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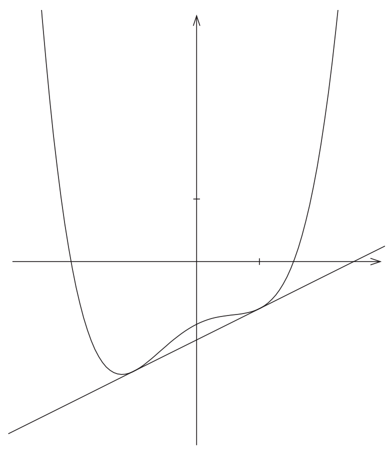
CHARACTERIZING CONVEX FUNCTIONS BY VARIATIONAL PROPERTIES

JEAN SAINT RAYMOND

Abstract. Let X be a reflexive Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be weakly l.s.c. We prove that if for every φ in a convex dense subset Y of the dual space X^* the set of points where the function $f - \varphi$ attains its minimum is convex then f is convex.

It is a common observation that if f is a non-convex continuous real function on the real line $\mathbb{R}$ which satisfies $\lim_{|x|\to\infty} \dfrac{f(x)}{|x|} = +\infty$ then there is an affine function $h \leq f$ such that the function $f - h$ vanishes at several points, and even on a non-convex set.

So it is a natural question to know whether this fact could characterize convex functions on an infinite-dimensional Banach space X.



It is shown in [5] that if $f : X \to \mathbb{R} \cup \{+\infty\}$ is a weakly l.s.c. function and if $f - \varphi$ attains its minimum on a convex non-empty subset of X for all φ in the dual space X^* then f is convex. It is worth noting that the hypothesis that $f - \varphi$ attains its minimum for every φ in X^* guarantees that the sublevels of f are all weakly compact in X.

Another result in the same direction, with a weaker assumption on the set of those φ in X^* for which $f - \varphi$ attains its minimum, can be found in [4] : if X is a reflexive Banach space and $f : X \to \mathbb{R}$ a coercive continuous and weakly l.s.c. function such that f is bounded on every bounded subset of X and that $f - \varphi$ has a unique global minimum for all φ in a convex dense subset Y of X^*, then f is convex.

In this paper we mix these two kinds of hypotheses : as in [5] we assume that f is weakly l.s.c. and that $f - \varphi$ attains its minimum for every $\varphi \in X^*$ in order to ensure the weak compactness of all sublevels of f (one can notice that if X is reflexive this condition is automatically satisfied for every coercive f) ; and as in [4] we shall assume something on the shape of the set M_φ where $f - \varphi$ attains its

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minimum (except that it is non-empty) only for φ in a convex dense subset Y of X^*.

Recall that if X is a Banach space a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *coercive* if $\lim_{\|x\|\to\infty} \dfrac{f(x)}{\|x\|} = +\infty$.

Definition 1. Let X be a Banach space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ will be said to be *quasi-coercive* if for every continuous linear functional φ on X the function $f - \varphi$ is bounded from below.

It is clear that a coercive function is quasi-coercive and well-known that the converse is false in every infinite-dimensional Banach space. A proof of the following lemma can be found in [5].

Lemma 2. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be bounded from below. Then the following three are equivalent*

- *f is quasi-coercive.*
- $\forall \varphi \in X^* \quad \liminf_{\|x\|\to\infty} \dfrac{f(x) - \varphi(x)}{\|x\|} > 0.$
- $\forall \varphi \in X^* \quad \lim_{\|x\|\to\infty} f(x) - \varphi(x) = +\infty.$

The following result generalizes Theorem 4 in [4].

Theorem 3. *Let X be a reflexive space and $f : X \to \mathbb{R}$ be quasi-coercive and weakly l.s.c. If Y is a convex dense subset of X^* and if $f - \varphi$ attains its minimum at a unique point for every $\varphi \in Y$, then f is convex.*

For the proof of this theorem we need several lemmas.

Lemma 4. *If Y is a convex dense subset of the Banach space E, $\varepsilon > 0$, $R \geq 0$ and $(a_1, a_2, \ldots, a_n)$ a finite family of points in E, then there exist a point ω in E and a finite-dimensional linear subspace V of E such that $\|\omega\| < \varepsilon$, $V \cap B(0, R) \subset Y - \omega$, $d(a_j, V) < \varepsilon$ for $j = 1, 2, \ldots, n$ and $d(x, V_0) < \varepsilon$ for all $x \in V \cap B(0, R)$, where V_0 denotes the linear span of $(a_1, a_2, \ldots, a_n)$.*

Proof. Denote by m the dimension of V_0, and choose $m+1$ points $(b_1, b_2, \ldots, b_{m+1})$ affinely independent in V_0 such that 0 be an interior point of the convex hull of $(b_1, b_2, \ldots, b_{m+1})$. We can for example take for $(b_1, b_2, \ldots, b_m)$ a basis of V_0 and $b_{m+1} = -\sum_1^m b_j$. Increasing R if necessary we can assume that $\sup_j \|a_j\| \leq R$. and let $a_0 = 0$.

Then there exists $\eta > 0$ such that $V_0 \cap B(0, \eta) \subset \mathrm{conv}(b_1, b_2, \ldots, b_{m+1})$, and we define $b'_j = \dfrac{R + \varepsilon}{\eta} b_j$. It is clear that $V_0 \cap B(0, R+\varepsilon) \subset \mathrm{conv}(b'_1, b'_2, \ldots, b'_{m+1})$. Since Y is dense, we can find $b''_j \in Y$ such that $\left\| b'_j - b''_j \right\| < \varepsilon/2$.

If $0 \leq i \leq n$, the point a_i belongs to $\mathrm{conv}(b'_1, b'_2, \ldots, b'_{m+1})$, and there are $\lambda_j \geq 0$ with sum 1 such that $a_i = \sum_j \lambda_j b'_j$. Then $a''_i = \sum_j \lambda_j b''_j$ belongs to $\mathrm{conv}(b''_1, b''_2, \ldots, b''_{m+1}) \subset Y$, and $\left\| a''_j - a_j \right\| \leq \sum_j \lambda_j \left\| b''_j - b'_j \right\| \leq \varepsilon/2$. Define $\omega =$

a_0'', then V as the linear span of the $(b_j'' - \omega)_{j \le m+1}$. We have $\|\omega\| < \varepsilon/2$ and $d(a_j, \omega + V) \le \varepsilon/2$, hence $d(a_j, V) < \varepsilon$.

The faces of the simplex S' of vertices $(b_1', b_2', \ldots, b_{m+1}')$ are in the boundary of this simplex, hence disjoint from the open ball $B(0, R+\varepsilon)$. In particular, if $a \in \operatorname{conv}(b_1'', b_2'', \ldots, b_m'')$, we have $a = \sum_{j \le m} \mu_j b_j''$ with $\mu_j \ge 0$ and $\sum_j \mu_j = 1$, thus $d(a, \sum_{j\le m} \mu_j b_j') < \varepsilon/2$ and since $\sum_{j \le m} \mu_j b_j' \notin B(0, R+\varepsilon)$, we have $\|a - \omega\| > R$. More generally, if $a - \omega$ belongs to some face of the simplex S'' of vertices $b_j'' - \omega$, we have $\|a\| > R$; it follows that $V \cap B(0,R)$ is a convex subset of V, hence is connected, contains 0 and is disjoint from the boundary of the simplex S'', thus that

$$V \cap B(0,R) \subset S'' = \operatorname{conv}(b_1'', b_2'', \ldots, b_{m+1}'') - \omega \subset Y - \omega \ .$$

Finally, if $x \in V \cap B(0,R)$, we have $x + \omega \in \operatorname{conv}(b_1'', b_2'', \ldots, b_{m+1}'')$, and there are non-negative real numbers μ_j with sum 1 such that $x + \omega = \sum_j \mu_j b_j''$; we then have $\left\| x + \omega - \sum_j \mu_j b_j' \right\| \le \sum_j \mu_j \left\| b_j'' - b_j' \right\| \le \varepsilon/2$, whence $d(x+\omega, V_0) \le \varepsilon/2$ and $d(x, V_0) < \varepsilon$ since all b_j''s belong to the space V_0. □

Recall that a multivalued mapping G defined on a topological space T with (non-empty) values in a locally convex linear space E is said to be *usco* if it is upper semi-continuous and takes compact values. It is said to be *cusco* if moreover its values are convex. It is an easy and well-known result that G is usco if its graph $\{(x,y) \in T \times E : y \in G(x)\}$ is compact in $T \times E$.

The following avatar of Brouwers's theorem is well-known : its proof can be found in [1] (Théorème 1, p. 523).

Lemma 5. *Let E be a finite-dimensional normed space and G a cusco mapping defined on the ball $B(0,R)$ of E to the dual space E^*. If $\langle \xi, x\rangle > 0$ for each x in the sphere $S(0,R)$ and all $\xi \in G(x)$, then there is a point $\hat{x} \in B(0,R)$ such that $0 \in G(\hat{x})$.*

Lemma 6. *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasi-coercive function and K a norm-compact subset of X^*. Assume $f(0) < +\infty$. Then there exist $\delta > 0$ and $M \in \mathbb{R}^+$ such that $\|c\| \le M$ holds for every $\varphi \in X^*$ satisfying $d(\varphi, K) < \delta$ and every c such that $f(c) \le \varphi(c) + f(0)$.*

Proof. It follows from lemma 2 that, for all $\xi \in K$, there are $\varepsilon_\xi > 0$ and $r_\xi \ge 0$ such that, for all $x \in X$, $f(x) - \langle \xi, x\rangle \ge f(0) + \varepsilon_\xi . \|x\| - r_\xi$ holds. Then the open balls $B(\xi, \varepsilon_\xi/3)$ cover K and there exists a finite subset J of K such that $K \subset \bigcup_{\xi \in J} B(\xi, \varepsilon_\xi/3)$.

Then put $\delta = \min_{\xi\in J} \dfrac{\varepsilon_\xi}{3}$ and $M = \max_{\xi \in J} \dfrac{3 r_\xi}{\varepsilon_\xi}$. If $d(\varphi, K) < \delta$ there exists $\xi_0 \in K$ such that $\|\varphi - \xi_0\| < \delta$ and $\xi \in J$ such that $\|\xi_0 - \xi\| < \dfrac{\varepsilon_\xi}{3}$. Thus we have $\|\varphi - \xi\| \le \delta + \dfrac{\varepsilon_\xi}{3} < \dfrac{2\varepsilon_\xi}{3}$. If moreover $f(c) \le \varphi(c) + f(0)$,

$$\frac{2\varepsilon_\xi}{3} \|c\| \ge \|\varphi - \xi\| . \|c\| \ge \langle \varphi, c\rangle - \langle \xi, c\rangle \ge f(c) - f(0) - \langle \xi, c\rangle \ge \varepsilon_\xi \|c\| - r_\xi$$

what shows that $\dfrac{\varepsilon_\xi}{3}\,\|c\| \le r_\xi$, hence that $\|c\| \le \dfrac{3r_\xi}{\varepsilon_\xi} \le M$. □

Corollary 7. *Let X be a Banach space, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper quasi-coercive function and K a norm-compact subset of X^*. Then there exist $\delta > 0$ and $M \in \mathbb{R}^+$ such that $\|c\| \le M$ holds for every $\varphi \in X^*$ satisfying $d(\varphi, K) < \delta$ and every c where $f - \varphi$ attains its global minimum on X.*

Proof. Since f is proper there exists some $a \in X$ such that $f(a) < +\infty$. Applying Lemma 6 to the function $f_1 : x \mapsto f(x + a)$ we get M_1 and δ. If $d(\varphi, K) < \delta$ and if $f - \varphi$ attains its minimum at c the function $f_1 - \varphi : x \mapsto f(x+a) - \varphi(x+a) + \varphi(a)$ attaints its minimum at $c-a$. So we have $f_1(c-a) - \varphi(c-a) \le f_1(0) - \varphi(0) = f_1(0)$. Thus it follows that $\|c - a\| \le M_1$ hence that $\|c\| \le M = M_1 + \|a\|$. □

Lemma 8. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasi-coercive weakly l.s.c. function. Assume that the proper domain of f is dense in X and contains 0. Then, for every finite-dimensional linear subspace V of X^* and every $\beta > 0$, there exist $R > 0$ and $\delta > 0$ such that $f^*(\varphi) \ge \beta.R$ for all $\varphi \in X^*$ satisfying both $d(\varphi, V) < \delta$ and $R - \delta \le \|\varphi\| \le R + \delta$.*

Proof. The convex l.s.c. function $f^* : \varphi \mapsto \sup_{x\in X} \langle \varphi, x\rangle - f(x)$ is finite at every point of X^*, hence norm-continuous on X^* and a fortiori on V. Let $\beta' = \beta + 1$. The set $C = \{\varphi \in V : f^*(\varphi) \le f^*(0) + 1 + \beta'\,\|\varphi\|\}$ is a closed neighborhood of 0 in V. If C is unbounded in V, there exist for all integer k a $\varphi_k \in V$ such that $\|\varphi_k\| = 1$ and a $\lambda_k \ge k$ such that $\lambda_k.\varphi_k \in C$. Passing to a subsequence one can assume that (φ_k) converges to some $\varphi \in V$. Then $\|\varphi\| = 1$. Moreover, the function τ_k defined on $\mathbb{R}^+$ by $\tau_k(t) = f^*(t\varphi_k) - f^*(0) - 1 - \beta' t$ is convex, negative at 0 and at λ_k, hence at $\ell \le \lambda_k$. In particular if ℓ is any integer, we have $\ell.\varphi_k \in C$ for all $k \ge \ell$. Thus $\ell.\varphi \in C$ for all $\ell \in \mathbb{N}$. Then for all $\ell \ge k$ we have $f^*(\ell\varphi_k) \le 1 + f^*(0) + \beta'\ell$, hence $f^*(\ell\varphi) \le 1 + f^*(0) + \beta'\ell$. For every $x \in D_f$ and every $\ell \in \mathbb{N}$:

$$\ell\varphi(x) - f(x) \le f^*(\ell\varphi) \le 1 + f^*(0) + \beta'.\ell$$

whence $\varphi(x) \le \beta' + \dfrac{1 + f^*(0) + f(x)}{\ell}$, and finally $\varphi(x) \le \beta'$. And this contradicts the hypothesis that the non-empty open set $\{x : \langle\varphi, x\rangle > \beta'\}$ meets D_f.

It follows that C is compact, and there exists R_1 such that $C \subset V \cap B(0, R_1)$. Then, if $\|\varphi\| > R_1$, we have $\varphi \notin C$ and

$$\begin{aligned} 1 + f^*(0) + \beta' R_1 < f^*\Big(\frac{R_1}{\|\varphi\|}\varphi\Big) &= f^*\Big(\frac{R_1}{\|\varphi\|}\varphi + \Big(1 - \frac{R_1}{\|\varphi\|}\Big).0\Big) \\ &\le \frac{R_1}{\|\varphi\|} f^*(\varphi) + \Big(1 - \frac{R_1}{\|\varphi\|}\Big) f^*(0) \le \frac{R_1}{\|\varphi\|} f^*(\varphi) + f^*(0) \end{aligned}$$

hence $f^*(\varphi) \ge \dfrac{1 + \beta' R_1}{R_1}.\,\|\varphi\| \ge \beta'\,\|\varphi\|$ whenever $\varphi \in V$.

Applying lemma 6 to the compact set $K = V \cap B(0, R_1 + 1)$, we find M and δ' such that if $d(\varphi, K) < \delta'$ and if $c \in X$ is a point such that $f(c) \le f(0) + \varphi(c)$, we have $\|c\| \le M$. Put $\delta = \min(\dfrac{1}{4}, \delta', \dfrac{1}{M + \beta'})$ and $R = R_1 + 2\delta$.

Then if $\psi \in X^*$ with $R-\delta \leq \|\psi\| \leq R+\delta$ and $d(\psi, V) < \delta$, there exists $\varphi \in V$ such that $\|\varphi - \psi\| < \delta$; we have $\|\varphi\| > R_1$ and $\|\varphi\| \leq R + 2\delta \leq R_1 + 4\delta \leq R_1 + 1$, hence $f^*(\varphi) \geq \beta' R_1$ and $\varphi \in K$.

Moreover there exists a sequence (c_n) in X such that $c_0 = 0$ and $f(c_n) - \varphi(c_n)$ decreases to $\inf_x f(x) - \varphi(x)$. Thus $f^*(\varphi) = \sup_n \big(\varphi(c_n) - f(c_n)\big)$ and we have $f(c_n) - \varphi(c_n) \leq f(c_0) - \varphi(c_0) = f(0)$ hence $\|c_n\| \leq M$. Then

$$\begin{aligned} f^*(\psi) &\geq \liminf_n \big(\psi(c_n) - f(c_n)\big) \geq \liminf_n \big(\varphi(c_n) - f(c_n)\big) - \sup_n \|c_n\| \,.\, \|\varphi - \psi\| \\ &\geq f^*(\varphi) - M\varepsilon \geq \beta' R_1 - M\varepsilon \geq \beta' R - (M+\beta')\varepsilon \geq (\beta'-1)R = \beta R \ , \end{aligned}$$

and that completes the proof of the lemma. □

Proof of Theorem 3.

In fact we will deduce theorem 3 from the more general following statement.

Theorem 9. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a quasi-coercive weakly l.s.c. function. Assume that the proper domain of f is dense in X and that $f - \varphi$ attains its minimum for all $\varphi \in X^*$. If Y is a convex dense subset of X^* and if the set $M_\varphi = \{x \in X : f - \varphi$ attains its minimum at $x\}$ is convex for all $\varphi \in Y$, then f is convex.*

Proof. Since $f - \varphi$ attains its minimum for all $\varphi \in X^*$ it follows from [5] that the sublevels $S_f(\alpha) = \{x \in X : f(x) \leq \alpha\}$ of f are weakly compact in X.

Assume that the function f is weakly l.s.c. and quasi-coercive but not convex. There exist points a, b_1, b_2 in X and $\lambda \in]0,1[$ such that $a = \lambda b_1 + (1-\lambda) b_2$ and $\lambda f(b_1) + (1-\lambda) f(b_2) < f(a)$.

By hypothesis the function f attains its minimum μ at some point b.

We replace f by $f_1 : x \mapsto f(x+b) - \mu$, a, b_1 and b_2 by $a-b$, $b_1 - b$ and $b_2 - b$, and Y by $Y_1 = Y - \psi$, which is convex and dense in X^*. And the proper domain of f_1, $D_{f_1} = D_f - b$, is dense in X : the function f now satisfies the hypotheses of the theorem with $\inf_x f(x) = f(0) = 0$, $\lambda b_1 + (1-\lambda) b_2 = a$ and $\gamma = f(a) - \lambda f(b_1) - (1-\lambda) f(b_2) > 0$.

Since f is weakly l.s.c., there exists a weak neighborhood W of a such that $f(x) > f(a) - \dfrac{\gamma}{3}$ on W. Thus one can find $\xi_1, \xi_2, \dots \xi_n$ in X^* and $\varepsilon > 0$ such that

$$\sup_{i \leq n} \langle \xi_i, x - a \rangle < \varepsilon \Longrightarrow x \in W$$

Denote then by V_0 the linear subspace of X^* spanned by $(\xi_1, \xi_2, \dots, \xi_n)$. Using lemma 8 we choose $R \geq 1$ and $\delta' \leq \varepsilon$ such that $f^*(\varphi) > R.(\|a\| + \varepsilon)$ for all φ such that $R - \delta' \leq \|\varphi\| \leq R + \delta'$ and $d(\varphi, V_0) < \delta'$.

Since f is quasi-coercive, the sublevel $S = S_f(f(a))$ is bounded in X and we denote by D the diameter of S. Then if $\|\xi_i' - \xi_i\| < \dfrac{\varepsilon}{2D}$, if $f(x) \leq f(a)$ and if $\langle \xi_i, x - a \rangle \geq \varepsilon$, we get $\|x - a\| \leq D$, hence $\langle \xi_i', x - a \rangle \geq \dfrac{\varepsilon}{2}$. It follows that if $\sup_{i \leq n} \langle \xi_i', x - a \rangle < \dfrac{\varepsilon}{2}$, we have

- either $\|x - a\| \leq D$ and $x \in W$ hence $f(x) > f(a) - \dfrac{\gamma}{3}$,

- or $\|x-a\| > D$ and $x \notin S$ hence $f(x) > f(a)$,

thus $f(x) > f(a) - \dfrac{\gamma}{3}$ in both cases.

Fix $K_0 = V_0 \cap B(0, R+\varepsilon)$, $\delta < \delta' \leq \varepsilon$ and $M \geq 1$ such that $\|c'\| \leq M$ whenever $f - \varphi$ attains its minimum at c' and $d(\varphi, K_0) < \delta$.

Applying lemma 4 one finds $\omega \in X^*$ and a finite-dimensional subspace V of X^* such that $\|\omega\| < \dfrac{\varepsilon'}{M_1}$, $\|\omega\| < \dfrac{\gamma}{3D}$, $V \cap B(0,R) \subset Y - \omega$ and $d(\varphi, K_0) < \delta$ if $\varphi \in V \cap B(0,R)$, and also points $\xi'_i \in V$ such that $\|\xi_i - \xi'_i\| < \dfrac{\varepsilon}{2D}$. The points ξ'_i span V. Denote by π the restriction to X of the canonical projection of X^{**} onto V^* : for $x \in X$ and $\varphi \in V$, $\langle \pi(x), \varphi \rangle = \varphi(x)$. Then, since $\xi'_i \in V$, we see that if $x \in X$ and if $\pi(x) = \pi(a)$, we have $\langle \xi'_i, x - a \rangle = 0$ for all i, and in particular $f(x) > f(a) - \dfrac{\gamma}{3}$. Moreover if $\varphi \in V$ and $x \in X$, we have $\langle \pi(x), \varphi \rangle = \varphi(x)$. Observe that if $\varphi \in V$ and $\|\varphi\| \leq R$ then $\omega + \varphi \in Y$.

If $\varphi \in V$, $\|\varphi\| = R$ and $f - (\omega + \varphi)$ attains its minimum at c, we then have

$$(\omega+\varphi)(c) = f(c) + f^*(\omega+\varphi) > f(c) + R.(\|a\| + \varepsilon) \geq R(\|a\| + \varepsilon) = \|\varphi\| \,.\, \|a\| + \varepsilon$$

since $\|\omega\| < \varepsilon'$, hence $\varphi(c) - \varphi(a) > \|\varphi\| \,.\, \|a\| + \varepsilon - \varphi(a) - M. \|\omega\| \geq 0$.

Consider the multivalued function $G : V \cap B(0,R) \rightrightarrows X$ defined by

$$x \in G(\varphi) = M_{\omega+\varphi} \iff f^*(\omega+\varphi) = (\omega+\varphi)(x) - f(x)$$

If $x \in G(\varphi)$ we have $\|x\| \leq M$ hence $f(x) - (\omega+\varphi)(x) \leq f(0) - (\omega+\varphi)(0) = 0$ and

$$f(x) \leq (\omega+\varphi)(x) \leq \|\omega+\varphi\| \,.\, \|x\| \leq \alpha := (\|\varphi\| + \|\omega\|).M$$

So the values of G are contained in the weakly compact subset $S_f(\alpha)$ of X. And the graph Γ of G is compact in $(V \cap B(0,R)) \times X$, if $Y \subset X^*$ is equipped with the norm topology and X with the weak topology ; indeed $\Gamma \subset (V \cap B(0,R)) \times S_f(\alpha)$ and

$$(\varphi, c) \in \Gamma \iff c \in G(\varphi) \iff \forall x \in X \quad \varphi(c) - f(c) \geq \varphi(x) - f(x)$$

thus G is usco from Y equipped with the norm topology into X with the weak topology. And since G takes convex values by hypothesis this shows that it is a cusco mapping.

If $\varphi \in V \cap B(0,R)$ and $c \in G(\varphi)$ we necessarily have

$$\begin{aligned} f(c) - (\omega+\varphi)(c) &\leq \min\Big(f(b_1) - (\omega+\varphi)(b_1)\,,\, f(b_2) - (\omega+\varphi)(b_2)\Big) \\ &\leq \lambda f(b_1) + (1-\lambda) f(b_2) - (\omega+\varphi)(\lambda b_1 + (1-\lambda) b_2) \\ &\leq (f(a) - \gamma) - (\omega+\varphi)(a) = f(a) - (\omega+\varphi)(a) - \gamma \end{aligned}$$

but if $\pi(x) = \pi(a)$ we have $\varphi(x) = \varphi(a)$ hence

$$\begin{aligned} f(x) - (\omega+\varphi)(x) = f(x) - \varphi(a) - \omega(x) &> f(a) - \frac{\gamma}{3} - (\omega+\varphi)(a) - \|\omega\| \,.\, \|x-a\| \\ &> f(a) - (\omega+\varphi)(a) - \frac{\gamma}{3} - D. \|\omega\| \geq f(a) - (\omega+\varphi)(a) - 2\frac{\gamma}{3} \end{aligned}$$

and that shows that $\pi(c) \neq \pi(a)$.

The multivalued function $G_1 : \varphi \mapsto \pi \circ G(\varphi) - \pi(a)$ takes convex compact values in V^* and is upper semi-continuous from $V \cap B(0,R)$ to V^*, since π is weakly continuous from X to V^* and it follows from what precedes that $G_1(\varphi)$ cannot contain 0 for any $\varphi \in Y \cap V$.

By the choice of R, we have $\langle z, \varphi\rangle = \varphi(c) - \varphi(a) > 0$, for all $\varphi \in V$ such that $\|\varphi\| = R$ and all $z = \pi(c) - \pi(a) \in G_1(\varphi)$. We deduce then from lemma 5 that the cusco mapping G_1 should contain 0 at some point of $V \cap B(0,R)$, in contradiction with what precedes. And this shows that f must be convex. $\square$

Observe that most of the arguments of the above proof do not require any hypothesis of weak compactness. The only point where such an hypothesis is used (by the weak compactness of the sublevels of f) is the proof that G is cusco (and that so is G_1). If we attempt to replace each $G_1(\varphi)$ by its closure $G_1'(\varphi) = \overline{G_1(\varphi)}$ in V^*, we get a multivalued mapping with convex compact values but we cannot ensure that this new multivalued mapping is upper semi-continuous. And if we try to replace the graph $\Gamma_1 = \{(\varphi, z) : z \in G_1(\varphi)\}$ by its closure in $B(0,R) \times V^*$, we get the graph of a new u.s.c. multivalued mapping G_1'' with compact values avoiding 0 but could loose the convexity of its values $G_1''(\varphi)$.

Theorem 10. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper weakly l.s.c. function. Assume that $f - \varphi$ attains its minimum for all $\varphi \in X^*$ and that the set $M_\varphi = \{x \in X : f - \varphi$ attains its minimum at $x\}$ is convex for all φ in a convex dense subset Y of X^*. Then f is convex.*

Proof. If the proper domain of f is dense in X, the statement follows from theorem 9. In the other case, it follows from [5] that the sublevels $S_f(\alpha)$ of f are all weakly compact in X. If K_n denotes the sublevel $S_f(2^n)$ and if $m_n = \sup\{\|x\| : x \in K_n\}$, the set $K = \sum \dfrac{2^{-n}}{1+m_n} K_n$ is weakly compact in X. It follows then from [2] that there exists a subspace Z of X containing K and a norm $|||.|||$ on Z such that Z be a reflexive Banach space for $|||.|||$ and that K be bounded in Z. The unit ball of Z is compact for the weak topology $\sigma(Z, Z^*)$ and a fortiori for $\sigma(Z, X^*)$.

Since $K_n \subset 2^n(1+m_n)K \subset Z$, we have $D_f = \bigcup_n K_n \subset Z$. And replacing X by the closure $\bar{Z}$ of Z in X, we can also assume Z to be dense in X.

For any positive integer q, define

$$f_q(x) = \inf_{y \in X} f(y) + q\,|||x - y|||^2$$

If $a \in X$ satisfies $f(a) < +\infty$, we have $a \in D_f \subset Z$ and

$$f_q(x) \le f(a) + q\,|||x - a|||^2 < +\infty$$

whenever $x \in Z$; this shows that the function f_q is everywhere finite on Z, hence $D_{f_q} \supset Z$ and D_{f_q} is dense in X. In fact, if $f(y) < +\infty$ and $\|x - y\| < +\infty$, we have $y \in Z$ and $x - y \in Z$, thus $x \in Z$, whence $D_{f_q} = Z$. One sees also that $f_q \le f$. By hypothesis f is bounded from below, and we define $\mu = \inf_{x \in X} f(x)$.

The epigraphs E_f of f and E_q of the function $n_q : y \mapsto q\,|||y|||^2$ are both weakly closed and the function $\sigma : \big((y,t),(z,s)\big) \mapsto (y+z, s+t)$ is weakly perfect from $E_f \times E_q$ into $X \times \mathbb{R}$: indeed if H is weakly compact in $X \times \mathbb{R}$, there exists $h \in \mathbb{R}$

such that $H \subset X\times]-\infty, h]$. Then if $(y,t) \in E_f$, $(z,s) \in E_q$ and $(y+z, s+t) \in H$, we must have $\mu + s \leq s+t \leq h$, thus $s \leq h-\mu$ and $|||z||| \leq r = \big(\dfrac{h-\mu}{q}\big)^{1/2}$. It follows that (z,s) belongs to the weakly compact set $L = \{z : |||z||| \leq r\} \times [0, h-\mu]$ and (y,t) belongs to the weakly compact set $H - L$. Finally since E_f and E_q are weakly closed, $H_1 = E_f \cap (H-L)$ and $H_2 = E_q \cap L$ are weakly compact and $(E_f \times E_q) \cap \sigma^{-1}(K)$ is closed in the compact set $H_1 \times H_2$.

Since the mapping $\sigma_{|E_f \times E_q}$ is perfect, $E_f + E_q$ is weakly closed in $X \times \mathbb{R}$ and it is the epigraph of f_q. If $\varphi \in X^*$ the minimum of $f_q - \varphi$ is the minimum of the function $(x,u) \mapsto u - \varphi(x)$ on $E_f + E_q$: thus it is attained at some point (x,u) if and only if $(x,u) = (y,t) + (z,s)$, with $t - \varphi(y) = \inf\{u - \varphi(w) : (w,u) \in E_f\}$ and $s - \varphi(z) = \inf\{u - \varphi(w) : (w,u) \in E_q\}$.

Since Z is reflexive,

$$\inf\{u - \varphi(w) : w \in E_q\} = \inf\{q.\, |||w|||^2 - \varphi(w) : w \in W\}$$

is attained at some point of Z for any $\varphi \in Z^*$, and a fortiori for any $\varphi \in X^*$; and since the fonction n_q is convex the set N_φ where the function $n_q - \varphi$ attains its minimum is convex.

By hypothesis $\inf\{u - \varphi(w) : (w,u) \in E_f\} = \inf_{w\in X} f(w) - \varphi(w)$ and the set M_φ where this minimum is attained is convex for all φ. It follows that for all $\varphi \in Y$ the set where $\inf_{w\in X} f_q(w) - \varphi(w)$ is attained is the convex set $N_\varphi + M_\varphi$.

One deduces from theorem 9 that f_q is convex on X for all integer q. If it is shown that $f = \sup_q f_q$, then the proof of the convexity of f will be complete. Let $x \in X$ and $\lambda < f(x)$. Since f is weakly l.s.c., one can find some weak neighborhood W of x such that $f(y) > \lambda$ whenever $y \in W$, then $\delta > 0$ such that $W \supset B_{|||.|||}(x, \delta)$. Choose then q such that $q\delta^2 > \lambda - \mu$; we have

$$f(y) + q.\, |||x-y|||^2 \geq \begin{cases} \lambda + q\, |||x-y|||^2 > \lambda & \text{if } y \in W \\ \mu + q\delta^2 > \lambda & \text{if } y \notin W \end{cases}$$

thus $f_q(x) \geq \lambda$. This completes the proof that $f = \sup_q f_q$ is convex. □

It is shown in [5] that f is strictly convex as soon as it satisfies the conditions of theorem 3 with $Y = X^*$. Nevertheless we shall prove that on every reflexive infinite-dimensional Banach space X, there exists a weakly l.s.c. function f, convex but not strictly convex, and a dense linear subspace Y of X^* such that $f - \varphi$ attains its minimum at a unique point for every $\varphi \in Y$.

Lemma 11. *Let X be a reflexive Banach space and $\alpha \in X^*$. There exists a coercive convex non strictly convex weakly l.s.c. function $f : X \to \mathbb{R}$ such that $f - \varphi$ attains its global minimum at a unique point for all $\varphi \in X^*$ distinct from α.*

Proof. Without loss of generality one can assume thanks to [3] that X is strictly convex. Let K be a strictly convex bounded closed subset of X containing more than one point, e.g. the unit ball. Define on X the coercive function f by :

$$f(x) = d(x,K)^2 + \langle \alpha, x \rangle$$

It is clear that the function

$$f_0 : x \mapsto d(x, K) = \sup_{\varphi \in X^*, \|\varphi\|=1} \varphi(x) - \mu_K(\varphi)$$

where $\mu_K(\varphi) = \sup_{y \in K} \varphi(y)$, is convex non-negative and weakly l.s.c. Thus so is f_0^2, and f is convex and weakly l.s.c. The function $f - \alpha$ is constant on K, which is not a singleton ; thus f is not strictly convex. It remains to show that, for all $\varphi \in Y$ distinct from α, the minimum of $f - \varphi$ is attained at a unique point, it is the minimum of $f_0 - \varphi$ is attained at a unique point whenever $\varphi \neq 0$.

First remark that the function $f_0^2 - \varphi : x \mapsto d(x, K)^2 - \varphi(x)$ attains its minimum at x if and only if the function $(z, t) \mapsto t - \varphi(z)$ attains at $(x, g(x))$ its minimum on the epigraph E of $g = f_0^2$. It is easily checked that E is the sum of the closed convex sets $K_0 = K \times [0, +\infty[$ and $E_0 = \{(z, t) : t \geq \|z\|^2\}$ because the mapping "sum" is weakly perfect from $K_0 \times E_0$ to $X \times \mathbb{R}$.

The set where the linear functional $\psi : (z, t) \mapsto t - \varphi(z)$ attains its minimum on $K_0 + E_0$ is the sum of the sets where this same linear functional attains its minimum on K_0 and on E_0 respectively. If ψ attains its minimum on K_0 at (t, z), we necessarily have $t = 0$ and $\varphi(z) = \sup_{y \in K} \varphi(y)$. Since K is strictly convex and φ is non-zero, this minimum is attained at a unique point z of K.

In the same way, if ψ attains its minimum on E_0 at the point (t, z), we must have $t = \|z\|^2$ and $\varphi(z) = \sup\{\varphi(y) : \|y\| \leq \|z\|\}$; thus the linear functional φ attains its maximum on the unit ball of X at $\dfrac{z}{\sqrt{t}}$.

We also have $(s^2 t, sz) \in E_0$ for $s > 0$, hence

$$(st)^2 - s\varphi(z) = \|sz\|^2 - \varphi(sz) \geq t^2 - \varphi(z)$$

and $(s-1)\Big((s+1)t^2 - \varphi(z)\Big) \geq 0$, it is $\varphi(z) = \|\varphi\| \,.\, \|z\| = 2t^2 = 2\|z\|^2$, so $\|\varphi\| = 2\|z\|$. And since X is strictly convex the point u of the unit ball of X where φ attains its maximum $\|\varphi\|$ is unique, and so is the pair (z, t) : indeed $\|z\| = \dfrac{1}{2}\|\varphi\|$, $z = \|z\| . u$ and $t = \dfrac{1}{4}\|\varphi\|^2$.

This completes the proof of uniqueness for the global minimum of $f_0^2 - \varphi$ as $\varphi \neq 0$, hence the uniqueness for the global minimum of $f - \varphi$ as $\varphi \neq \alpha$. □

Theorem 12. *Let X be a reflexive infinite-dimensional Banach space. There exists a dense linear subspace Y of X^* and a coercive weakly l.s.c. convex and non stricly convex function $f : X \to \mathbb{R}$ such that $f - \varphi$ attains its global minimum at a unique point for every $\varphi \in Y$.*

Proof. Following the previous lemma, it is enough to find a dense linear subspace Y of X^* distinct from X^* and to choose $\alpha \in X^* \setminus Y$.

If X is separable, so is X^* ; it is enough to take a dense sequence (y_n^*) in X^* and to define Y as the linear subspace spanned by the (y_n^*)'s.

In the general case, if (a_n) is a linearly independent sequence in X, the closed linear subspace V spanned by the (a_n)'s is reflexive separable and infinite-dimensional, and so is V^*. The canonical projection $\pi : X^* \to V^*$ is onto and continuous, hence open. Then, if Z is a dense linear subspace of V^* distinct from V^*, the linear subspace $Y = \pi^{-1}(Z)$ is distinct from X^* and dense : indeed if U is a non-empty open set of X^*, $\pi(U)$ is a non-empty open set of V^* and meets Z. Thus there exists $y \in U$ such that $\pi(y) \in Z$, it is $y \in Y \cap U$. $\square$

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Jean Saint Raymond

Analyse Fonctionnelle, Institut de Mathématique de Jussieu, Université Pierre et Marie Curie (Paris 6), Boîte 186 - 4 place Jussieu , F- 75252 Paris Cedex 05, France

E-mail address: raymond@math.jussieu.fr