{n,i}) J x_n\big)\Big)
\end{aligned}
$$

for some $g \in G$ and for all $1 \le i \le n$. This implies that

$$
\begin{aligned}
&\phi(u, u_n) \le \phi(u, y_n) = \phi(u, W_n x_n) = \phi(u, U_{n,0} x_n) \\
&= \|u\|^2 - 2\beta_{n,0}\langle u, J(S_0 U_{n,1} x_n)\rangle - 2(1-\beta_{n,0})\langle u, J x_n\rangle \\
&\quad + \|\beta_{n,0} J(S_0 U_{n,1} x_n) + (1-\beta_{n,0}) J x_n\|^2 \\
&\le \beta_{n,0}\phi(u, U_{n,1} x_n) + (1-\beta_{n,0})\phi(u, x_n) \\
&\quad - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} x_n) - J x_n\|) \\
&\le \beta_{n,0}\Big\{\beta_{n,1}\phi(u, U_{n,2} x_n) + (1-\beta_{n,1})\phi(u, x_n) \\
&\quad - \beta_{n,1}(1-\beta_{n,1}) g(\|J(S_1 U_{n,2} x_n) - J x_n\|) \\
&\quad - \phi\Big(U_{n,1} x_n, J^{-1}\big(\beta_{n,1} J(S_1 U_{n,2} x_n) + (1-\beta_{n,1}) J x_n\big)\Big)\Big\} \\
&\quad + (1-\beta_{n,0})\phi(u, x_n) - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} x_n) - J x_n\|) \\
&\le \cdots \\
&\le \phi(u, x_n) - \beta_{n,0}(1-\beta_{n,0}) g(\|J(S_0 U_{n,1} x_n) - J x_n\|) \\
&\quad - \beta_{n,0}\beta_{n,1}(1-\beta_{n,1}) g(\|J(S_1 U_{n,2} x_n) - J x_n\|) - \cdots \\
&\quad - \beta_{n,0}\beta_{n,1}\cdots\beta_{n,n}(1-\beta_{n,n}) g(\|J(S_n U_{n,n+1} x_n) - J x_n\|) \\
&\quad - \beta_{n,0}\phi\Big(U_{n,1} x_n, J^{-1}\big(\beta_{n,1} J(S_1 U_{n,2} x_n) + \big(1-\beta_{n,1}) J x_n\big)\Big) - \cdots \\
&\quad - \beta_{n,0}\beta_{n,1}\cdots\beta_{n,n-1} \\
&\qquad \times \phi\Big(U_{n,n} x_n, J^{-1}\big(\beta_{n,n} J(S_n U_{n,n+1} x_n) + \big(1-\beta_{n,n}) J x_n\big)\Big)
\end{aligned}
\tag{3.10}
$$

for all $n \ge 0$. From (3.9), (3.10) and $\liminf_{n\to\infty} \beta_{n,i}(1-\beta_{n,i}) > 0$ we obtain

$$\lim_{n\to\infty} g(\|J(S_i U_{n,i+1} x_n) - J x_n\|) = 0,$$

$$\lim_{n\to\infty} \phi\Big(U_{n,i+1}x_n, J^{-1}\big(\beta_{n,i+1}J(S_{i+1}U_{n,i+2}x_n) + (1-\beta_{n,i+1})Jx_n\big)\Big) = 0$$

for all $i \geq 0$. By the definition of g and Proposition 2.2, we have

$$\lim_{n\to\infty} \|J(S_iU_{n,i+1}x_n) - Jx_n\| = 0, \tag{3.11}$$

$$\lim_{n\to\infty} \big\|U_{n,i+1}x_n - J^{-1}\big(\beta_{n,i+1}J(S_{i+1}U_{n,i+2}x_n) + (1-\beta_{n,i+1})Jx_n\big)\big\| = 0. \tag{3.12}$$

From (3.11) we obtain

$$\begin{aligned}&\lim_{n\to\infty} \|\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n - Jx_n\|\\ &= \lim_{n\to\infty} \beta_{n,i}\|J(S_iU_{n,i+1}x_n) - Jx_n\| = 0.\end{aligned} \tag{3.13}$$

Since J^{-1} is also norm-to-norm continuous on bounded sets, from (3.11) and (3.13) we have

$$\lim_{n\to\infty} \|S_iU_{n,i+1}x_n - x_n\| = 0, \tag{3.14}$$

$$\lim_{n\to\infty} \big\|J^{-1}\big(\beta_{n,i}J(S_iU_{n,i+1}x_n) + (1-\beta_{n,i})Jx_n\big) - x_n\big\| = 0 \tag{3.15}$$

for all $i \geq 0$. From (3.12) and (3.15) we obtain

$$\lim_{n\to\infty} \|U_{n,i+1}x_n - x_n\| = 0 \tag{3.16}$$

for all $i \geq 0$. Since $x_{n_k} \rightharpoonup p$, we have $U_{n_k,i+1}x_{n_k} \rightharpoonup p$ for all $i \geq 0$. From (3.14) and (3.16) we obtain

$$\lim_{n\to\infty} \|S_iU_{n,i+1}x_n - U_{n,i+1}x_n\| = 0$$

for each $i \geq 0$. Since $U_{n_k,i+1}x_{n_k} \rightharpoonup p$ and S_i is relatively nonexpansive, we have $p \in \hat{F}(S_i) = F(S_i)$ for all $i \geq 0$. Hence $p \in \bigcap_{i=0}^{\infty} F(S_i)$. Now we shall prove that $p \in EP(f)$. From (3.2), (3.9) and Proposition 2.10 we have

$$\begin{aligned}\phi(u_n, y_n) = \phi(T_{\gamma_n}y_n, y_n) &\leq \phi(u, y_n) - \phi(u, T_{\gamma_n}y_n)\\ &\leq \phi(u, x_n) - \phi(u, u_n) \to 0.\end{aligned}$$

It follows from Proposition 2.2 that

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.17}$$

Since $x_{n_k} \rightharpoonup p$, it follows from (3.6) and (3.17) that $u_{n_k} \rightharpoonup p$ and $y_{n_k} \rightharpoonup p$. Since J is uniformly continuous on any bounded set of E, from (3.17) we obtain $\|Ju_n - Jy_n\| \to 0$. By the assumption that $\gamma_n \geq r$, we have

$$\lim_{n\to\infty} \frac{1}{\gamma_n}\|Ju_n - Jy_n\| = 0. \tag{3.18}$$

Since $u_n = T_{\gamma_n}y_n$, we obtain

$$f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \tag{3.19}$$

for all $y \in C$. Replacing n by n_k in (3.19), from (A_2) we deduce

$$\frac{1}{\gamma_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k}\rangle \geq -f(u_{n_k}, y) \geq f(y, u_{n_k}) \tag{3.20}$$

for all $y \in C$. Since $y \mapsto f(x,y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in (3.20), from (3.18) and (A_4) we have $f(y,p) \leq 0$ for all $y \in C$. For $t \in (0,1]$ and $y \in C$, letting $y_t = ty + (1-t)p$, then $y_t \in C$ and $f(y_t, p) \leq 0$. From (A_1) and (A_4) we obtain

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

Dividing by t, we have $f(y_t, y) \geq 0$ for all $y \in C$. Letting $t \downarrow 0$, from (A_3) we obtain $f(p,y) \geq 0$. Therefore $p \in EP(f)$, and so $p \in F$. This shows that $\omega(\{x_n\}) \subset F$.

Finally, we have prove that $\omega(\{x_n\})$ is a singleton and $x_n \to \Pi_F x_0$. Let $w = \Pi_F x_0$. Since $w \in F \subset C_n \cap Q_n$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$, we have

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0)$$

for all $n \geq 0$. Since the norm is weakly lower semicontinuous, this implies that

$$\begin{aligned} \phi(p, x_0) &= \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 \\ &\leq \liminf_{k\to\infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2) \\ &= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\ &\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0). \end{aligned} \tag{3.21}$$

It follows from the definition of w and (3.21) that $p = w$. This implies that $\omega(\{x_n\})$ is a singleton and $\phi(x_{n_k}, x_0) \to \phi(w, x_0)$. Therefore

$$\begin{aligned} 0 &= \lim_{k\to\infty} \big(\phi(x_{n_k}, x_0) - \phi(w, x_0)\big) \\ &= \lim_{k\to\infty} (\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0\rangle) \\ &= \lim_{k\to\infty} \|x_{n_k}\|^2 - \|w\|^2, \end{aligned}$$

that is,

$$\lim_{k\to\infty} \|x_{n_k}\|^2 = \|w\|^2. \tag{3.22}$$

Since E is uniformly convex, it has the Kadec-Klee property. It follows from (3.22) and $x_{n_k} \rightharpoonup w$ that $x_{n_k} \to w = \Pi_F x_0$. Since $\omega(\{x_n\})$ is a singleton, we have $x_n \to \Pi_F x_0$. □

The following theorems can be obtained by Theorem 3.1.

Theorem 3.2 ([19]). *Let E be a uniformly smooth and uniformly convex Banach space, C a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A_1)–(A_4) and S a relatively nonexpansive mapping from C into itself such that $F := F(S) \cap EP(f) \neq \emptyset$. Let $\{\alpha_n\} \subset [0,1]$ be a sequence real numbers* *such that* $\lim_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *Let* $\{x_n\}$ *be the sequence generated by*

$$(3.23)\qquad \begin{cases} x_0 \in C, \\ y_n = J^{-1}(\alpha_n JSx_n + (1-\alpha_n)Jx_n), \\ u_n \in C \text{ such that } f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \text{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases}$$

for $n \geq 0$, *where* $\Pi_{C_n\cap Q_n}$ *is the generalized projection of* E *onto* $C_n \cap Q_n$ *and* $\{\gamma_n\} \subset [r, \infty)$ *for some* $r > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* Π_F *is the generalized projection of* E *onto* F.

Proof. Let $S_n = S$, $\beta_{n,0} = \alpha_n$ and $\{\beta_{n,i}\}_{i=1}^n = \{0\}$ for all $n \geq 0$ in Theorem 3.1. This shows that (3.1) is equivalent to (3.23). Therefore, the conclusion of Theorem 3.2 can be deduced from Theorem 3.1. □

Theorem 3.3. *Let* E *be a uniformly smooth and uniformly convex Banach space,* C *a nonempty closed convex subset of* E. *Let* A *be an* α*-inverse strongly monotone operator of* C *into* E^*, $f : C \times C \to \mathbb{R}$ *a bifunction satisfying* (A_1)–(A_4) *and* $\{S_i\}_{i=0}^\infty$ *an infinite family of relatively nonexpansive mappings of* C *into itself such that* $F := \bigcap_{i=0}^\infty F(S_i) \cap EP \neq \emptyset$. *Let* $\{\beta_{n,i}\}_{i=0}^n$ *be a sequence real numbers such that* $\liminf_{n\to\infty} \beta_{n,i}(1-\beta_{n,i}) > 0$, W_n *the* W*-mapping generated by* $\{S_i\}_{i=0}^n$ *and* $\{\beta_{n,i}\}_{i=0}^n$. *Let* $\{x_n\}$ *be the sequence generated by*

$$(3.24)\qquad \begin{cases} x_0 \in C, \\ y_n = W_n x_n, \\ u_n \in K_{\gamma_n} y_n, \ \text{ that is,} \\ \quad f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \text{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases}$$

for $n \geq 0$, *where* $\Pi_{C_n\cap Q_n}$ *is the generalized projection of* E *onto* $C_n \cap Q_n$ *and* $\{\gamma_n\} \subset [r, \infty)$ *for some* $r > 0$. *Then* $\{x_n\}$ *converges strongly to* $\Pi_F x_0$, *where* Π_F *is the generalized projection of* E *onto* F.

Proof. Let $g(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n\rangle$. By Propositions 2.11 and 2.12, (3.24) is equivalent to (3.1) in Theorem 3.1. Therefore, the conclusion of Theorem 3.3 can be deduced from Theorem 3.1. □

4. Strong convergence theorems of convex combinations

In this section, we prove strong convergence theorems of convex combinations for finding a common element of the set of solutions for a generalized equilibrium problem and the set of common fixed points of infinite relatively nonexpansive mappings in a Banach space.

Theorem 4.1. *Let E be a uniformly smooth and uniformly convex Banach space, C a nonempty closed convex subset of E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A_1)–(A_4) and $\{S_i\}_{i=0}^{\infty}$ an infinite family of relatively nonexpansive mappings of C into itself such that $F := \bigcap_{i=0}^{\infty} F(S_i) \cap EP(f) \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=0}^{n} \subset [0, 1)$ be a sequence real numbers such that $\sum_{i=0}^{n} \lambda_{n,i} = 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \lambda_{n,i} > 0$ for each $i \geq 0$, and V_n the mapping defined by (2.3). Let $\{x_n\}$ be the sequence generated by*

$$(4.1) \qquad \begin{cases} x_0 \in C, \\ y_n = V_n x_n, \\ u_n \in T_{\gamma_n} y_n, \text{ that is, } f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \text{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases}$$

for $n \geq 0$, where $\Pi_{C_n\cap Q_n}$ is the generalized projection of E onto $C_n \cap Q_n$ and $\{\gamma_n\} \subset [r, \infty)$ for some $r > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. First we prove that $C_n \cap Q_n \subset C$ is closed convex subset for all $n \geq 0$. In fact, it is obvious that C_n is closed, and Q_n is closed and convex for all $n \geq 0$. It follows that C_n is convex for all $n \geq 0$ because $\phi(z, u_n) \leq \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Ju_n\rangle \leq \|x_n\|^2 - \|u_n\|^2.$$

Thus $C_n \cap Q_n$ is closed and convex for all $n \geq 0$.

Next we prove that $F \subset C_n \cap Q_n$ for all $n \geq 0$. Let $u_n = T_{\gamma_n} y_n$ for all $n \geq 0$ and $u \in F$. It follows from Proposition 2.8 (i) and Proposition 2.9 (iii) that $u \in \bigcap_{n=0}^{\infty} F(V_n) \cap F(T_{\gamma_n})$. We have T_{γ_n} is relatively nonexpansive by Proposition 2.9. By Proposition 2.8 (ii), we have

$$(4.2) \qquad \phi(u, u_n) = \phi(u, T_{\gamma_n} y_n) \leq \phi(u, y_n) = \phi(u, V_n x_n) \leq \phi(u, x_n).$$

This implies that $u \in C_n$ and so $F \subset C_n$ for all $n \geq 0$. By induction, now we prove that $F \subset C_n \cap Q_n$ for all $n \geq 0$. In fact, since $Q_0 = C$, we have $F \subset C_0 \cap Q_0$. Suppose that $F \subset C_k \cap Q_k$ for some $k \geq 0$. Then there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = \Pi_{C_k\cap Q_k} x_0$. By the definition of x_{k+1}, we have

$$(4.3) \qquad \langle x_{k+1} - z, Jx_0 - Jx_{k+1}\rangle \geq 0$$

for all $z \in C_k \cap Q_k$. Since $F \subset C_k \cap Q_k$, we have (4.3) for all $z \in F$. This shows that $z \in Q_{k+1}$, and so $F \subset Q_{k+1}$. Therefore $F \subset C_n \cap Q_n$ for all $n \geq 0$.

Next we prove that $\{x_n\}$ is bounded. By the definition of Q_n, we have $x_n = \Pi_{Q_n} x_0$ for all $n \geq 0$. Hence, by Proposition 2.5,

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(u, x_0) - \phi(u, \Pi_{Q_n} x_0) \leq \phi(u, x_0)$$

for all $u \in F \subset Q_n$ and $n \geq 0$. This implies that $\{\phi(x_n, x_0)\}$ is bounded, and so $\{x_n\}$ and $\{u_n\}$ are bounded in C. Since $x_{n+1} = \Pi_{C_n\cap Q_n} x_0$ and $x_n = \Pi_{Q_n} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$$

for all $n \geq 0$. This implies that $\{\phi(x_n, x_0)\}$ is nondecreasing, and so there exists the limit $\lim_{n\to\infty} \phi(x_n, x_0)$. By Proposition 2.5, we deduce

$$\begin{aligned}\phi(x_{n+1}, x_n) = \phi(x_{n+1}, \Pi_{Q_n} x_0) &\leq \phi(x_{n+1}, x_0) - \phi(\Pi_{Q_n} x_0, x_0)\\ &= \phi(x_{n+1}, x_0) - \phi(x_n, x_0)\end{aligned}$$

for all $n \geq 0$. This implies that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{4.4}$$

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$, by the definition of C_n, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n). \tag{4.5}$$

Since E is smooth and uniformly convex, from (4.4), (4.5) and Proposition 2.2 we obtain

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$$

and

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{4.6}$$

Since J is uniformly norm-to-norm continuous on bounded subsets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{4.7}$$

Next we prove that $\|S_l x_n - x_n\| \to 0$ for all $l \geq 0$. By the definition of $\lambda_{n,i}$, we have $1-\lambda_{n,l} = \sum_{i=0,1,\ldots,n, i\neq l} \lambda_{n,i}$. For large enough $n \geq 0$ and $0 \leq l \leq n$, Proposition 2.1 implies that

$$\begin{aligned}
&\phi(u, u_n) \leq \phi(u, y_n) = \phi(u, V_n x_n)\\
&= \|u\|^2 - 2\sum_{i=0}^{n} \lambda_{n,i} \langle u, J(S_i x_n)\rangle + \left\| J^{-1} \sum_{i=0}^{n} \lambda_{n,i} J(S_i x_n)\right\|^2\\
&= \|u\|^2 - 2\sum_{i=0}^{n} \lambda_{n,i} \langle u, J(S_i x_n)\rangle\\
&\quad + \left\| \lambda_{n,l} J(S_l x_n) + (1-\lambda_{n,l}) \frac{\sum_{i=0,1,\ldots,n,i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\|^2\\
&\leq \|u\|^2 - 2\sum_{i=0}^{n} \lambda_{n,i} \langle u, J(S_i x_n)\rangle + \lambda_{n,l} \|S_l x_n\|^2\\
&\quad + (1-\lambda_{n,l}) \left\| \frac{\sum_{i=0,1,\ldots,n,i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\|^2\\
&\quad - \lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{i=0,1,\ldots,n,i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}} \right\| \right)\\
&= \|u\|^2 - 2\sum_{i=0}^{n} \lambda_{n,i} \langle u, J(S_i x_n)\rangle + \sum_{i=0}^{n} \lambda_{n,i} \|S_i x_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{i=0,1,\dots,n, i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}}\right\|\right) \\
&= \phi(u, S_i x_n) - \lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{i=0,1,\dots,n, i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}}\right\|\right) \\
&\le \phi(u, x_n) - \lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{i=0,1,\dots,n, i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}}\right\|\right)
\end{aligned}
$$

for some $g \in G$. Thus

$$
\begin{aligned}
&\lambda_{n,l}(1-\lambda_{n,l}) g\left(\left\| J(S_l x_n) - \frac{\sum_{i=0,1,\dots,n, i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}}\right\|\right) \\
&\le \phi(u, x_n) - \phi(u, u_n) \\
&= \|x_n\|^2 - \|u_n\|^2 + 2\langle u, Ju_n - Jx_n\rangle \\
&\le 2\|u\| \cdot \|Ju_n - Jx_n\| + (\|x_n\| + \|u_n\|)\|x_n - u_n\|.
\end{aligned}
$$

This implies that, together with (4.6) and (4.7),

$$
\lim_{n\to\infty} \left\| J(S_l x_n) - \frac{\sum_{i=0,1,\dots,n, i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}}\right\| = 0 \tag{4.8}
$$

for all $l \ge 0$. From (4.2), (4.6), (4.7) and Proposition 2.10 we have

$$
\begin{aligned}
\phi(u_n, y_n) &= \phi(T_{\gamma_n} y_n, y_n) \\
&\le \phi(u, y_n) - \phi(u, T_{\gamma_n} y_n) \\
&\le \phi(u, x_n) - \phi(u, u_n) \\
&\le \|x_n - u_n\|(\|x_n\| + \|y_n\|) + 2\|u\|\|Ju_n - Jx_n\| \to 0.
\end{aligned}
$$

This implies that

$$
\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{4.9}
$$

From (4.6) and (4.9) we obtain

$$
\lim_{n\to\infty} \|x_n - y_n\| \le \lim_{n\to\infty} \{\|x_n - u_n\| + \|u_n - y_n\|\} = 0.
$$

Since J is uniformly norm-to-norm continuous on bounded subsets, we have

$$
\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{4.10}
$$

Since

$$
\begin{aligned}
\|Jx_n - J(S_l x_n)\| &\le \|Jx_n - J(V_n x_n)\| + \|J(S_l x_n) - J(V_n x_n)\| \\
&= \|Jx_n - Jy_n\| + \left\| J(S_l x_n) - \sum_{i=0}^{n} \lambda_{n,i} J(S_i x_n)\right\| \\
&= \|Jx_n - Jy_n\| \\
&\quad + (1-\lambda_{n,l})\left\| J(S_l x_n) - \frac{\sum_{i=0,1,\dots,n, i\neq l} \lambda_{n,i} J(S_i x_n)}{1-\lambda_{n,l}}\right\|
\end{aligned}
$$

for large enough $n \geq 0$, from (4.8) and (4.10) we obtain

$$\lim_{n\to\infty} \|Jx_n - J(S_l x_n)\| = 0.$$

Since J^{-1} is also uniformly norm-to-norm continuous on bounded subsets, we have

$$\lim_{n\to\infty} \|x_n - S_l x_n\| = 0 \tag{4.11}$$

for all $l \geq 0$.

Next we prove that $\omega(\{x_n\}) \subset F$, where $\omega(\{x_n\})$ is the set consisting all of the weak limits points of $\{x_n\}$. In fact, for any $p \in \omega(\{x_n\})$, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \rightharpoonup p$. Since S_i is relatively nonexpansive, (4.11) implies $p \in \bigcap_{i=0}^{\infty} \hat{F}(S_i) = \bigcap_{i=0}^{\infty} F(S_i)$. Now we prove that $p \in EP(f)$. Since $x_{n_k} \rightharpoonup p$, it follows from (4.6) and (4.9) that $u_{n_k} \rightharpoonup p$ and $y_{n_k} \rightharpoonup p$. Since J is uniformly continuous on any bounded set of E, from (4.9) we have $\|Ju_n - Jy_n\| \to 0$. By the assumption that $\gamma_n > r$, we have

$$\lim_{n\to\infty} \frac{1}{\gamma_n}\|Ju_n - Jy_n\| = 0. \tag{4.12}$$

Since $u_n = T_{\gamma_n} y_n$, we obtain

$$f(u_n, y) + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \tag{4.13}$$

for all $y \in C$. Replacing n by n_k in (4.13), from (A_2) we have

$$\frac{1}{\gamma_{n_k}}\langle y - u_{n_k}, Ju_{n_k} - Jy_{n_k}\rangle \geq -f(u_{n_k}, y) \geq f(y, u_{n_k}) \tag{4.14}$$

for all $y \in C$. Since $y \mapsto f(x, y)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. Letting $n_k \to \infty$ in (4.14), from (4.12) and (A_4) we obtain $f(y, p) \leq 0$ for all $y \in C$. For $t \in (0, 1]$ and $y \in C$, letting $y_t = ty + (1-t)p$, then $y_t \in C$ and $f(y_t, p) \leq 0$. From (A_1) and (A_4) we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y).$$

Dividing by t, we obtain $f(y_t, y) \geq 0$ for all $y \in C$. Letting $t \downarrow 0$, from (A_3) we have $f(p, y) \geq 0$ for all $y \in C$. Therefore $p \in EP(f)$, and so $p \in F$. This shows that $\omega(\{x_n\}) \subset F$.

Finally, we have prove that $\omega(\{x_n\})$ is a singleton and $x_n \to \Pi_F x_0$. Let $w = \Pi_F x_0$. From $w \in F \subset C_n \cap Q_n$ and $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$ we have

$$\phi(x_{n+1}, x_0) \leq \phi(w, x_0)$$

for all $n \geq 0$. Since the norm is weakly lower semicontinuous, this implies that

$$\begin{aligned}
\phi(p, x_0) &= \|p\|^2 - 2\langle p, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{k\to\infty} (\|x_{n_k}\|^2 - 2\langle x_{n_k}, Jx_0\rangle + \|x_0\|^2) \\
&= \liminf_{k\to\infty} \phi(x_{n_k}, x_0) \\
&\leq \limsup_{k\to\infty} \phi(x_{n_k}, x_0) \leq \phi(w, x_0). \qquad (4.15)
\end{aligned}$$

It follows from the definition of w and (4.15) that $p = w$. This implies that $\omega(\{x_n\})$ is a singleton and $\phi(x_{n_k}, x_0) \to \phi(w, x_0)$. Therefore

$$\begin{aligned} 0 &= \lim_{k\to\infty} \big(\phi(x_{n_k}, x_0) - \phi(w, x_0)\big) \\ &= \lim_{k\to\infty} (\|x_{n_k}\|^2 - \|w\|^2 - 2\langle x_{n_k} - w, Jx_0\rangle) \\ &= \lim_{k\to\infty} \|x_{n_k}\|^2 - \|w\|^2, \end{aligned}$$

that is,

$$\lim_{k\to\infty} \|x_{n_k}\|^2 = \|w\|^2. \tag{4.16}$$

Since E is uniformly convex, it has the Kadec-Klee property. It follows from (4.16) and $x_{n_k} \rightharpoonup w$ that $x_{n_k} \to w = \Pi_F x_0$. Since $\omega(\{x_n\})$ is a singleton, we have $x_n \to \Pi_F x_0$. □

The following theorem can be obtained by Theorem 4.1.

Theorem 4.2. *Let E be a uniformly smooth and uniformly convex Banach space, C a nonempty closed convex subset of E. Let A be an α-inverse strongly monotone operator of C into E^*, $f : C \times C \to \mathbb{R}$ a bifunction satisfying (A_1)–(A_4) and $\{S_i\}_{i=0}^{\infty}$ an infinite family of relatively nonexpansive mappings of C into itself such that $F := \bigcap_{i=0}^{\infty} F(S_i) \cap EP \neq \emptyset$. Let $\{\lambda_{n,i}\}_{i=0}^{n} \subset [0,1)$ be a sequence real numbers such that $\sum_{i=0}^{n} \lambda_{n,i} = 1$ for all $n \geq 0$ and $\lim_{n\to\infty} \lambda_{n,i} > 0$ for each $i \geq 0$, and V_n the mapping defined by (2.3). Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in C, \\ y_n = V_n x_n, \\ u_n \in K_{\gamma_n} y_n, \ \text{that is,} \\ \qquad f(u_n, y) + \langle Au_n, y - u_n\rangle + \frac{1}{\gamma_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \text{ for all } y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}; \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}; \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \end{cases} \tag{4.17}$$

for $n \geq 0$, where $\Pi_{C_n \cap Q_n}$ is the generalized projection of E onto $C_n \cap Q_n$ and $\{\gamma_n\} \subset [r, \infty)$ for some $r > 0$. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. Let $g(u_n, y) = f(u_n, y) + \langle Au_n, y - u_n\rangle$. By Propositions 2.11 and 2.12, (4.17) is equivalent to (4.1) in Theorem 4.1. Therefore, the conclusion of Theorem 4.2 can be deduced from Theorem 4.1. □

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