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ON SOME QUASILINEAR EQUATIONS OF KIRCHHOFF TYPE: A GENERALIZED ORLICZ-SOBOLEV SPACES SETTING

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ABSTRACT. In this paper we study some nonlocal quasilinear problems with a nonhomogeneous divergence operator on the whole space $\mathbb{R}^N$ and involving variable exponents. Employing variational (Ricceri's theorems) and topological (topological degree theory for (S_+) type mappings) techniques, we prove existence and multiplicity results of nonnegative solutions in generalized Orlicz-Sobolev spaces setting.

1. INTRODUCTION AND PRELIMINARIES

In the present paper, we are concerned with some nonhomogeneous quasilinear problem of Kirchhoff type. More precisely, we deal with problems of the model

$$(M)\quad -\Lambda_0\left(\int_{\mathbb{R}^N}\Phi\left(x,|\nabla u(x)|\right)dx,\int_{\mathbb{R}^N}\Phi\left(x,|u(x)|\right)dx\right)\left(\operatorname{div}\left(a\left(x,|\nabla u(x)|\right)\nabla u\right)-a\left(x,|u(x)|\right)u\right)$$
$$=\Lambda_1\left(\int_{\mathbb{R}^N}\Phi\left(x,|\nabla u(x)|\right)dx,\int_{\mathbb{R}^N}\Phi\left(x,|u(x)|\right)dx,\int_{\mathbb{R}^N}\left(\int_0^{u(x)}f(x,t)dt\right)dx\right)f(x,u)$$
$$+h(x)\text{ in }\mathbb{R}^N,\quad N\geq 3,$$

where $a(\cdot,\cdot):\mathbb{R}^N\times[0,+\infty[\to\mathbb{R}$ is some function such that: for all $x\in\mathbb{R}^N$, the mapping $\varphi(x,\cdot):\mathbb{R}\to\mathbb{R}$ defined by $\varphi(x,t)=\begin{cases} a(x,|t|)t & \text{if } t\neq 0\\ 0 & \text{if } t=0\end{cases}$ is odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$ and $\Phi(\cdot,\cdot):\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is defined by $\Phi(x,t)=\int_0^t\varphi(x,s)ds$. Moreover, the function $f(\cdot,\cdot):\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is assumed to be a Carathéodory function.

Problems involving nonhomogeneous differential operators have be given a great interest in recent years. This can be explained by the fact that such type of problems has various applications in many fields of mathematics, such as mathematical physics, approximation theory, differential geometry, stochastic analysis, prediction analysis (see [23-28, 31]).

In the particular case when in (M) we have $a(x,t)=\alpha(x,t)\ t^{q(x)-2}$ with $\alpha(\cdot,\cdot):\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ and $q(\cdot):\mathbb{R}^N\to\mathbb{R}$ are two continuous functions, we deal with equation

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involving variable exponents growth condition. Quasilinear equations of this type of non-standard growth conditions have been a very interesting topic of research in the recent years. Let us just quote [14-17, 22]. This great interest could be mainly motivated by their physical applications. In fact, such kind of problems can describe various phenomena which arise from studying elastic mechanics, electrorheological fluids (sometimes referred as “smart” fluids), image restoration (see [5, 7, 34, 40]). Another important phenomenon which could also be modelled by an equation involving variable exponents is the motion of a compressible or incompressible fluid in a nonhomogeneous and anisotropic medium, that is a medium whose characteristics may vary in dependence on directions and points. In fact, the continuity equation in this case (we have supposed that the Darcy law holds) has the following form

$$\operatorname{div}\left(\eta_0(x,p)\,|\nabla p|^{\lambda(x)-2}\,\nabla p\right)=\eta_1(x,p,\nabla p),$$

where p denotes the pressure of the fluid. For more details concerning this phenomenon, see [5].

On the other hand, the study of Kirchhoff type problems has been receiving considerable attention in recent years; see for instance [3, 4, 8, 9, 21]. This interest for the study of such problems with various proposed coefficients could be explained by their contributions to the modelling of many physical and biological phenomena.

Let us first mention here that quasilinear equations of the model

$$-M\left(\int_\Omega |\nabla u|^p\,dx\right)\left(\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)\right)=f(x,u)\quad \text{in } \Omega,$$

where Ω is a domain of $\mathbb{R}^N$ is essentially related to the stationary analog of the Kirchhoff equation

$$u_{tt}-M\left(\int_\Omega |\nabla u|^2\,dx\right)\Delta u=f(x,t),$$

where $M(s)=a\ s+b,\ a,\ b>0$. This last equation was proposed by Kirchhoff [18] as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. The Kirchhoff model takes into account the length changes of the string produced by transverse vibrations. On the other hand, equation of the model

$$-M\left(\int_\Omega |u|^p\,dx\right)\left(\operatorname{div}\left(|\nabla u|^{p-2}\,\nabla u\right)\right)=f(x,u)\quad \text{in } \Omega$$

arises in numerous physical phenomena such as systems of particles in thermodynamical equilibrium via gravitational potential, thermal runaway in Ohmic Heating, shear bands in metal deformed under high strain rates (see [38] and references therein). We have also to cite here two others phenomena which could be modelized by quasilinear equations containing nonlocal terms and which could be found in the very interesting thesis of B. Lovat untitled “ Etudes de quelques problèmes paraboliques non locaux” (see [20]). The first one concerns the heat propagation in a domain Ω whose precised description leads us to study an equation of the type

$$\frac{\partial u}{\partial t}-a\left(\int_\Omega u\right)\Delta u=f(x,u)\quad \text{in } \Omega\times(0,T).$$

The second phenomenon arises in the biological studies. In fact the evolution of the density of a population living in a domain Ω could be described by equation of the model

$$\begin{cases} \frac{\partial u}{\partial t} - p(t)\Delta u &= f(x,u) \text{ in } \Omega \times (0,T) \\ u(x,0) &= u_0(x). \end{cases}$$

Here we are mainly concerned with finding an equilibred solution $u(x,t)$; that is its distribution $\frac{u(x,t)}{\int_\Omega u(x,t)dx}$ is independent of the time. As it was proved in [20, Theorem 0.1], a necessary condition to find such a solution is to assume that $p(t) = \frac{C}{\int_\Omega u(x,t)dx}$ where C is some constant. Consequently, we are again led to solve a problem of the model

$$\frac{\partial u}{\partial t} - a\left(\int_\Omega u\right)\Delta = f \quad \text{in } \Omega \times (0,T).$$

It is important to mention that, to our knowledge, there is not a great number of papers which have dealt with nonlocal p(x)-Laplacian equations. We can cite [6, 10-12, 14, 37]. In our present work, we discuss nonlocal problems for a more large class of nonhomogeneous divergence operators. We emphasize also on the fact that we deal with equations on the whole space $\mathbb{R}^N$ and many of embeddings compactness arguments used in the case of bounded domains do not hold any more.

Returning to our functions φ and Φ in the model (M), we notice that Φ verifies that: for all $x \in \mathbb{R}^N$ we have

- $\Phi(x,t) = 0$ if and only if $t = 0$,
- $\Phi(x,\cdot)$ is convex,
- $\lim\limits_{t\to+\infty} \dfrac{\Phi(x,t)}{t} = +\infty$ and $\lim\limits_{t\to 0} \dfrac{\Phi(x,t)}{t} = 0.$

Thus, Φ is a generalized N-function (see [13]). Denote now by $L^0(\mathbb{R}^N)$ the space of all $\mathbb{R}-$valued measurable functions on $\mathbb{R}^N$. We define the mapping $\rho_\Phi : L^0(\mathbb{R}^N) \to [0,+\infty]$ by

$$\rho_\Phi(u) = \int_{\mathbb{R}^N} \Phi\left(x, |u(x)|\right) dx, \quad u \in L^0(\mathbb{R}^N).$$

Obviously the mapping ρ_Φ is a modular (see [13]). Thus, we can introduce its corresponding modular space, that is

$$\begin{aligned} L^\Phi(\mathbb{R}^N) &= \left\{u \in L^0(\mathbb{R}^N, \lim_{\lambda\to 0^+} \rho_\Phi(\lambda u) = 0\right\} \\ &= \left\{u \in L^0(\mathbb{R}^N),\ \rho_\Phi(\lambda u) < +\infty \text{ for some } \lambda > 0\right\}. \end{aligned}$$

This space becomes a Banach space if we equip it with the Luxemburg norm

$$|u|_\Phi = \text{Inf}\left\{\lambda > 0,\ \int_{\mathbb{R}^N} \Phi\left(x, \frac{|u(x)|}{\lambda}\right) \le 1\right\}$$

or the equivalent Orlicz norm

$$|u|_{(\Phi)} = \text{Sup}\left\{\left|\int_{\mathbb{R}^N} uvdx\right|;\ u \in L^{\Phi^*}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} \Phi^*(x,|v(x)|)dx \le 1\right\}$$

where Φ^* denotes the conjugate Young function of Φ, that is

$$\Phi^*(x,s)=\int_0^s(\varphi(x,\ \cdot))^{-1}(t)dt;\quad (x,s)\in\mathbb{R}^N\times\mathbb{R}$$

or

$$\Phi^*(x,s)=\sup_{t\geq 0}\{ts-\Phi(x,t)\},\quad \forall\ x\in\mathbb{R}^N \text{ and } \forall\ t\geq 0.$$

Observe here that $\Phi^*(\cdot,\cdot)$ is also a generalized N-function and the following Hölder's inequality holds true

$$\left|\int_{\mathbb{R}^N}uvdx\right|\leq 2\,|u|_{\Phi}\,|v|_{\Phi^*}\quad \forall\ u\in L^{\Phi}(\mathbb{R}^N),\ v\in L^{\Phi^*}(\mathbb{R}^N) \tag{1.1}$$

(see [13, Lemma 2.6.5]). Throughout this paper, we assume that

(H_1) $1<\varphi_0\leq\frac{t\varphi(x,t)}{\Phi(x,t)}\leq\varphi^0<+\infty$ $\forall\ x\in\mathbb{R}^N$ and $\forall\ t\geq 0$ where φ_0 and φ^0 are positive constants.

With assumption (H_1), we are assured that the function Φ satisfies the global Δ_2-condition (see [24, Proposition 2.3]), that is

$$\Phi(x,2t)\leq K\Phi(x,t),\quad \forall\ x\in\mathbb{R}^N,\ \forall\ t\geq 0,$$

where K is a positive constant. This Δ_2-condition implies that

$$L^{\Phi}(\mathbb{R}^N)=\left\{u\in L^0(\mathbb{R}^N),\ \int_{\mathbb{R}^N}\Phi(x,|u(x)|)dx<+\infty\right\}.$$

By (H_1), we can easily establish that

$$\sigma^{\varphi_0}\Phi(x,t)\leq\Phi(x,\sigma t)\leq\sigma^{\varphi^0}\Phi(x,t)\quad \forall\ x\in\mathbb{R}^N,\ \forall\ t\geq 0,\ \forall\ \sigma>1, \tag{1.2}$$

$$\sigma^{\varphi^0}\Phi(x,t)\leq\Phi(x,\sigma t)\leq\sigma^{\varphi_0}\Phi(x,t)\quad \forall\ x\in\mathbb{R}^N,\ \forall\ t\geq 0,\ \forall\ 0<\sigma<1. \tag{1.3}$$

We introduce now the Orlicz-Sobolev space

$$W^{1,\Phi}(\mathbb{R}^N)=\left\{u\in L^{\Phi}(\mathbb{R}^N),\ \frac{\partial u}{\partial x_i}\in L^{\Phi}(\mathbb{R}^N),\ i=1,\cdots,N\right\}.$$

This space equipped with the following norm

$$\|u\|=\mathrm{Inf}\left\{\lambda>0,\ \int_{\mathbb{R}^N}\left(\Phi\left(x,\frac{|\nabla u(x)|}{\lambda}\right)+\Phi\left(x,\frac{|u(x)|}{\lambda}\right)\right)dx\leq 1\right\}$$

is a Banach space. Furthermore, in the present work we shall assume that Φ satisfies the following condition

(H_2) for each $x\in\mathbb{R}^N$, the function defined on $[0,+\infty[$ by $t\longmapsto\Phi(x,\sqrt{t})$ is convex.

Conditions (H_1) and (H_2) assure that the spaces $L^{\Phi}(\mathbb{R}^N)$ and $W^{1,\Phi}(\mathbb{R}^N)$ are uniformly convex and by consequence a reflexive spaces (see [24, Proposition 2.2]). Taking into account inequalities (1.2) and (1.3) we have (see [24, Proposition 2.5])

$$\|u\|^{\varphi_0}\leq\rho_{\Phi}(|\nabla u|)+\rho_{\Phi}(|u|)\leq\|u\|^{\varphi^0},\quad \forall\ u\in W^{1,\Phi}(\mathbb{R}^N) \text{ with } \|u\|>1 \tag{1.4}$$

$$\|u\|^{\varphi^0} \leq \rho_\Phi\left(|\nabla u|\right) + \rho_\Phi\left(|u|\right) \leq \|u\|^{\varphi_0}, \quad \forall\, u \in W^{1,\Phi}(\mathbb{R}^N) \text{ with } \|u\| < 1. \tag{1.5}$$

Moreover, for $(u_n),\ u \in W^{1,\Phi}(\mathbb{R}^N)$ we have

$$\|u_n - u\| \;\to\; 0 \Leftrightarrow \rho_\Phi\left(|\nabla u_n - \nabla u|\right) + \rho_\Phi\left(|u_n - u|\right) \to 0, \tag{1.6}$$

$$\|u_n\| \;\to\; +\infty \Leftrightarrow \rho_\Phi\left(|\nabla u_n|\right) + \rho_\Phi\left(|u_n|\right) \to +\infty. \tag{1.7}$$

For more details concerning Orlicz-Sobolev spaces and its properties, we refer to [1, 2, 13, 19, 29, 30, 31].

Denote now by $C_+(\mathbb{R}^N)$ the set

$$C_+(\mathbb{R}^N) = \left\{ v \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \inf_{x \in \mathbb{R}^N} v(x) > 1 \right\}.$$

For each $p \in C_+(\mathbb{R}^N)$, we define $p^+ = \sup_{x \in \mathbb{R}^N} p(x)$ and $p^- = \inf_{x \in \mathbb{R}^N} p(x)$ and we introduce the variable exponent Lebesgue space

$$L^{p(\cdot)}(\mathbb{R}^N) = \left\{ u \in L^0(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u(x)|^{p(x)}\, dx < +\infty \right\}.$$

Obviously, $L^{p(\cdot)}(\mathbb{R}^N)$ is a particular case of the generalized Orlicz space. In fact, it suffices to take $\Phi(x,t) = |t|^{p(x)}$. Thus, this space becomes a Banach space with respect of the Luxemburg norm, that is

$$|u|_{L^{p(\cdot)}(\mathbb{R}^N)} = \mathrm{Inf}\left\{ \lambda > 0,\ \int_{\mathbb{R}^N} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}$$

and reflexive provided that $1 < p^- < p^+ < +\infty$. By the virtue of inequality (1.1), we easily see that

$$\left| \int_{\mathbb{R}^N} uv dx \right| \leq 2\, |u|_{L^{\cdot)}(\mathbb{R}^N)}\, |v|_{L^{p'(\cdot)}(\mathbb{R}^N)} \tag{1.8}$$

for any $u \in L^{p(\cdot)}(\mathbb{R}^N)$ and $v \in L^{p'(\cdot)}(\mathbb{R}^N)$ where $p'(\cdot) \in C_+(\mathbb{R}^N)$ is such that $\frac{1}{p'(x)} + \frac{1}{p(x)} = 1 \quad \forall\, x \in \mathbb{R}^N$. Similarly, if $\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1 \quad \forall\, x \in \mathbb{R}^N$, then for any $u \in L^{p_1(\cdot)}(\mathbb{R}^N),\ v \in L^{p_2(\cdot)}(\mathbb{R}^N)$ and $w \in L^{p_3(\cdot)}(\mathbb{R}^N)$,

$$\left| \int_{\mathbb{R}^N} uvw dx \right| \leq 3\, |u|_{L^{p_1(\cdot)}(\mathbb{R}^N)}\, |v|_{L^{p_2(\cdot)}(\mathbb{R}^N)}\, |w|_{L^{p_3(\cdot)}(\mathbb{R}^N)}. \tag{1.9}$$

Next, for $p \in C_+(\mathbb{R}^N)$, we define the variable exponent Sobolev space $W^{1,p(\cdot)}(\mathbb{R}^N)$ as the set

$$W^{1,p(\cdot)}(\mathbb{R}^N) = \left\{ u \in L^{p(\cdot)}(\mathbb{R}^N),\ \frac{\partial u}{\partial x_i} \in L^{p(\cdot)}(\mathbb{R}^N),\ i = 1, \cdots, N \right\}.$$

It is clear that $W^{1,p(\cdot)}(\mathbb{R}^N)$ is a particular case of the generalized Orlicz-Sobolev space. Provided that $1 < p^- \leq p^+ < +\infty$, $W^{1,p(\cdot)}(\mathbb{R}^N)$ equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} = \mathrm{Inf}\left\{ \lambda > 0,\ \int_{\mathbb{R}^N} \left(\frac{|\nabla u|^{p(x)} + |u|^{p(x)}}{\lambda^{p(x)}} \right) dx \leq 1 \right\}$$

is a Banach reflexive and separable space. For more properties of variable exponent Lebesgue and Sobolev space, we refer to [13, 17, 29].

We have to precise here that we shall study nonlocal quasilinear problems of model (M) when the function Φ satisfies the following additional condition

(H_3) there exist $p(\cdot) \in C_+(\mathbb{R}^N)$ and $c_0 > 0$ such that $1 < p^- \leq p^+ < N$ and

$$\Phi(x,t) \geq c_0\, t^{p(x)} \quad \forall\, x \in \mathbb{R}^N \quad \text{and } \forall\, t \geq 0.$$

According to [2, Pragraph 8.4], condition (H_3) means that for all $x \in \mathbb{R}^N$, $\Phi(x,\cdot)$ dominates globally the function $t \longmapsto t^{p(x)}$ which implies (see [2, Theorem 8.12]) that the following continuous embedding holds true

$$W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow W^{1,p(\cdot)}(\mathbb{R}^N). \tag{1.10}$$

We finish this paragraph by giving some examples of functions $\varphi(\cdot,\cdot)$ such that the corresponding $\Phi(\cdot,\cdot)$ satisfies conditions (H_{1-3}).

Example 1. $\varphi(x,t) = \left(1 + \text{Log}\left(1 + |t|^{\alpha(x)}\right)\right) |t|^{p(x)-2}\, t,\ t \in \mathbb{R}$ where $p(\cdot) \in C_+(\mathbb{R}^N)$, $p(x) > 2\ \forall\, x \in \mathbb{R}^N$ and $\alpha(\cdot) \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N),\ \alpha(x) > 0\ \forall\, x \in \mathbb{R}^N$.

Example 2. $\varphi(x,t) = (\pi + \text{Arctg}\,(|t|))\, |t|^{p(x)-2}\, t,\ t \in \mathbb{R}$ where $p(\cdot) \in C_+(\mathbb{R}^N,\ p(x) > 2\ \forall\, x \in \mathbb{R}^N$.

Example 3. $\varphi(x,t) = \sqrt{t^2+1}\ \ |t|^{p(x)-2}\, t,\ t \in \mathbb{R}$ where $p(\cdot) \in C_+(\mathbb{R}^N),\ p(x) > 2\ \forall\, x \in \mathbb{R}^N$.

2. Existence result

In this first part, we investigate existence result for the following nonlocal quasilinear problem

$$(I) \quad -A\left(\|u\|\right)\left(\text{div}\left(a(x,|\nabla u|)\,\nabla u - a\left(x,|u|\right) u\right) = B\left(\|u\|\right) f(x,u) + h \quad \text{in } \mathbb{R}^N,\ N \geq 3.$$

Here $A : [0,+\infty[\to [0,+\infty[$ and $B : [0,+\infty[\to \mathbb{R}$ are two continuous functions. Problem (I) should be taken under the following hypotheses:

(K_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying that

$$|f(x,s)| \leq g(x)\, |s|^{\beta(x)} \quad \text{a.e } x \in \mathbb{R}^N \quad \text{and} \quad \forall\, s \in \mathbb{R},$$

with $\beta(\cdot) \in C_+(\mathbb{R}^N),\ \beta(x) < p^*(x) = \frac{Np(x)}{N-p(x)}\ \ \forall\, x \in \mathbb{R}^N (p(\cdot)$ is defined by $(H_3)),\ g(\cdot) \in L^{r(\cdot)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ where $r(\cdot) \in C_+(\mathbb{R}^N)$ and there exists $\eta(\cdot) \in C_+(\mathbb{R}^N)$ such that

$$\eta(x) > \beta(x),\ p(x) \leq \eta(x) \leq p^*(x),\ \frac{1}{p(x)} + \frac{\beta(x)}{\eta(x)} = 1 \quad \forall\, x \in \mathbb{R}^N.$$

We also assume that $f(x,s) = 0$ a.e $x \in \mathbb{R}^N$ and $\forall\, s \leq 0$.

(K_2) Assume that

- there exist $\lambda_0,\ \lambda_1 \in \mathbb{R},\ a_0 > 0,\ b_0 > 0$ and $M > 0$ such that

$$A(s) \geq a_0\ s^{\lambda_0} \quad \text{and} \quad |B(s)| \leq b_0\ s^{\lambda_1} \quad \forall\ s > M,$$

with $\lambda_0 + \varphi_0 > \sup\left(\lambda_1 + \beta^+,\ 1\right)$,
- if $A(s) = 0$, then $B(s) = 0$,
- if $s \neq 0$, then $A(s) \neq 0$.

(K_2') Assume that

- there exist $\lambda_0,\ \lambda_1 \in \mathbb{R},\ a_0 > 0,\ b_0 > 0$ and $M > 0$ such that

$$A(s) \geq a_0\ s^{\lambda_0} \quad \text{and} \quad |B(s)| \leq b_0\ s^{\lambda_1} \quad \forall\ 0 < s < M,$$

- if $A(s) = 0$, then $B(s) = 0$,
- if $s \neq 0$, then $A(s) \neq 0$.

(K_3) $h \in \left(W^{1,\Phi}(\mathbb{R}^N)\right)'$ (dual space of $W^{1,\Phi}(\mathbb{R}^N)$), $h \neq 0$. Moreover, we assume that

$$\langle h, v\rangle \geq 0 \quad \text{for all } v \in W^{1,\Phi}(\mathbb{R}^N) \quad \text{with } v \geq 0.$$

Here $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{1,\Phi}(\mathbb{R}^N)$ and its dual space $\left(W^{1,\Phi}(\mathbb{R}^N)\right)'$.

Definition 2.1. A function $u \in W^{1,\Phi}(\mathbb{R}^N)$ is said to be a weak solution of the problem (I), if it satisfies

$$\begin{aligned} &A\left(\|u\|\right)\left(\int_{\mathbb{R}^N} a\left(x, |\nabla u|\right)\nabla u \cdot \nabla v dx + \int_{\mathbb{R}^N} a\left(x, |u|\right) u \cdot v dx\right) \\ &= B\left(\|u\|\right)\int_{\mathbb{R}^N} f(x,u) v dx + \langle h, v\rangle \quad \forall\ v \in W^{1,\Phi}(\mathbb{R}^N). \end{aligned}$$

We state here our main result in this section.

Theorem 2.2. (i) *Assume that (H_{1-3}) hold true. If hypotheses (K_1) and (K_2) hold true, then for all $h \in \left(W^{1,\Phi}(\mathbb{R}^N)\right)'$ satisfying (K_3), the problem (I) has at least one nonnegative and nontrivial solution.*

(ii) *Assume that (H_{1-3}) hold true. If (K_1) and (K_2') hold true, then we have*

- *if $\lambda_0 + \varphi^0 < \inf\left(1,\ \lambda_1 + \beta^-\right)$, then, for all $h \in \left(W^{1,\Phi}(\mathbb{R}^N)\right)'$ satisfying (K_3), the problem (I) has at least one nonnegative and nontrivial weak solution,*
- *if $1 < \lambda_0 + \varphi^0 < \lambda_1 + \beta^-$, then there exists a positive constant M_0 with the following property: for all $h \in \left(W^{1,\Phi}(\mathbb{R}^N)\right)'$ satisfying (K_3) and such that*

$$\|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'} = \sup_{\|v\|\leq 1} |\langle h, v\rangle| \leq M_0$$

the problem (I) has at least one nonnegative and nontrivial weak solution.

Proof. The proof relies essentially on the topological degree theory for (S_+) type mappings(see [36, 39]). Define the following operator $L : W^{1,\Phi}(\mathbb{R}^N) \to \left(W^{1,\Phi}(\mathbb{R}^N)\right)'$

by

$$\langle L(u), v\rangle = A(\|u\|)\left(\int_{\mathbb{R}^N} a(x,|\nabla u|)\nabla u\cdot\nabla v dx + \int_{\mathbb{R}^N} a(x,|u|)uvdx\right)$$
$$- B(\|u\|)\int_{\mathbb{R}^N} f(x,u)vdx - \langle h, v\rangle; \quad u,\ v\ \in W^{1,\Phi}(\mathbb{R}^N).$$

It is clear that L is a bounded operator. On the other hand, L is demicontinuous, that is: for all sequence $(u_n)\subset W^{1,\Phi}(\mathbb{R}^N)$ such that $u_n\to u$ strongly in $W^{1,\Phi}(\mathbb{R}^N)$, $L(u_n)\rightharpoonup L(u)$ weakly in $\left(W^{1,\Phi}(\mathbb{R}^N)\right)'$. Now, we show that L is of (S_+) type. Let $(u_n)\subset W^{1,\Phi}(\mathbb{R}^N)$ and $\limsup\limits_{n\to+\infty}\langle L(u_n),\ u_n-u\rangle\le 0$, we claim that $u_n\to u$ strongly in $W^{1,\Phi}(\mathbb{R}^N)$. Let E be a measurable subset of $\mathbb{R}^N$, by (K_1) and (1.9), we have

$$\int_E |f(x,u_n)|\ |u_n-u|\,dx \le 3\,|g|_{L^{r(\cdot)}(E)}\left||u_n|^{\beta(\cdot)-1}\right|_{L^{\frac{\eta(\cdot)}{\beta(\cdot)-1}}(E)}|u_n-u|_{L^{\eta(\cdot)}(E)}\,. \tag{2.1}$$

Since $g\in L^{r(\cdot)}(\mathbb{R}^N)$ and (u_n) is bounded in $W^{1,\Phi}(\mathbb{R}^N)$ (and then in $W^{1,p(\cdot)}(\mathbb{R}^N)$ by the virtue of (1.10)), by (2.1) the integral $\int_E |f(x,u_n)|\ |u_n-u|\,dx$ is small uniformly in n when the measure of E is small. Let now $R>0$ and $B_R=\left\{x\in\mathbb{R}^N,\ |x|<R\right\}$; we have

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} &|f(x,u_n)|\ |u_n-u|\,dx \\ &\le 3\,|g|_{L^{r(\cdot)}(\mathbb{R}^N\setminus B_R)}\left||u_n|^{\beta(\cdot)-1}\right|_{L^{\frac{\eta(\cdot)}{\beta(\cdot)-1}}(\mathbb{R}^N)}|u_n-u|_{L^{\eta(\cdot)}(\mathbb{R}^N)}\,.\end{aligned} \tag{2.2}$$

Taking again into account that $g\in L^{r(\cdot)}(\mathbb{R}^N)$, then by (2.2), for all $\epsilon>0$ there exists $R_\epsilon>0$ large enough such that $|g|_{L^{r(\cdot)}(\mathbb{R}^N\setminus B_{R_\epsilon})}<\epsilon$. Hence, we get the equi-integrability of the sequence $(f(\cdot,u_n)(u_n-u))$. By the virtue of Vitali's theorem, we deduce that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n-u)dx = 0. \tag{2.3}$$

Define now the operator $\Lambda: W^{1,\Phi}(\mathbb{R}^N)\to\left(W^{1,\Phi}(\mathbb{R}^N)\right)'$ by

$$\langle \Lambda(u), v\rangle = \int_{\mathbb{R}^N} a(x,|\nabla u|)\nabla u\cdot\nabla v dx + \int_{\mathbb{R}^N} a(x,|u|)uvdx;\quad u,\ v\in W^{1,\Phi}(\mathbb{R}^N).$$

According to [24, Proposition 4.5], the operator Λ is of (S_+) type. Using (2.3) together with the fact that $\limsup\limits_{n\to+\infty}\langle L(u_n), u_n-u\rangle\le 0$, it yields

$$\limsup_{n\to+\infty} A(\|u_n\|)\langle\Lambda(u_n), u_n-u\rangle\le 0. \tag{2.4}$$

If $\|u_n\|\to 0$, then $u_n\to 0$ strongly in $W^{1,\Phi}(\mathbb{R}^N)$ and there is nothing to prove. Otherwise, i.e. if $\|u_n\|\to t\ne 0$, then $A(\|u_n\|)\to A(t)\ne 0$. In this case, (2.4) implies that $\limsup\limits_{n\to+\infty}\langle\Lambda(u_n), u_n-u\rangle\le 0$ and since Λ is of (S_+) type, it follows that $u_n\to u$ strongly in $W^{1,\Phi}(\mathbb{R}^N)$.

(i) Assume that (K_2) holds true. For $\|u\| > \sup(1, M)$, by (H_1) and (1.4), we have

$$\begin{aligned}\langle L(u), u\rangle &= A(\|u\|)\left(\int_{\mathbb{R}^N} \varphi(x, |\nabla u|)|\nabla u|\,dx + \int_{\mathbb{R}^N} \varphi(x, |u|)|u|\,dx\right)\\ &\quad - B(\|u\|)\int_{\mathbb{R}^N} f(x,u)u dx - \langle h, u\rangle\\ &\geq \varphi_0 A(\|u\|)(\rho_\Phi(|\nabla u|) + \rho_\Phi(|u|))\\ &\quad - c_1 |B(\|u\|)|\ \|u\|^{\beta^+} - \|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'}\|u\|\\ &\geq c_2\ \|u\|^{\lambda_0+\varphi_0} - c_3\ \|u\|^{\lambda_1+\beta^+} - \|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'}\|u\|.\end{aligned} \tag{2.5}$$

Since $\lambda_0 + \varphi_0 > \sup(\lambda_1 + \beta^+, 1)$, then from (2.5) we deduce that there exists $R > 0$ large enough such that $\langle L(u), u\rangle > 0$ for $\|u\| = R$. Therefore, by the topological degree theory for (S_+) type mappings (see again [36, 39]), we have $deg(L, B(0, R), 0) = 1$ and consequently there exits $u \in B(0, R)$ such that $L(u) = 0$ in $\left(W^{1,\Phi}(\mathbb{R}^N)\right)'$, that is, the problem (I) has a solution $u \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|u\| < R$. Define now $u^- = \min(u, 0)$; since $f(x, u^-) = 0$ a.e. x in $\mathbb{R}^N$, $\langle h, u^-\rangle \leq 0$ and $\langle L(u), u^-\rangle = 0$, it immediately follows that $u^- = 0$ and $u \geq 0$. Moreover, since $h \neq 0$, then $u \neq 0$.

(ii) Assume now that (K_2') holds true. For $\|u\| < \inf(1, M)$, by (H_1) and (1.5) we have:

- if $\lambda_0 + \varphi^0 < \inf(1, \lambda_1 + \beta^-)$, then

$$\begin{aligned}\langle L(u), u\rangle &\geq \varphi_0 A(\|u\|)(\rho_\Phi(|\nabla u|) + \rho_\Phi(|u|)) - c_4\|u\|^{\beta^-+\lambda_1} - \|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'}\|u\|\\ &\geq \|u\|^{\lambda_0+\varphi^0}\left(c_5 - c_4\|u\|^{\lambda_1+\beta^- -\lambda_0-\varphi^0} - \|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'}\|u\|^{1-\lambda_0-\varphi^0}\right).\end{aligned} \tag{2.6}$$

  Since $\left(\lambda_1 + \beta^- - \lambda_0 - \varphi^0\right) > 0$ and $\left(1 - \lambda_0 - \varphi^0\right) > 0$, then from (2.6) we deduce that there exits $0 < R_1 < 1$ small enough such that $\langle L(u), u\rangle > 0$ for $\|u\| = R_1$. Hence $deg(L, B(0, R_1), 0) = 1$ and the existence of a weak solution for the problem (I) then follows.
- If $1 < \lambda_0 + \varphi^0 < \lambda_1 + \beta^-$, then

$$\begin{aligned}\langle L(u), u\rangle &\geq c_5\|u\|^{\lambda_0+\varphi^0} - c_4\|u\|^{\lambda_1+\beta^-} - \|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'}\|u\|\\ &\geq \|u\|\left(c_5\ \|u\|^{\lambda_0+\varphi^0-1} - c_4\|u\|^{\lambda_1+\beta^- -1} - \|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'}\right).\end{aligned} \tag{2.7}$$

  Since $0 < \lambda_0 + \varphi^0 - 1 < \lambda_1 + \beta^- - 1$, then we deduce from (2.7) that there exists $0 < R_2 < 1$ small enough such that

$$\langle L(u), u\rangle \geq R_2\left(c_6\ R_2^{\lambda_0+\varphi^0-1} - \|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'}\right).$$

  Hence, for $\|h\|_{(W^{1,\Phi}(\mathbb{R}^N))'}$ small enough, it yields $\langle L(u), u\rangle > 0$ for $\|u\| = R_2$. Therefore, $deg(L, B(0, R_2), 0) = 1$ and consequently the problem (I) admits a weak solution $u \in W^{1,\Phi}(\mathbb{R}^N)$ with $\|u\| < R_2$. This ends the proof of Theorem 2.2.

□

3. Multiplicity results

Using technical arguments based on two versions of three-critical-points theorem of B. Ricceri (see [32, 33]), we prove in this section two theorems establishing the existence of at least three solutions for the following nonlocal quasilinear problem:

$$(II)\quad \begin{aligned} &-A'\left(\rho_\Phi\left(|\nabla u|\right)+\rho_\Phi\left(|u|\right)\right)\left(\operatorname{div}\left(a\left(x,|\nabla u|\right)\nabla u\right)-a\left(x,|u|\right)u\right)\\ &=\lambda B'\left(\int_{\mathbb{R}^N}F(x,u)dx\right)f(x,u)+\mu\ h \quad \text{in } \mathbb{R}^N \end{aligned}$$

where $F(x,s)=\int_0^s f(x,t)dt$, $A'(\cdot)$ and $B'(\cdot)$ denote the derivatives of two C^1-functions $A(\cdot)$ and $B(\cdot)$ and $\lambda,\ \mu$ are two real numbers.

First, we shall consider the problem (II) under the following hypotheses:

(J_1) $A:[0,+\infty[\to\mathbb{R}$ is a C^1-function satisfying that $A(0)=0$ and
- there exist $a_0>0,\ C_0>0$ and $M_0>0$ such that $A(s)\ge C_0\ s^{a_0}\quad\forall\ s>M_0$,
- there exist $a_1>0$ and $C_1>0$ such that $A(s)\ge C_1\ s^{a_1}\quad\forall\ 0\le s\le 1$,
- A is increasing strictly convex and bounded on each bounded subset of $[0,+\infty[$,
- if $s>0$, then $A'(s)>0$,
- $\liminf_{s\to+\infty}\left(A'(s)\ s^{1-\frac{1}{\varphi_0}}\right)=+\infty$.

(J_2) $B:\mathbb{R}\to\mathbb{R}$ is a C^1-function satisfying that
- there exist $C_2>0$ and $\gamma_0>0$ such that $|B(s)|\le C_2\left(1+|s|^{\gamma_0}\right)\quad\forall\ s\in\mathbb{R}$,
- there exist $C_3>0$ and $\gamma_1>0$ such that $|B(s)|\le C_3\,|s|^{\gamma_1}\quad\forall\ 0\le s\le 1$.

(J_3) There exist $R_0>0$ and $t_0>0$ such that
- $\displaystyle\int_{\{|x|<R_0\}}F(x,t_0)dx>0$,
- $F(x,t)\ge 0\quad$ a.e $R_0\le|x|\le R_0+1$ and $\forall\ 0\le t\le t_0$.

Definition 3.1. A function $u\in W^{1,\Phi}(\mathbb{R}^N)$ is said to be a weak solution of the problem (II) if it satisfies

$$\begin{aligned} &A'\left(\rho_\Phi\left(|\nabla u|\right)+\rho_\Phi\left(|u|\right)\right)\left(\int_{\mathbb{R}^N}a\left(x,|\nabla u|\right)\nabla u\cdot\nabla v dx+\int_{\mathbb{R}^N}a\left(x,|u|\right)uvdx\right)\\ &=\lambda B'\left(\int_{\mathbb{R}^N}F(x,u)dx\right)\int_{\mathbb{R}^N}f(x,u)vdx+\mu\left\langle h,v\right\rangle,\quad\forall\ v\in W^{1,\Phi}(\mathbb{R}^N). \end{aligned}$$

Theorem 3.2. *Assume that (H_{1-3}), (J_{1-3}), (K_1) and (K_3) hold true. Assume also that $Inf\left(\frac{\beta^-\gamma_1}{a_1\varphi^0},\ \frac{a_0\varphi_0}{\gamma_0\beta^+}\right)>1$, then there exist a nonempty set $E\subset[0,+\infty[$ and a positive number r with the following property: for each $\lambda\in E$, there exists $\delta=\delta(\lambda,h)\ge 0$ such that, for each $\mu\in[0,\delta]$ the problem (II) has at least three nonnegative solutions whose norms in $W^{1,\Phi}(\mathbb{R}^N)$ are less than r.*

The proof of Theorem 3.2 is essentially based on the following Theorem due to B. Ricceri:

Theorem 3.3 (see [32])**.** *Let X be a reflexive real Banach space; $I \subset \mathbb{R}$ an interval; $\phi : X \to \mathbb{R}$ a sequentially weakly lower semicontinuous C^1-functional whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a C^1-functional with compact derivative. In addition, ϕ is bounded on each bounded subset of X. Assume that*

$$\lim_{\|x\| \to +\infty} (\phi(x) + \lambda J(x)) = +\infty, \tag{3.1}$$

for all $\lambda \in I$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{x \in X} (\phi(x) + \lambda(J(x) + \rho)) < \inf_{x \in X} \sup_{\lambda \in I} (\phi(x) + \lambda(J(x) + \rho)). \tag{3.2}$$

Then, there exist a nonempty open set $E \subset I$ and a positive real number r with the following property: for every $\lambda \in E$ and every C^1-functional $\psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation

$$\phi'(x) + \lambda J'(x) + \mu \psi'(x) = 0$$

has at least three solutions in X whose norms are less than r.

In order to prove Theorem 3.2, we have to introduce the following functionals:

$$\chi : W^{1,\Phi}(\mathbb{R}^N) \to \mathbb{R},\ \chi(u) = \rho_\Phi (|\nabla u|) + \rho_\Phi (|u|),$$

$$\phi : W^{1,\Phi}(\mathbb{R}^N) \to \mathbb{R},\ \phi(u) = A(\chi(u)),$$

$$J : W^{1,\Phi}(\mathbb{R}^N) \to \mathbb{R},\ J(u) = -B\left(\int_{\mathbb{R}^N} F(x,u)dx\right).$$

According to [24, Lemma 4.2] and using the strict convexity of $A(\cdot)$, it is obvious that $\phi \in C^1\left(W^{1,\Phi}(\mathbb{R}^N),\ \mathbb{R}\right)$ and that ϕ is strictly convex. So, ϕ is sequentially weakly lower semi-continuous and $\phi'(\cdot) = A'(\chi(\cdot))\Lambda(\cdot)$ (where $\Lambda(\cdot)$ is defined in section 2) is a strictly monotone operator. On the other hand, since $\liminf_{s \to +\infty} \dfrac{A'(s)s}{s^{\frac{1}{\varphi_0}}} = +\infty$, then by (1.4) we have $\lim_{\|u\| \to +\infty} \dfrac{\langle \phi'(u), u\rangle}{\|u\|} = +\infty$. Hence, the operator ϕ' is coercive. By the Minty-Browder's Theorem [39, Theorem 26.A], we know that ϕ' is inversible. It remains to show that $(\phi')^{-1}$ is continuous in $\left(W^{1,\Phi}(\mathbb{R}^N)\right)'$. Observe first that the operator ϕ' is of (S_+) type. Indeed, let $(u_n) \subset W^{1,\Phi}(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ weakly in $W^{1,\Phi}(\mathbb{R}^N)$ and $\liminf_{n \to +\infty} \left\langle A'(\chi(u_n))\Lambda(u_n), u_n - u\right\rangle \le 0$. If $\chi(u_n) \to 0$ as $n \to +\infty$, then $u_n \to 0$ and there is nothing to prove. Otherwise, $\chi(u_n) \to t > 0$ and $A'(t) > 0$. Having in mind that $\Lambda(\cdot)$ is of (S_+) type, then the result follows. Let now $w_n \to w$ strongly in $\left(W^{1,\Phi}(\mathbb{R}^N)\right)'$ and $u_n = (\phi')^{-1}(w_n),\ u = (\phi')^{-1}(w)$. Since $w_n = \phi'(u_n)$ and ϕ' is coercive, then necessarily (u_n) is bounded. Without loss of generality, we can assume that $u_n \rightharpoonup u_0$ weakly in $W^{1,\Phi}(\mathbb{R}^N)$. Having in mind that $w_n \to w$, then

$$\lim_{n \to +\infty} \left\langle \phi'(u_n) - \phi'(u), u_n - u\right\rangle = \lim_{n \to +\infty} \langle w_n - w, u_n - u\rangle = 0.$$

Taking into account that ϕ' is of (S_+) type, it follows that $u_n \to u_0$ and $u_0 = u$. We conclude that $(\phi')^{-1}$ is continuous. Furthermore, by (1.10) and [15, Lemma 3.2] $J \in C^1\left(W^{1,\Phi}(\mathbb{R}^N), \mathbb{R}\right)$ and J' is compact.

The following lemmas are needed.

Lemma 3.4. *For all $\lambda \geq 0$, we have* $\lim_{\|u\|\to+\infty} (\phi(u) + \lambda J(u)) = +\infty$.

Proof. By (1.4), we have

$$\chi(u) = \int_{\mathbb{R}^N} \Phi(x, |\nabla u|)\, dx + \int_{\mathbb{R}^N} \Phi(x, |u|)\, dx \geq \|u\|^{\varphi_0} \quad \text{for } \|u\| > 1.$$

Thus, for $\|u\| > \sup\left(1, M_0^{\frac{1}{\varphi_0}}\right)$, it yields

$$\phi(u) = A(\chi(u)) \geq A(\|u\|^{\varphi_0}) \geq C_0 \ \|u\|^{a_0\varphi_0}. \tag{3.3}$$

On the other hand, for $\lambda \geq 0$ and $\|u\| > 1$, we have

$$\begin{aligned} \lambda J(u) &= -\lambda B\left(\int_{\mathbb{R}^N} F(x,u)dx\right) \\ &\geq -\lambda C_2\left(\left|\int_{\mathbb{R}^N} F(x,u)dx\right|^{\gamma_0} + 1\right) \\ &\geq -\lambda C_3\left(\|u\|^{\gamma_0\beta^+} + 1\right). \end{aligned} \tag{3.4}$$

Combining (3.3) with (3.4), we obtain that

$$\phi(u) + \lambda J(u) \geq C_0 \|u\|^{a_0\varphi_0} - \lambda C_3\left(\|u\|^{\gamma_0\beta^+} + 1\right) \quad \forall\ \|u\| > \sup\left(1, M_0^{\frac{1}{\varphi_0}}\right).$$

Since $a_0\varphi_0 > \gamma_0\beta^+$, then we conclude that

$$\lim_{\|u\|\to+\infty} (\Phi(u) + \lambda J(u)) = +\infty \quad \forall\ \lambda \geq 0.$$

□

Lemma 3.5. *Assume that there are $\gamma > 0$, u_0, $u_1 \in W^{1,\Phi}(\mathbb{R}^N)$ such that $\phi(u_0) = J(u_0) = 0$, $\phi(u_1) > \gamma$, $-J(u_1) > 0$ and*

$$\sup_{u\in\phi^{-1}(]-\infty,\gamma])} (-J(u)) < \gamma\left(-\frac{J(u_1)}{\phi(u_1)}\right),$$

then for each ρ satisfying

$$\sup_{u\in\phi^{-1}(]-\infty,\gamma])} (-J(u)) < \rho < \gamma\left(-\frac{J(u_1)}{\phi(u_1)}\right)$$

one has that (3.2) holds with $I = [0, +\infty[$.

Proof. Since $W^{1,\Phi}(\mathbb{R}^N) \hookrightarrow W^{1,p(\cdot)}(\mathbb{R}^N)$, then the proof of Lemma 3.5 is quite similar to [35, Proposition 2.6]. □

Proof of Theorem 3.2 completed. By Lemma 3.4, condition (3.1) of Theorem 3.3 holds true with $I = [0, +\infty[$. Observe now that $\phi(0) = J(0) = 0$. On the other hand, define

$$u_1(x) = \begin{cases} t_0 & \text{if} \quad |x| < R_0 \\ t_0\left(R_0 + 1 - |x|\right) & \text{if} \quad R_0 \le |x| \le R_0 + 1 \\ 0 & \text{if} \quad |x| \ge R_0 + 1. \end{cases}$$

It is clear that $u_1 \in W^{1,\Phi}(\mathbb{R}^N)$. By (J_3), we have $-J(u_1) = B\left(\int_{\mathbb{R}^N} F(x, u_1)dx\right) > 0$ and $\phi(u_1) > 0$. By the virtue of Lemma 3.5 and in order to prove condition (3.2) of Theorem 3.3, it sufficies to show that

$$\lim_{\gamma \to 0} \frac{\sup_{u \in \phi^{-1}(]-\infty,\gamma])} \left(|J(u)|\right)}{\gamma} = 0. \tag{3.5}$$

Observe first that for $0 < \gamma < 1$ small enough, we have $\phi^{-1}\left(]-\infty, \gamma]\right) \subset \left\{u \in W^{1,\Phi}(\mathbb{R}^N),\ \|u\| < 1\right\}$. Next, by (J_1) and for $\|u\| < 1$, it yields

$$\phi(u) \ge A\left(\|u\|^{\varphi^0}\right) \ge C_1 \|u\|^{a_1 \varphi^0}. \tag{3.6}$$

For $\phi(u) \le \gamma$, by (3.6) it follows that $\|u\| \le \left(\frac{\gamma}{C_1}\right)^{\frac{1}{a_1 \varphi^0}}$. On the other hand, for $0 < \gamma < 1$ chosen small enough and for $\|u\| \le \left(\frac{\gamma}{C_1}\right)^{\frac{1}{a_1\varphi^0}}$ we get $\left|\int_{\mathbb{R}^N} F(x,u)dx\right| \le C_4 \|u\|^{\beta^-}$. Hence, by (J_2) we obtain

$$|J(u)| \le C_5 \|u\|^{\gamma_1 \beta^-} \le C_5 \left(\frac{\gamma}{C_1}\right)^{\frac{\gamma_1 \beta^-}{a_1 \varphi^0}}. \tag{3.7}$$

From (3.7), we deduce that

$$\frac{\sup_{u \in \phi^{-1}(]-\infty,\gamma])} \left(|J(u)|\right)}{\gamma} \le C_6 \frac{\gamma^{\frac{\gamma_1 \beta^-}{a_1 \varphi^0}}}{\gamma}.$$

Taking into account that $\frac{\gamma_1 \beta^-}{a_1 \varphi^0} > 1$, then (3.5) follows. This ends the proof of Theorem 3.2. $\square$

Now, changing the assumptions taken on the terms $A(\cdot)$, $B(\cdot)$ and $f(\cdot,\cdot)$ and using a more recent version of the three-critical-points theorem of B. Ricceri (see [33]), we shall also prove that the problem (II) has at least three weak solutions.

Equation (II) is now considered under the following hypotheses:

(J_1') $A : [0, +\infty[\to [0, +\infty[$ is of class C^1 satisfying that $A(0) = 0$ and
- A is increasing strictly convex and bounded on each bounded subset of $[0, +\infty[$,
- $\liminf\limits_{s \to +\infty} \dfrac{A'(s)s}{s^{\frac{1}{\varphi_0}}} = +\infty$,
- if $s > 0$, then $A'(s) > 0$,
- $\liminf\limits_{s \to +\infty} A(s) = +\infty$,

- there exist $C_7>0$ and $\theta_0>0$ such that $A(s)\geq C_7\ s^{\theta_0}\quad \forall\ 0\leq s\leq 1$.

(J_2') $B:\mathbb{R}\to\mathbb{R}$ is of class C^1 satisfying that $B(0)=0$ and
- B is convex,
- $B(s)\leq A(s)\quad \forall\ s\geq 0$,
- there exist $C_8>0$ and $\theta>\frac{\theta_0 p^+}{(p^*)^-}$ such that $B(s)\leq C_8 s^{\theta}\quad \forall\ 0\leq s\leq 1$.

(J_3') $\limsup\limits_{|s|\to+\infty}\dfrac{F(x,s)}{|s|^{p(x)}}\leq 0$, uniformly in $x\in\mathbb{R}^N$.

(J_4') $\limsup\limits_{s\to 0}\dfrac{F(x,s)}{|s|^{p(x)}}\leq 0$, uniformly in $x\in\mathbb{R}^N$.

(J_5') For all compact $K\subset\mathbb{R}$, there exists a function $\psi_K\in L^1(\mathbb{R}^N)$ such that

$$F(x,s)\leq\psi_K(x)\quad \text{a.e. } x \text{ in } \mathbb{R}^N \quad \text{and} \quad \forall\ s\in K.$$

Set $\xi=\sup\limits_{u\in W^{1,\Phi}(\mathbb{R}^N)\setminus\{0\}}\left(\dfrac{(-J)(u)}{\phi(u)}\right)$ where

$$(-J)(u)=B\left(\int_{\mathbb{R}^N}F(x,u)dx\right) \text{ and } \phi(u)=A\left(\rho_\Phi\left(|\nabla u|\right)+\rho_\Phi\left(|u|\right)\right),\ u\in W^{1,\Phi}(\mathbb{R}^N).$$

Theorem 3.6. *Assume that (H_{1-3}), $\left(J_{1-5}'\right)$, (J_3), (K_1) and (K_3) hold true. Then, for each compact interval $[a,b]\subset\left(\frac{1}{\xi},+\infty\right)$ there exists $r>0$ with the following property: for every $\lambda\in[a,b]$, there exists $\delta=\delta(\lambda,h)>0$ such that for each $\mu\in[0,\delta]$ the equation (II) has at least three nonegative weak solutions whose norms are less than r.*

The main tool used in the proof of Theorem 3.6 is the following result due to B. Ricceri (see [33, Theorem 2]):

Theorem 3.7. *Let X be a separable and reflexive real Banach space; $\phi:X\to\mathbb{R}$ a coercive, sequentially weakly lower semicontinuous C^1 functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse on X^* (the dual of X) and blonging to W_X, that is, if (u_n) is a sequence in X converging weakly to $u\in X$ and $\liminf\limits_{n\to+\infty}\phi(u_n)\leq\phi(u)$, then (u_n) has a subsequence converging strongly to u. Let $(-J):X\to\mathbb{R}$ be a C^1 functional with compact derivative. Assume that Φ has a strict local minimum x_0 with $\phi(x_0)=(-J)(x_0)=0$. Finally, setting*

$$\alpha=\max\left\{0,\limsup_{\|x\|\to+\infty}\frac{(-J)(x)}{\phi(x)},\ \limsup_{x\to x_0}\frac{(-J)(x)}{\phi(x)}\right\}$$

$$\beta=\sup_{x\in\phi^{-1}(]0,+\infty[)}\frac{(-J)(x)}{\Phi(x)}$$

assume that $\alpha < \beta$. Then, for each compact interval $[a,b] \subset \left(\frac{1}{\beta}, \frac{1}{\alpha}\right)$ (with the convention $\frac{1}{0} = +\infty$, $\frac{1}{+\infty} = 0$) there exists $r > 0$ with the following property: for every $\lambda \in [a,b]$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation $\phi'(x) = \lambda(-J)'(x) + \mu\Psi'(x)$ has at least three solutions whose norms are less than r.

Remark 3.8. Theorem 3.3 gives non further information on the size and location of the set E. So, Theorem 3.7 has brought more precision concerning these points.

Proof of Theorem 3.6. Proceeding exactly as in the proof of Theorem 3.2 and using (J_1') and (K_1), it is immediate that we have:

- the functional ϕ is coercive, sequentially weakly lower semicontinuous, of class C^1, bounded on each bounded subset of $W^{1,\Phi}(\mathbb{R}^N)$ and whose derivative is a homeomorphism. Moreover $\phi \in W_{W^{1,\Phi}(\mathbb{R}^N)}$,
- the functional $(-J) \in C^1(W^{1,\Phi}(\mathbb{R}^N), \mathbb{R})$ with compact derivative.

Observe now that $\phi(u) > 0$ for every $u \in W^{1,\Phi}(\mathbb{R}^N)\backslash\{0\}$. Then 0 is a strict local (even global) minimum with $\phi(0) = (-J)(0) = 0$. Let $0 < \epsilon < 1$, by (J_3') and (J_5') we have

$$F(x,s) \leq \epsilon\frac{|s|^{p(x)}}{p(x)} + \psi_\epsilon(x) \quad \text{a.e. } x \text{ in } \mathbb{R}^N \quad \text{and} \quad \forall\, s \in \mathbb{R}$$

where $\psi_\epsilon \in L^1(\mathbb{R}^N)$. This implies that

$$\int_{\mathbb{R}^N} F(x,u)dx \leq \epsilon \int_{\mathbb{R}^N} \frac{|u|^{p(x)}}{p(x)}dx + \int_{\mathbb{R}^N} \psi_\epsilon(x)dx, \quad u \in W^{1,\Phi}(\mathbb{R}^N). \tag{3.8}$$

Since $B(\cdot)$ is convex, by (3.8), (H_3) and (J_2') we get

$$\begin{aligned} B\left(\int_{\mathbb{R}^N} F(x,u)dx\right) &\leq \epsilon B\left(\int_{\mathbb{R}^N} \frac{|u|^{p(x)}}{p(x)}dx\right) + (1-\epsilon)B\left(\frac{1}{1-\epsilon}\int_{\mathbb{R}^N} \psi_\epsilon(x)dx\right) \\ &\leq \epsilon A\left(\int_{\mathbb{R}^N} \frac{|u|^{p(x)}}{p(x)}dx\right) + (1-\epsilon)B\left(\frac{1}{1-\epsilon}\int_{\mathbb{R}^N} \psi_\epsilon(x)dx\right) \\ &\leq C_9\epsilon\phi(u) + (1-\epsilon)B\left(\frac{1}{1-\epsilon}\int_{\mathbb{R}^N} \psi_\epsilon(x)dx\right). \end{aligned} \tag{3.9}$$

By (3.9), it yields $\limsup_{\|u\|\to+\infty} \dfrac{(-J)(u)}{\phi(u)} \leq C_9\epsilon$. Since ϵ is arbitrary, the following inequality holds true

$$\limsup_{\|u\|\to+\infty} \frac{(-J)(u)}{\phi(u)} \leq 0. \tag{3.10}$$

On the other hand, by (J_3') and (J_4'), for every $0 < \epsilon < 1$, there exists $c_\epsilon > 0$ such that

$$F(x,s) \leq \epsilon\frac{|s|^{p(x)}}{p(x)} + c_\epsilon\,|s|^{p^*(x)} \quad \text{a.e. } x \text{ in } \mathbb{R}^N \quad \text{and} \quad \forall\, s \in \mathbb{R}.$$

Then,

$$\int_{\mathbb{R}^N} F(x,u)dx \leq \epsilon \int_{\mathbb{R}^N} \frac{|u|^{p(x)}}{p(x)}dx + c_\epsilon \int_{\mathbb{R}^N} |u|^{p^*(x)}\,dx, \quad u \in W^{1,\Phi}(\mathbb{R}^N). \tag{3.11}$$

Again by the convexity of $B(\cdot)$ and (J_2'), we deduce from (3.11) that for $\|u\| < 1$ small enough, we have

$$\begin{aligned} B\left(\int_{\mathbb{R}^N} F(x,u)dx\right) &\leq \epsilon B\left(\int_{\mathbb{R}^N} \frac{|u|^{p(x)}}{p(x)}dx\right) + (1-\epsilon)B\left(\frac{c_\epsilon}{1-\epsilon}\int_{\mathbb{R}^N} |u|^{p^*(x)}\,dx\right) \\ &\leq C_9\epsilon\phi(u) + C_{10}(1-\epsilon)\left(\frac{c_\epsilon}{1-\epsilon}\right)^\theta \|u\|^{\theta(p^*)^-}. \end{aligned}$$

Hence,

$$\frac{(-J)(u)}{\phi(u)} \leq C_9\epsilon + C_{10}(1-\epsilon)\left(\frac{c_\epsilon}{1-\epsilon}\right)^\theta \frac{\|u\|^{\theta(p^*)^-}}{\|u\|^{\theta_0 p^+}}. \tag{3.12}$$

Having in mind that $\frac{\theta(p^*)^-}{\theta_0 p^+} > 1$, then by (3.12) we obtain: $\limsup\limits_{s\to+\infty} \dfrac{(-J)(u)}{\phi(u)} \leq \epsilon$. Since ϵ is arbitrary, we deduce that

$$\limsup_{\|u\|\to 0} \frac{(-J)(u)}{\phi(u)} \leq 0. \tag{3.13}$$

Taking now assumption (J_3) into account, it follows from (3.10) and (3.13) that

$$\begin{aligned} &\max\left\{0,\ \limsup_{\|u\|\to+\infty} \frac{(-J)(u)}{\phi(u)},\ \limsup_{\|u\|\to 0} \frac{(-J)(u)}{\phi(u)}\right\} \\ &= 0 < \sup_{u\in\phi^{-1}(]0,+\infty[)} \frac{(-J)(u)}{\phi(u)}. \end{aligned}$$

Therefore, all the conditions of Theorem 3.7 are fulfilled and the conclusion of Theorem 3.6 then follows. □

References

[1] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory, Grundlehren der Mathematischen Wissenschaften*, vol. 314, Springer-Verlag, Berlin, 1996.

[2] R. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[3] C. O. Alves, F. J. S. A. Corrêa and T. F. Ma, *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. **49** (2005), 85–93.

[4] G. Anello, *On a perturbed Dirichlet problem for a nonlocal differential equations of Kirchhoff type*, Bound. Value Probl. Vol. 2011, Article ID 891430.

[5] S. Antontsev and S. Shmarev,*Handbook of Differential Equations, Stationary Partial Differential Equations*, Volume III, Chap. 1, 2006.

[6] S. Aouaoui, *Multiplicity result for some nonlocal anisotropic equation via nonsmooth critical point theory approach*, Appl. Math. Comput. **218** (2011), 532–541.

[7] Y. Chen, S. Levine and R. Rao, *Variable exponent, linear growth functionals in image processing*, SIAM J. Appl. Math. **66** (2006), 1383–1406.

[8] M. Chipot and B. Lovat, *Some remarks on nonlocal elliptic and parabolic problems*, Nonlinear Anal. **30** (1997), 4619–4627.

[9] F. J. S. A. Corrêa and G. M. Figueiredo, *On an elliptic equation of p-Kirchhoff type via variational methods*, Bull. Aust. Math. Soc. **74** (2006), 263–277.

[10] G. Dai, *Infinitely many solutions for a $p(x)$-Laplacian equation in $\mathbb{R}^N$*, Nonlinear Anal. **71** (2009), 1133–1139.

[11] G. Dai and J. Wei, *Infinitely many non-negative solutions for a $p(x)$-Kirchhoff-type problem with Dirichlet boundary condition*, Nonlinear Anal. **73** (2010), 3420–3430.

[12] G. Dai and R. Hao, *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. **359** (2009), 275–284.

[13] L. Diening, P. Harjulehto, P. Hästö and M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2011.

[14] X. Fan, *On nonlocal $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal. **72** (2010), 3314–3323.

[15] X. Fan and X. Han, *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$*, Nonlinear Anal. **59** (2004), 173–188.

[16] Y. Fu and X. Zhang, *A multiplicity result for $p(x)$- Laplacian problem in $\mathbb{R}^N$*, Nonlinear Anal. **70** (2009), 2261–2269.

[17] P. Harjulehto, P. Hästö, U. V. Lê and M. Nuortio, *Overview of differential equations with non-standard growth*, Nonlinear Anal. **72** (2010), 4551–4574.

[18] G. Kirchhoff, *Mechanik*, Teubner, Leibzig, 1883.

[19] M. A. Krasnosel'skiĭ and Y. B. Rutickiĭ, *Convex functions and Orlicz spaces*, P. Noordhoff LTD. 1961.

[20] B. Lovat, *Etudes de quelques problèmes paraboliques non locaux*, Thèse, Université de Metz, 1995.

[21] T. F. Ma, *Remarks on an elliptic equation of Kirchhoff type*, Nonlinear Anal. **63** (2005), 1967–1977.

[22] M. Mihăilescu and V. Rădulescu, *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Por. Roy. Soc. London Der. A **462** (2006), 2625–2641.

[23] M. Mihăilescu and V. Rădulescu, *Eigenvalue problems associated with nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Anal. Appl. **6** (2008), 83–98.

[24] M. Mihăilescu and V. Rădulescu, *Neumann problems associated to nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Ann. Inst. Fourier **58** (2008), 2087–2111.

[25] M. Mihăilesecu and V. Rădulescu, *A continuous spectrum for nonhomogeneous differential operators in Orlicz-Sobolev spaces*, Math. Scand. **104** (2009), 132–146.

[26] M. Mihăilescu, A. Kristály and V. Rădulescu, *Two non-trivial solutions for a non-homogeneous Neumann problem: an Orlicz-Sobolev space setting*, Pro. R. Soc. of Edinburgh **139A** (2009), 1–13.

[27] M. Mihăilescu and V. Rădulescu, *Existence and multiplicity of solutions for quasilinear nonhomogeneous problems: an Orlicz-Sobolev spaces*, J. Math. Anal. Appl. **330** (2007), 416–432.

[28] M. Mihăilescu, G. Morosanu and V. Rădulescu,*Eigenvalue problems in anisotropic Orlicz-Sobolev spaces*, C.R. Mathématique **347** (2009), 521–526.

[29] J. Musielak, *Orlicz spaces and Modular spaces*, volume 1034 of Lecture notes in Mathematics, Springer-Verlag, Berlin, 1983.

[30] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker Inc., New York, 1991.

[31] M. M. Rao and Z. D. Ren, *Applications of Orlicz spaces*, Marcel Dekker Inc., New York, 2002.

[32] B. Ricceri, *Three critical points theorem revisited*, Nonlinear Anal. **70** (2009), 3084–3089.

[33] B. Ricceri, *A further three critical points theorem*, Nonlinear Anal. **71** (2009), 4151–4157.

[34] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2000.

[35] X. Shi and J. Liu, *Existence of three solutions for a class of quasilinear elliptic systems involving the $(p(x), q(x))-$ Laplacian*, Nonlinear Anal.**71** (2009), 550–557.

[36] I. V. Skrypnik, *Methods for Analysis of Nonlinear Elliptic Boundary Value Problems*, Nauka, Moscow, 1990(in Russian); Translation of Mathematical Monographs, Vol. 139, American Mathematical Society, Providence, Rhode Island, 1994 (English translation).

[37] L. Vilasi and F. Cammaroto, *Multiple solutions for a Kirchhoff-type problem involving the $p(x)-$Laplacian operator*, Nonlinear Anal. **74** (2011), 1841–1852.
[38] F. Wang and Y. An, *Existence of nontrivial solution for a nonlocal elliptic equation with nonlinear boundary condition*, Bound. Value Probl., Vol. 2009, Article ID 540360.
[39] E. Zeidler, —it Nonlinear Functional Analysis and its Applications, II/B: Nonlinear Monotone Operators, Springer, New York, 1990.
[40] V. Zhikov, *Averaging of functionals in the calculus of variations and elasticity*, Math. USSR Izv. **29** (1987), 33–66.

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