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ON THE H-DIFFERENTIABILITY OF LÖWNER FUNCTION WITH APPLICATION IN SYMMETRIC CONE COMPLEMENTARITY PROBLEM

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Abstract. Let $\mathcal{K}$ be the symmetric cone in a Jordan algebra $\mathbb{V}$. For any function f from $\mathbb{R}$ to $\mathbb{R}$, one can define the corresponding Löwner function $f^{\mathrm{sc}}(x)$ on $\mathbb{V}$ by the spectral decomposition of $x \in \mathbb{V}$ with respect to $\mathcal{K}$. In this paper, we study the relationship regarding H-differentiability between f^{sc} and f. The class of H-differentiable functions is known to be wider than the class of semismooth functions. Therefore, our result will contribute to solution analysis and solution methods for solving more general symmetric cone programs (SCP) and symmetric cone complementarity problems (SCCP). Besides, we also study a merit function approach for SCCP under H-differentiable condition. In particular, for such class of complementarity problems, we provide conditions to guarantee every stationary point of the associated merit function to be a solution.

1. Introduction and preliminary

Let $\mathbb{V}$ be an n-dimensional vector space over the real field $\mathbb{R}$, endowed with a bilinear mapping $(x, y) \mapsto x \circ y$ from $\mathbb{V} \times \mathbb{V}$ into $\mathbb{V}$. The pair $(\mathbb{V}, \circ)$ is called a *Jordan algebra* [9, 16] if the followings hold.

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$,
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$.

Note that a Jordan algebra is not necessarily associative, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$ may not hold for all $x, y, z \in \mathbb{V}$. We call an element $e \in \mathbb{V}$ the *identity* element if $x \circ e = e \circ x = x$ for all $x \in \mathbb{V}$. A Jordan algebra $(\mathbb{V}, \circ)$ with an identity element e is called a *Euclidean Jordan algebra* if there is an inner product $\langle \cdot, \cdot \rangle_{\mathbb{V}}$ such that

(iii) $\langle x \circ y, z \rangle_{\mathbb{V}} = \langle y, x \circ z \rangle_{\mathbb{V}}$ for all $x, y, z \in \mathbb{V}$.

Given a Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \circ, \langle \cdot, \cdot \rangle_{\mathbb{V}})$, we denote the set of squares as

$$\mathcal{K} := \left\{ x^2 \mid x \in \mathbb{V} \right\}.$$

From [9, Theorem III.2.1], $\mathcal{K}$ is a symmetric cone which means that $\mathcal{K}$ is a self-dual closed convex cone with nonempty interior and for any two elements $x, y \in \mathbf{int}\mathcal{K}$, there exists an invertible linear transformation $\mathcal{T} : \mathbb{V} \to \mathbb{V}$ such that $\mathcal{T}(\mathcal{K}) = \mathcal{K}$ and $\mathcal{T}(x) = y$. In the following, we present three examples of Euclidean Jordan algebras.

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Example 1.1. Consider $\mathbb{R}^n$ with the (usual) inner product and Jordan product defined respectively as

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i \quad \text{and} \quad x \circ y = x * y \qquad \forall x, y \in \mathbb{R}^n$$

where x_i denotes the ith component of x, etc., and $x * y$ denotes the componentwise product of vectors x and y. Then, $\mathbb{R}^n$ is a Euclidean Jordan algebra with the nonnegative orthant $\mathbb{R}^n_+$ as its cone of squares.

Example 1.2. The algebra $\mathscr{S}_n$ of $n \times n$ real symmetric matrices. Let $\mathbb{S}^{n\times n}$ be the space of all $n \times n$ real symmetric matrices with the trace inner product and Jordan product, respectively, defined by

$$\langle X, Y\rangle_{\mathrm{T}} := \mathrm{Tr}(XY) \quad \text{and} \quad X \circ Y := \frac{1}{2}(XY + YX) \qquad \forall X, Y \in \mathbb{S}^{n\times n}.$$

Then, $(\mathbb{S}^{n\times n}, \circ, \langle\cdot,\cdot\rangle_{\mathrm{T}})$ is a Euclidean Jordan algebra, and we write it as $\mathscr{S}_n$. The cone of squares $\mathbb{S}^{n\times n}_+$ in $\mathscr{S}_n$ is the set of all positive semidefinite matrices in $\mathbb{S}^{n\times n}$.

Example 1.3. The Jordan spin algebra $\mathscr{L}_n$. Consider $\mathbb{R}^n$ $(n > 1)$ with the inner product $\langle\cdot,\cdot\rangle$ and Jordan product

$$x \circ y := \begin{bmatrix} \langle x, y\rangle \\ x_0\bar{y} + y_0\bar{x} \end{bmatrix}$$

for any $x = (x_0; \bar{x}), y = (y_0; \bar{y}) \in \mathbb{R}\times\mathbb{R}^{n-1}$. We denote the Euclidean Jordan algebra $(\mathbb{R}^n, \circ, \langle\cdot,\cdot\rangle)$ by $\mathscr{L}_n$. The cone of squares, called the Lorentz cone (or second-order cone), is given by $\mathscr{L}_n^+ := \left\{(x_0; \bar{x}) \in \mathbb{R}\times\mathbb{R}^{n-1} \mid x_0 \geq \|\bar{x}\|\right\}$.

For any given $x \in \mathbb{A}$, let $\zeta(x)$ be the degree of the minimal polynomial of x, i.e.,

$$\zeta(x) := \min\left\{k : \{e, x, x^2, \ldots, x^k\} \text{ are linearly dependent}\right\}.$$

Then, the *rank* of $\mathbb{A}$ is defined as $\max\{\zeta(x) : x \in \mathbb{V}\}$. In this paper, we use r to denote the rank of the underlying Euclidean Jordan algebra. Recall that an element $c \in \mathbb{V}$ is *idempotent* if $c^2 = c$. Two idempotents c_i and c_j are said to be *orthogonal* if $c_i \circ c_j = 0$. One says that $\{c_1, c_2, \ldots, c_k\}$ is a *complete system of orthogonal idempotents* if

$$c_j^2 = c_j, \quad c_j \circ c_i = 0 \text{ if } j \neq i \text{ for all } j, i = 1, 2, \ldots, k \quad \text{and} \quad \textstyle\sum_{j=1}^{k} c_j = e.$$

An idempotent is *primitive* if it is nonzero and cannot be written as the sum of two other nonzero idempotents. We call a complete system of orthogonal primitive idempotents a *Jordan frame*. Now we state a version of the spectral decomposition theorem which is important for subsequent analysis.

Theorem 1.1 ([9, Theorem III.1.2]). *Suppose that $\mathbb{A}$ is a Euclidean Jordan algebra with rank r. Then, for any $x \in \mathbb{V}$, there exists a Jordan frame $\{c_1, \ldots, c_r\}$ and real numbers $\lambda_1(x), \ldots, \lambda_r(x)$, arranged in the decreasing order $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_r(x)$, such that*

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

The numbers $\lambda_j(x)$ (counting multiplicities), which are uniquely determined by x, are called the eigenvalues and $\mathrm{tr}(x) = \sum_{j=1}^r \lambda_j(x)$ *the trace of x.*

Since, by [9, Prop. III.1.5], a Jordan algebra $(\mathbb{V}, \circ)$ with an identity element $e \in \mathbb{V}$ is Euclidean if and only if the symmetric bilinear form $\mathrm{tr}(x \circ y)$ is positive definite, we may define another inner product on $\mathbb{V}$ by $\langle x, y\rangle := \mathrm{tr}(x \circ y)$ for any $x, y \in \mathbb{V}$. The inner product $\langle \cdot, \cdot\rangle$ is associative by [9, Prop. II. 4.3], i.e., $\langle x, y \circ z\rangle = \langle y, x \circ z\rangle$ for any $x, y, z \in \mathbb{V}$. Every Euclidean Jordan algebra can be written as a direct sum of so-called **simple** ones. In finite dimensions, the simple Euclidean Jordan algebras come in four families with infinite cases, together with one exceptional case:

Theorem 1.2 ([9, Theorem V.3.7])**.** *Suppose that $\mathbb{A} = (\mathbb{V}, \circ, \langle\cdot,\cdot\rangle_{\mathbb{V}})$ is a simple Euclidean Jordan algebra of rank $r \geq 3$. Then, $\mathbb{A}$ is isomorphic to one of the following:*

(i) *The algebra $\mathscr{S}_n$ of $n \times n$ real symmetric matrices given by Example 1.2;*
(ii) *The algebra $\mathscr{H}_n$ of all $n \times n$ complex Hermitian matrices with trace inner product $\langle x, y\rangle_{\mathrm{T}} := \mathrm{IRTr}(xy^*)$ and Jordan product $x \circ y := \frac{1}{2}(xy + yx)$ for any $x, y \in \mathbb{H}^{n\times n}$;*
(iii) *The algebra $\mathscr{Q}_n$ of all $n \times n$ quaternionic Hermitian matrices with trace inner product $\langle x, y\rangle_{\mathrm{T}} := \mathrm{IRTr}(xy^*)$ and Jordan product $x \circ y := \frac{1}{2}(xy + yx)$ for any $x, y \in \mathcal{Q}^{n\times n}$;*
(iv) *The algebra $\mathscr{O}_3$ of all 3×3 octonionic Hermitian matrices with trace inner product $\langle x, y\rangle_{\mathrm{T}} := \mathrm{IRTr}(xy^*)$ and Jordan product $x \circ y := \frac{1}{2}(xy + yx)$ for any $x, y \in \emptyset^{3\times 3}$;*
(v) *The Jordan spin algebra $\mathscr{L}_n$ given by Example 1.3.*

where the notation "$*$" means the conjugate transpose, $\mathrm{Tr}(xy)$ denotes the trace of xy which is the multiplication of matrices x and y, and $\mathrm{IR}a$ means the real part of a.

Given an n-dimensional Euclidean Jordan algebra $\mathbb{A} = (\mathbb{V}, \langle\cdot,\cdot\rangle, \circ)$ with $\mathcal{K}$ being its corresponding symmetric cone in $\mathbb{V}$. For any scalar function $f : \mathrm{IR} \to \mathrm{IR}$, we define a vector-valued function $f^{\mathrm{sc}}(x)$ (called Löwner function) on $\mathbb{V}$ as

$$f^{\mathrm{sc}}(x) = f(\lambda_1(x))c_1 + f(\lambda_2(x))c_2 + \cdots + f(\lambda_r(x))c_r \tag{1.1}$$

where $x \in \mathbb{V}$ has the spectral decomposition

$$x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r.$$

When $\mathbb{V}$ is the Jordan spin algebra $\mathscr{L}_n$ in which $\mathcal{K}$ corresponds the second-order cone (SOC), which is defined as

$$\mathcal{K}^n := \{(x_1, x_2) \in \mathrm{IR} \times \mathrm{IR}^{n-1} \mid \|x_2\| \leq x_1\},$$

the function f^{sc} reduces to so-called SOC-function f^{soc} studied in [2, 3, 4, 5]. More specifically, under such case, the spectral decomposition for any $x = (x_1, x_2) \in \mathrm{IR} \times \mathrm{IR}^{n-1}$ becomes

$$x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}, \tag{1.2}$$

where $\lambda_1(x)$, $\lambda_2(x)$, $u_x^{(1)}$ and $u_x^{(2)}$ with respect to $\mathcal{K}^n$ are given by

$$\lambda_i(x) \quad = \quad x_1 + (-1)^i\|x_2\|,$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}\Big(1, (-1)^i \dfrac{x_2}{\|x_2\|}\Big) & \text{if } x_2 \neq 0, \\ \frac{1}{2}\Big(1, (-1)^i w\Big) & \text{if } x_2 = 0, \end{cases}$$

for $i = 1, 2$, with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition (1.2) is unique. With this spectral decomposition, for any function $f : \mathbb{R} \to \mathbb{R}$, the Löwner function f^{sc} associated with $\mathcal{K}^n$ reduces to f^{soc} as below:

$$f^{\text{soc}}(x) = f(\lambda_1(x))u_x^{(1)} + f(\lambda_2(x))u_x^{(2)} \qquad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{1.3}$$

For SOC case, Chen, Chen and Tseng in [5] show that the Löwner function f^{soc} inherits from f the properties of continuity, Lipschitz continuity, directional differentiability, Fréchet differentiability, continuous differentiability, as well as semismoothness. The Hölder continuity of f^{soc} and f is recently shown by the authors in [2]. Sun and Sun [23] extend some of the aforementioned results to more general symmetric cone case regarding f^{sc}. In addition, the Hölder continuity about f^{sc} and f for symmetric cone case is investigated by Lu and Huang in [17]. These results are useful in the design and analysis of smoothing and nonsmooth methods for solving symmetric cone programs (SCP) and symmetric cone complementarity problems (SCCP), see [4, 6, 19, 20] and references therein.

The concepts of H-differentiability and H-differential were introduced in [13] to study the injectivity on nonsmooth functions. As remarked in [13, 25, 26, 27, 28], the Fréchet derivative of a Fréchet differentiable function, the Clarke generalized Jacobian of a locally Lipschitz continuous function, the Bouligand subdifferential of a semismooth function, and the C-differential of C-differentiable function are all examples of H-differentials. It is known that any superset of an H-differential is an H-differential, H-differentiability implies continuity, and H-differentials satisfy simple sum, product and chain rules. Furthermore, an H-differentiable function need not to be locally Lipschitz continuous nor directionally differentiable. With the above facts, the class of H-differentiable functions is wider than the class of semismooth functions.

In this paper, we study whether the H-differentiability of the Löwner function f^{sc} can be also inherited from f or not. Since the class of H-differentiable functions is known as wider than the class of semismooth functions, we believe that this result will contribute to solution analysis and solution methods towards more general SCP and SCCP. Besides, we also study a merit function approach for SCCP under H-differentiable condition. In particular, for such class of complementarity problems, we provide conditions to guarantee every stationary point of the associated merit function to be a solution.

2. The relationship on H-differentiabilities between f^{sc} and f

In this section, we first review several concepts related to H-differentiability. Then, we present our first main result which says the H-differentiability of the vector-valued Löwner function f^{sc} implies that of f.

The concepts of H-differentiability and H-differential of a function were first proposed by Gowda and Ravindran in [13]. Their motivation was to study a generalization (to nonsmooth case) of a result of Gale and Nikaido [11] which asserts that if the Jacobian matrix of a differentiable function f from a closed rectangle $K \subseteq \mathbb{R}^n$ into $\mathbb{R}^n$ is an P-matrix at each point of K, then f is one-to-one on K. More issues about H-differentiability have been studied in [12, 25, 26].

Definition 2.1. Given a function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, where Ω is an open set in $\mathbb{R}^n$ and $x^* \in \Omega$, we say that a nonempty subset $T(x^*)$, also denoted by $T_F(x^*)$, of $\mathbb{R}^{m\times n}$ is an H-differential of F at x^* if for every sequence $x^k \in \Omega$ converging to x^*, there exist a subsequence x^{k_j} and a matrix $A \in T(x^*)$ such that

$$F(x^{k_j}) - F(x^*) - A(x^{k_j} - x^*) = o(\|x^{k_j} - x^*\|).$$

We say that F is H-differentiable at x^* if F has an H-differential at x^*.

A useful equivalent definition of an H-differential $T_F(x^*)$ is: for any sequence $x^k := x^* + t_k d^k$ with $t_k \downarrow 0$ and $\|d^k\| = 1$ for all k, there exist convergent subsequences $t_{k_j} \downarrow 0$ and $d^{k_j} \to d$, and $A \in T_F(x^*)$ such that

$$\lim_{j\to\infty} \frac{F(x^* + t_{k_j} d^{k_j}) - F(x^*)}{t_{k_j}} = Ad.$$

Here are summaries of some well-known facts about H-differentiability, for more details please refer to [13, 25, 26, 27, 28].

Remark 2.2. (i) Any superset of an H-differential is an H-differential.
(ii) H-differentiability implies continuity.
(iii) If a function $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is H-differentiable at a point $\bar{x}$, then there exist a constant $L > 0$ and a neighborhood $B(\bar{x}, \delta)$ of $\bar{x}$ with

$$\|F(x) - F(\bar{x})\| \le L\|x - \bar{x}\| \quad \forall x \in B(\bar{x}, \delta). \tag{2.1}$$

Conversely, if condition (2.1) holds, then $T(\bar{x}) := \mathbb{R}^{m\times n}$ can be taken as an H-differential of F at $\bar{x}$.
(iv) Let $f : \Omega \to \mathbb{R}$ be a real-valued function defined on an open set $\Omega \subseteq \mathbb{R}^n$. Suppose that f is not H-differentiable at $\bar{x} \in \Omega$. Then there exists a sequence $\{x^k\}$ in Ω converging to $\bar{x}$ and for all subsequence x^{k_j}, there is no $a \in \mathbb{R}$ such that

$$\frac{f(x^{k_j}) - f(\bar{x})}{\|x^{k_j} - \bar{x}\|} \to a.$$

Hence the set $\left\{ \frac{f(x^k)-f(\bar{x})}{\|x^k-\bar{x}\|} \right\}$ is unbounded by Bolzano-Weierstrass Theorem. After taking subsequence, this is equivalent to saying that there exists a sequence $\{x^k\}$ in Ω converging to $\bar{x}$ such that

$$\frac{f(x^k) - f(\bar{x})}{\|x^k - \bar{x}\|} \to \infty \text{ or } -\infty.$$

The following lemma gives a sufficient condition for H-differentiability. It is just a direct consequence of Bolzano-Weierstrass Theorem, we omit its proof.

Lemma 2.3. *Let $f : \Omega \to \mathbb{R}$ be a real-valued function defined on an open set $\Omega \subseteq \mathbb{R}$. Define subset $A(\bar{x})$ with $\bar{x} \in \Omega$ as*

$$A(\bar{x}) = \left\{ \frac{f(x^k) - f(\bar{x})}{\|x^k - \bar{x}\|} : \quad \text{for all sequence } \{x^k\} \text{ in } \Omega \text{ converging to } \bar{x} \right\}$$

where we use the convention $\frac{0}{0} = 1$. Suppose $A(\bar{x})$ is bounded. Then, the function f is H-differentiable at $\bar{x}$.

In the following, we present our first main result which says the H-differentiability of the Löwner function f^{sc} implies that of f, and we also give a counter-example to show that the converse may not be true in general.

Theorem 2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ and f^{sc} be the corresponding Löwner function defined in (1.1). Suppose f^{sc} is H-differentiable at x with $x = \lambda_1(x)c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r$. Then, f is H-differentiable at $\lambda_i(x)$ for $1 \le i \le r$.*

Proof. We argue it by contradiction. Suppose that f is not H-differentiable at $\lambda_1 = \lambda_1(x)$. From Remark 2.2(iv), there exists a sequence λ_1^k converging to λ_1 such that

$$m_k = \frac{f(\lambda_1^k) - f(\lambda_1)}{\lambda_1^k - \lambda_1} \to \infty \text{ or } -\infty. \tag{2.2}$$

Define $x^k = \lambda_1^k c_1 + \lambda_2(x)c_2 + \cdots + \lambda_r(x)c_r$. Then, we know $x^k \to x$. By direct computation, we also have

$$f^{\mathrm{sc}}(x^k) - f^{\mathrm{sc}}(x) = m_k(x^k - x)$$

where $x^k - x = (\lambda_1^k - \lambda_1)c_1$. Because f^{sc} is H-differentiable at x, there exist a subsequence x^{k_j} and $A \in \mathbb{R}^{n\times n}$ such that

$$\frac{m_{k_j}(x^{k_j} - x) - A(x^{k_j} - x)}{\|x^{k_j} - x\|} \to 0. \tag{2.3}$$

For simplicity, we denote $y_j = \frac{x^{k_j} - x}{\|x^{k_j} - x\|}$. In addition, by noting that the norm of $\frac{x^{k_j} - x}{\|x^{k_j} - x\|}$ is 1, without lost of generality, we may assume that the sequence y_j converges to a y. Now, with $Ay_j \to Ay$ and (2.3), we obtain

$$m_{k_j} y_j \to Ay$$

which contradicts (2.2). Similar arguments apply for the other $\lambda_i(x)$. Thus, the proof is complete. $\square$

It is natural to ask whether the reverse implication holds or not. Unfortunately, the answer is uncertain. Here is a counter-example in SOC case. Consider a point $x = (x_1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $h = (h_1, h_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$. First, we write down the difference

$$\begin{aligned} & f^{\mathrm{soc}}(x+h) - f^{\mathrm{soc}}(x) \\ (2.4) \quad = \; & \Big[f(x_1 + h_1 - \|h_2\|) - f(x_1) \Big] \, v^{(1)} + \Big[f(x_1 + h_1 + \|h_2\|) - f(x_1) \Big] \, v^{(2)} \end{aligned}$$

where $v^{(i)} = \frac{1}{2}(1, (-1)^i h_2/\|h_2\|)$ for $i = 1, 2$. Suppose f is H-differentiable at $x_1 = \lambda_1(x) = \lambda_2(x)$ and given an arbitrary sequence $h^k = (h_1^k, h_2^k)$ converging to 0. By the definition of H-differentiability, there exist subsequence h^{k_j} and real numbers $a_1, a_2 \in T_f(x_1)$ such that

$$f\left(x_1 + h_1^{k_j} - \|h_2^{k_j}\|\right) - f(x_1) - a_1\left(h_1^{k_j} - \|h_2^{k_j}\|\right) = o\left(h_1^{k_j} - \|h_2^{k_j}\|\right),$$

$$f\left(x_1 + h_1^{k_j} + \|h_2^{k_j}\|\right) - f(x_1) - a_2\left(h_1^{k_j} + \|h_2^{k_j}\|\right) = o\left(h_1^{k_j} + \|h_2^{k_j}\|\right).$$

From the inequality $\sqrt{a^2 + b^2} \geq \frac{1}{2}|a \pm b|$, it is not hard to verify that $o(h_1^{k_j} \pm \|h_2^{k_j}\|)$ are also small "o" function $o(\|h^{k_j}\|)$ of $\|h^{k_j}\|$. Plugging these into equation (2.4), we have

$$\begin{aligned}
& f^{\mathrm{soc}}(x + h^{k_j}) - f^{\mathrm{soc}}(x) \\
= & \left[f(x_1 + h_1^{k_j} - \|h_2^{k_j}\|) - f(x_1)\right] v^{(1)} + \left[f(x_1 + h_1^{k_j} + \|h_2^{k_j}\|) - f(x_1)\right] v^{(2)} \\
= & \frac{a_1 + a_2}{2}\left(h_1^{k_j}, h_2^{k_j}\right) + \frac{a_2 - a_1}{2}\left(\|h_2^{k_j}\|, h_1 \frac{h_2^{k_j}}{\|h_2^{k_j}\|}\right) + o(\|h^{k_j}\|).
\end{aligned}$$

The first term is linear with respect to $h^{k_j} = (h_1^{k_j}, h_2^{k_j})$, but the second term is the trouble one which is nonlinear in general when $a_1 \neq a_2$. This is the trouble place that the reverse implication cannot be guaranteed at present.

3. A merit function for SCCP with H-differentiable functions

Recently, applications to nonlinear complementarity problems (NCP) and variational inequalities under H-differentiability have been considered in [25, 27, 28]. In this section, similar applications are extended to SCCP under H-differentiable condition which is a wider class of SCCPs than traditional SCCPs.

The formulations of SCCPs is to find $x, y \in \mathbb{V}$ and $\zeta \in \mathbb{V}$ such that

$$\langle x, y\rangle = 0, \quad x \in \mathcal{K}, \quad y \in \mathcal{K}, \tag{3.1}$$

$$x = F(\zeta), \quad y = G(\zeta), \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, $F : \mathbb{V} \to \mathbb{V}$ and $G : \mathbb{V} \to \mathbb{V}$ are H-differentiable mappings, $\mathbb{V}$ is the Cartesian product of simple Jordan algebras, and $\mathcal{K}$ is the Cartesian product of corresponding symmetric cones, i.e.,

$$\mathbb{V} = \mathbb{V}_1 \times \cdots \times \mathbb{V}_N \quad \text{and} \quad \mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_N.$$

Here each n_i-dimensional space $\mathbb{V}_i$ is a simple Jordan algebra with $n_1, \ldots, n_N \geq 1$, $n_1 + \cdots + n_N = n$, and

$$\mathcal{K}_i := \left\{x_i^2 \mid x_i \in \mathbb{V}_i\right\}.$$

For any $x, y \in \mathbb{V}$, we write $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N)$ with $x_i, y_i \in \mathbb{V}_i$. Then, $x \circ y = (x_1 \circ y_1, \ldots, x_N \circ y_N)$ and $\langle x, y\rangle = \langle x_1, y_1\rangle + \cdots + \langle x_N, y_N\rangle$. Therefore, the SCCP is equivalent to finding an $\zeta \in \mathbb{V}$ such that

$$F_i(\zeta) \in \mathcal{K}_i, \quad G_i(\zeta) \in \mathcal{K}_i, \quad \langle F_i(\zeta), G_i(\zeta)\rangle = 0, \quad i = 1, 2, \ldots, N. \tag{3.3}$$

An important special case of SCCP corresponds to $G(\zeta) = \zeta$ for all $\zeta \in \mathbb{V}$, namely, (3.1)-(3.2) reduces to

$$\langle F(\zeta), \zeta \rangle = 0, \quad F(\zeta) \in \mathcal{K}, \quad \zeta \in \mathcal{K}. \tag{3.4}$$

Next, we turn into the merit function approach for SCCP under H-differentiable condition. To this end, we recall that a smooth function $\psi : \mathbb{V} \times \mathbb{V} \to \mathbb{R}_+$ is called a merit function if

$$\psi(x, y) = 0 \iff (x, y) \text{ satisfies } (3.1).$$

A popular merit function is

$$\psi_{\mathrm{FB}}(x, y) = \frac{1}{2}\|\phi_{\mathrm{FB}}(x, y)\|^2 \tag{3.5}$$

where $\phi_{\mathrm{FB}} : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is the well-known Fisher-Burmeister (FB) complementarity function defined by

$$\phi_{\mathrm{FB}}(x, y) = (x^2 + y^2)^{\frac{1}{2}} - x - y. \tag{3.6}$$

It is known that $\phi_{\mathrm{FB}}(x, y) = 0$ if and only if (x, y) satisfies (3.1) by [14, Proposition 6]. With this fact, the SCCP can be expressed as an unconstrained (global) minimization problem associated with the merit function ψ_{FB}:

$$\min_{\zeta \in \mathbb{V}} f(\zeta) := \psi_{\mathrm{FB}}(F(\zeta), G(\zeta)). \tag{3.7}$$

The following proposition describes what we just mentioned.

Proposition 3.1 ([18, Lemma 2.2])**.** *Let ϕ_{FB} and ψ_{FB} be defined as in (3.6) and (3.5), respectively. Then, ψ_{FB} is continuously differentiable everywhere. Furthermore, $\nabla_x \psi_{\mathrm{FB}}(0, 0) = \nabla_y \psi_{\mathrm{FB}}(0, 0) = 0$; and if $(x, y) \neq (0, 0)$,*

$$\nabla_x \psi_{\mathrm{FB}}(x, y) = \left(L_x L^{-1}_{(x^2+y^2)^{1/2}} - I \right) \phi_{\mathrm{FB}}(x, y),$$
$$\nabla_y \psi_{\mathrm{FB}}(x, y) = \left(L_y L^{-1}_{(x^2+y^2)^{1/2}} - I \right) \phi_{\mathrm{FB}}(x, y),$$

where I denotes the identity operator from $\mathbb{V}$ to $\mathbb{V}$.

Lemma 3.2 ([18, Proposition 3.3])**.** *Let ϕ_{FB} and ψ_{FB} be defined as in (3.6) and (3.5), respectively. Then, for any $(x, y) \in \mathbb{V}$, the following hold.*

(a) $\langle \nabla_x \psi_{\mathrm{FB}}(x, y), \nabla_y \psi_{\mathrm{FB}}(x, y) \rangle \geq 0$, *with equality holding if and only if* $\phi_{\mathrm{FB}}(x, y) = 0$.

(b) $\psi_{\mathrm{FB}}(x, y) = 0 \iff \nabla_x \psi_{\mathrm{FB}}(x, y) = 0 \iff \nabla_y \psi_{\mathrm{FB}}(x, y) = 0$.

Let $T_F(\zeta)$ and $T_G(\zeta)$ denote the H-differentials of F and G, respectively. Since a Fréchet differentiable function is H-differentiable, by Proposition 3.1 and using the chain rule for H-differentiable functions, the H-differential of f defined as in (3.7) can be written as

$$\begin{aligned} &T_f(\zeta) \\ &= \{ M \nabla_x \psi_{\mathrm{FB}}(F(\zeta), G(\zeta)) + N \nabla_y \psi_{\mathrm{FB}}(F(\zeta), G(\zeta)) \mid M \in T_F(\zeta), N \in T_G(\zeta) \}. \end{aligned} \tag{3.8}$$

Now, we present the main result for merit function approach which indicates under what condition every stationary point of (3.7) is a solution of the SCCP with H-differentiable condition. This is answered in Proposition 3.3 whereas Proposition

3.5 provides a descent direction for non-stationary point. To establish it, we need the definition of the Cartesian P_0-property for a linear transformation from $\mathbb{V}$ to $\mathbb{V}$. Specifically, a linear transformation $\Upsilon : \mathbb{V} \to \mathbb{V}$ is said to have the Cartesian P_0-property if for any $0 \neq \zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{V}$, there exists an index $\nu \in \{1, 2, \ldots, N\}$ such that $\zeta_\nu \neq 0$ and $\langle \zeta_\nu, (\Upsilon\zeta)_\nu \rangle \geq 0$.

Proposition 3.3. *Let $\phi_{\rm FB}$ and $\psi_{\rm FB}$ be defined as in (3.6) and (3.5), respectively, and f be given by (3.7). Suppose F and G are H-differentiable and the H-differentials of F and G satisfy one of the following conditions:*

(i) *for every $\zeta \in \mathbb{V}$, $\forall M \in T_F(\zeta)$, $N \in T_G(\zeta)$, $M, -N$ are column monotone, i.e., for any $u, v \in \mathbb{V}$,*

$$Mu + (-N)v = 0 \implies \langle u, v \rangle \geq 0. \tag{3.9}$$

(ii) *for every $\zeta \in \mathbb{V}$, $\forall M \in T_F(\zeta)$, $N \in T_G(\zeta)$, N is invertible and $N^{-1}M$ has the Cartesian P_0-property.*

Then, there hold

$$0 \in T_f(\zeta) \quad \Longleftrightarrow \quad \psi_{\rm FB}(F(\zeta), G(\zeta)) = 0$$

Proof. "$\Longleftarrow$" This direction is easy to verify. To see this, from Proposition 3.1(b), it is clear that $\psi_{\rm FB}(F(\zeta), G(\zeta)) = 0$ implies

$$(\nabla_x \psi_{\rm FB}(F(\zeta), G(\zeta)),\ \nabla_y \psi_{\rm FB}(F(\zeta), G(\zeta))) = 0,$$

which yields $T_f(\zeta) = \{0\}$ by applying (3.8).

"$\Longrightarrow$" For this direction, suppose that $0 \in T_f(\zeta)$. From (3.8), there exist $M \in T_F(\zeta)$ and $N \in T_G(\zeta)$ such that

$$M\nabla_x \psi_{\rm FB}(F(\zeta), G(\zeta)) + N\nabla_y \psi_{\rm FB}(F(\zeta), G(\zeta)) = 0. \tag{3.10}$$

(a) If condition (i) is satisfied, from the column monotonicity of M and $-N$, we know

$$\langle \nabla_x \psi_{\rm FB}(F(\zeta), G(\zeta)), \nabla_y \psi_{\rm FB}(F(\zeta), G(\zeta)) \rangle \leq 0.$$

This together with Lemma 3.2(b) implies $\psi_{\rm FB}(F(\zeta), G(\zeta)) = 0$.

(b) If condition (ii) is satisfied. From (3.10), we have

$$N^{-1}M\nabla_x \psi_{\rm FB}(F(\zeta), G(\zeta)) + \nabla_y \psi_{\rm FB}(F(\zeta), G(\zeta)) = 0. \tag{3.11}$$

For any $u = (u_1, \ldots, u_N), v = (v_1, \ldots, v_N) \in \mathbb{V}$ with $u_i, v_i \in \mathbb{V}_i$, we write

$$\nabla_x \psi_{\rm FB}(u, v) = \left(\nabla_{x_1} \psi_{\rm FB}(u_1, v_1), \ldots, \nabla_{x_N} \psi_{\rm FB}(u_N, v_N)\right),$$
$$\nabla_y \psi_{\rm FB}(u, v) = \left(\nabla_{y_1} \psi_{\rm FB}(u_1, v_1), \ldots, \nabla_{y_N} \psi_{\rm FB}(u_N, v_N)\right).$$

Assume that ζ is not a solution of $\psi_{\rm FB}(F(\zeta), G(\zeta)) = 0$, applying Lemma 3.2(b) gives

$$\nabla_x \psi_{\rm FB}(F(\zeta), G(\zeta)) \neq 0 \quad \text{and} \quad \nabla_y \psi_{\rm FB}(F(\zeta), G(\zeta)) \neq 0. \tag{3.12}$$

By the Cartesian P_0-property of $N^{-1}M$, there exists an index $\nu \in \{1, 2, \ldots, N\}$ such that $\nabla_{x_\nu} \psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta)) \neq 0$ and

$$\left\langle \nabla_{x_\nu} \psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta)), \left[N^{-1}M\nabla_x \psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta))\right]_\nu \right\rangle \geq 0.$$

On the other hand, it follows from (3.11) that

$$\left\langle \nabla_{x_\nu} \psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta)), \left[N^{-1}M\nabla_x \psi_{\rm FB}(F_\nu(\zeta), G_\nu(\zeta))\right]_\nu \right\rangle$$

$$= \; -\left\langle \nabla_{x_\nu}\psi_{\rm FB}(F_\nu(\zeta),G_\nu(\zeta)),\nabla_{y_\nu}\psi_{\rm FB}(F_\nu(\zeta),G_\nu(\zeta))\right\rangle.$$

Combining last two equations and using Lemma 3.2(a) yield

$$\langle \nabla_{x_\nu}\psi_{\rm FB}(F_\nu(\zeta),G_\nu(\zeta)),\nabla_{y_\nu}\psi_{\rm FB}(F_\nu(\zeta),G_\nu(\zeta))\rangle = 0.$$

Hence, $\phi_{\rm FB}(F_\nu(\zeta),G_\nu(\zeta))=0$ which contradicts (3.12). Therefore, under condition (ii), we prove that $0\in T_f(\zeta)$ implies $\psi_{\rm FB}(F(\zeta),G(\zeta))=0$. □

Remark 3.4. A different merit function for the the SCCP with F and G being H-differentiable is considered in [24]. In fact, [24, Theorem 4.1] also provides a condition, under which any stationary point of merit function is a solution of the SCCP. However, that condition is stricter than condition (ii) in Proposition 3.3.

Proposition 3.5. *Let $\phi_{\rm FB}$ and $\psi_{\rm FB}$ be defined as in (3.6) and (3.5), respectively, and f be given by (3.7). Suppose F and G are H-differentiable and the H-differentials of F and G satisfy assumption (3.9). In the case of $0\notin T_f(\zeta)$, if there exists $\bar N\in T_G(\zeta)$ which is invertible, then*

$$d_{\rm FB}(\zeta) := -(\bar N^{-1})^T\nabla_x\psi_{\rm FB}(F(\zeta),G(\zeta))$$

is a descent direction of f at ζ.

Proof. From the definition of a descent direction for an H-differentiable function at a point, it is sufficient to prove that for some $M\in T_F(\zeta)$ and $N\in T_G(\zeta)$, there holds

$$\langle d_{\rm FB}(\zeta), M\nabla_x\psi_{\rm FB}(F(\zeta),G(\zeta))+N\nabla_y\psi_{\rm FB}(F(\zeta),G(\zeta))\rangle<0. \tag{3.13}$$

In fact, for any $M\in T_F(\zeta)$ (dropping the argument "$(F(\zeta),G(\zeta))$" for simplicity), we have

$$\begin{aligned}\left\langle d_{\rm FB}(\zeta), M\nabla_x\psi_{\rm FB}+\bar N\nabla_y\psi_{\rm FB}\right\rangle &= \left\langle -(\bar N^{-1})^T\nabla_x\psi_{\rm FB}, M\nabla_x\psi_{\rm FB}+\bar N\nabla_y\psi_{\rm FB}\right\rangle\\ &= -\left\langle \nabla_x\psi_{\rm FB},\bar N^{-1}M\nabla_x\psi_{\rm FB}\right\rangle-\langle\nabla_x\psi_{\rm FB},\nabla_y\psi_{\rm FB}\rangle\\ &\le -\left\langle\nabla_x\psi_{\rm FB},\nabla_y\psi_{\rm FB}\right\rangle,\end{aligned}$$

where the inequality follows from the fact $\bar N^{-1}M$ is a positive semi-definite matrix (this is guaranteed from the invertibility of $\bar N$ and assumption (3.9)). By Lemma 3.2(b), the right-hand side is non-positive and equals to zero if and only if $\psi_{\rm FB}(F(\zeta),G(\zeta))=0$. On the other hand, applying Proposition 3.3 gives

$$\psi_{\rm FB}(F(\zeta),G(\zeta))=0 \quad\Longleftrightarrow\quad 0\in T_f(\zeta).$$

Therefore, in the case of $0\notin T_f(\zeta)$, the right-hand cannot equal zero, so it must be negative. Thus, (3.13) is satisfied which says $d_{\rm FB}(\zeta)$ is a descent direction. □

It is known that the SCCP is closely related to the Karush-Kuhn-Tucker (KKT) optimality conditions for the convex symmetric cone program (CSCP):

$$\begin{array}{ll}\min & g(x)\\ \text{s.t.} & Ax=b,\quad x\in\mathcal{K}\end{array} \tag{3.14}$$

where $A:\mathbb{V}\to\mathbb{R}^m$ is a linear operator, $b\in\mathbb{R}^m$ and $g:\mathbb{V}\to\mathbb{R}$ is a convex and smooth (continuously differentiable) function with its gradient mapping $\nabla g:\mathbb{V}\to\mathbb{V}$ being H-differentiable. Especially, when $\mathcal{K}$ is a second order cone, the assumption of

g being a convex and smooth function is equivalent to the condition $\nabla g : \mathbb{R}^n \to \mathbb{R}^n$ being a monotone function. Furthermore, if ∇g satisfies either

(a) ∇g is Fréchet differentiable on $\mathbb{R}^n$, or
(b) ∇g is locally Lipschitzian on $\mathbb{R}^n$,

then ∇g is H-differentiable at any $x \in \mathbb{R}^n$. For each case, the H-differential of ∇g is the set $\{\nabla^2 g(x)\}$ and the generalized Jacobian

$$\partial(\nabla g)(x) = \operatorname{conv}\left\{ \lim_{k\to\infty} \nabla^2 g(x^k) \mid x^k \in \mathcal{D}_{\nabla g}, x^k \to x \right\},$$

respectively, where $\mathcal{D}_{\nabla g}$ denotes the set Fréchet differentiable points of ∇g in $\mathbb{R}^n$. In particular, under each of the aforementioned two cases, it is well known that the convexity of g (or the monotonicity of ∇g) is equivalent to the conclusion that the H-differentials of ∇g consist of positive semi-definite (p.s.d.) matrices, see [15, Proposition 2.3]. In summary, under the smoothness of g,

$$\begin{aligned} & g \text{ is convex} \\ \text{(3.15)} \iff & \nabla g \text{ is monotone} \\ \iff & \text{the } H\text{-differential of } \nabla g \text{ consists of p.s.d. matrices for case (a) or (b).} \end{aligned}$$

The above discussions in SOC case raise the motivation of investigating such equivalences in general case, i.e., is (3.15) true in general? In fact, for the general case without convexity of g, one direction is known true, i.e., if the H-differential of ∇g consists of positive semi-definite matrices then ∇g is a monotone function, see the following theorem.

Theorem 3.6 ([13, Theorem 4])**.** *If $h : \mathbb{R}^n \to \mathbb{R}^n$ is H-differentiable at each point $x \in \mathbb{R}^n$ with $T_h(x)$ consisting of positive semi-definite (positive definite) matrices, then h is a monotone (strictly monotone) function.*

Theorem 3.6 indicates that the H-differential of an H-differentiable function consisting of positive semi-definite matrices provides a sufficient condition for monotonicity of this function. However, the H-differential consisting of positive semi-definite matrices is not a necessary condition, namely, we don't know whether the opposite side of Theorem 3.6 is true or not. What can we achieve for necessary case? Below are results describing the opposite direction of Theorem 3.6.

Proposition 3.7. *Suppose $h : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone and H-differentiable mapping.*

(a) *For $n = 1$, if $a \in T_h(x)$ for a point $x \in \mathbb{R}$, then $a \geq 0$.*
(b) *For $n \geq 2$, if $A \in T_h(x)$ for a point $x \in \mathbb{R}^n$, then A is positive semi-definite in the subspace $\mathcal{E} \subseteq \mathbb{R}^n$ where*

$$\mathcal{E} = \{d \in \mathbb{R}^n \mid d \text{ satisfies } (3.17)\}. \tag{3.16}$$

In particular, from the equivalent definition of Definition 2.1, for $A \in T_h(x)$, there exists some $d \in \mathbb{R}^n$ satisfying

$$\lim_{j\to\infty} \frac{h(x + t_{k_j} d^{k_j}) - h(x)}{t_{k_j}} = Ad \tag{3.17}$$

for some sequence $x^k := x + t_k d^k$ *with* $t_k \downarrow 0$ *and* $\|d^k\| = 1$ *for all* k*, and for any convergent subsequence* $t_{k_j} \downarrow 0$ *and* $d^{k_j} \to d$*.*

Proof. (a) From Definition 2.1, we have

$$h(x + t_{k_j} d^{k_j}) - h(x) = a t_{k_j} d^{k_j} + o\left(t_{k_j}\right)$$

for some sequence $x^k := x + t_k d^k$ with $t_k \downarrow 0$ and $\|d^k\| = 1$ for all k, and for any convergent subsequence $t_{k_j} \downarrow 0$ and $d^{k_j} \to d$, where $d = 1$ or -1 in this case. By the monotonicity of h,

$$0 \leq \Big(h(x + t_{k_j} d^{k_j}) - h(x)\Big) t_{k_j} d^{k_j} = a t_{k_j}^2 \left(d^{k_j}\right)^2 + o\left(t_{k_j}^2\right) = a t_{k_j}^2 + o\left(t_{k_j}^2\right)$$

which implies $a \geq 0$.
(b) From equation (3.17), we know

$$h(x + t_{k_j} d^{k_j}) - h(x) = A t_{k_j} d^{k_j} + o\left(t_{k_j}\right).$$

By the monotonicity of h again,

$$0 \leq \langle h(x + t_{k_j} d^{k_j}) - h(x), t_{k_j} d^{k_j} \rangle = t_{k_j}^2 (d^{k_j})^T A d^{k_j} + o\left(t_{k_j}^2\right),$$

and hence

$$d^T A d = \lim_{j \to \infty} \frac{t_{k_j}^2 (d^{k_j})^T A d^{k_j} + o\left(t_{k_j}^2\right)}{t_{k_j}^2} \geq 0$$

which says A is positive semi-definite in the subspace $\mathcal{E}$ defined in (3.16). □

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