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NONLINEAR ERGODIC THEOREMS WITHOUT CONVEXITY FOR NONEXPANSIVE SEMIGROUPS IN HILBERT SPACES

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Abstract. In this paper, we introduce the concept of common attractive points of a nonexpansive semigroup in a Hilbert space and study fundamental properties for the points. We also prove a nonlinear mean convergence theorem of Baillon's type [3] without convexity for nonexpansive semigroups. Using this result, we obtain new and well-known nonlinear mean convergence theorems in a Hilbert space.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty subset of H. For a mapping $T : C \to C$, we denote by $F(T)$ the set of *fixed points* of T and by $A(T)$ the set of *attractive points* [15] of T, i.e.,

(i) $F(T) = \{z \in C : Tz = z\}$;
(ii) $A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \ \forall x \in C\}$.

A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 1975, Baillon [3] proved the following first nonlinear ergodic theorem in a Hilbert space: Let C be a nonempty bounded closed convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, for any $x \in C$, $S_n x = \frac{1}{n} \sum_{i=0}^{n-1} T^i x$ converges weakly to a fixed point of T (see also [14]).

Recently, Kocourek, Takahashi and Yao [5] introduced a broad class of nonlinear mappings called *generalized hybrid* which containing nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. They proved a mean convergence theorem for generalized hybrid mappings which generalizes Baillon's nonlinear ergodic theorem. Motivated by Baillon [3], and Kocourek, Takahashi and Yao [5], Takahashi and Takeuchi [15] introduced the concept of attractive points of a nonlinear mapping in a Hilbert space and they proved a mean convergence theorem of Baillon's type without convexity for generalized hybrid mappings.

In this paper, we introduce the concept of common attractive points of a nonexpansive semigroup in a Hilbert space and study fundamental properties for the points. We also prove a nonlinear mean convergence theorem of Baillon's type without convexity for nonexpansive semigroups. Using this result, we obtain new and well-known nonlinear mean convergence theorems in a Hilbert space.

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2. Preliminaries and notations

Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all positive integers and the set of all real numbers, respectively. We also denote by $\mathbb{Z}^+$ and $\mathbb{R}^+$ the set of all nonnegative integers and the set of all nonnegative real numbers, respectively. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We know the following basic equality from [14]. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

Furthermore, we obtain that for all $x, y, w \in H$,

$$\langle (x-y) + (x-w), y-w \rangle = \|x-w\|^2 - \|x-y\|^2. \tag{2.2}$$

In fact, we have that

$$\begin{aligned} &\langle (x-y) + (x-w), y-w \rangle \\ &= \langle (x-y) + (x-w), (y-x) + (x-w) \rangle \\ &= \|x-w\|^2 - \|x-y\|^2 + \langle x-y, x-w \rangle + \langle x-w, y-x \rangle \\ &= \|x-w\|^2 - \|x-y\|^2. \end{aligned}$$

Let C be a closed and convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. The mapping P_C is called the *metric projection* of H onto C. It is characterized by

$$\langle P_C x - y, x - P_C x \rangle \geq 0$$

for all $y \in C$. See [14] for more details. The following result is well-known; see [14].

Lemma 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let T be a nonexpansive mapping of C into itself. Then, $F(T) \neq \emptyset$.*

We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges strongly to x. We also write $x_n \rightharpoonup x$ (or w-$\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{x_n\}$ of vectors in H converges weakly to x. In a Hilbert space, it is well known that $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. We say that a Banach space E satisfies *Opial's condition* [9] if for each sequence $\{x_n\}$ in E which converges weakly to x,

$$\varliminf_{n\to\infty} \|x_n - x\| < \varliminf_{n\to\infty} \|x_n - y\| \tag{2.3}$$

for each $y \in E$ with $y \neq x$. In a reflexive Banach space, this condition is equivalent to the analogous condition for a bounded net which has been introduced in [7]. It is also known that this condition is equivalent to the analogous condition of $\varlimsup$ (see [2]). It is known that Hilbert spaces satisfy Opial's condition (see [9, 14]).

Let S be a semitopological semigroup, i.e., S is a semigroup with a Hausdorff topology such that for each $a \in S$ the mappings $s \mapsto a \cdot s$ and $s \mapsto s \cdot a$ from S to S are continuous. In the case when S is commutative, we denote st by $s + t$. A commutative semigroup S is a directed system when the binary relation is defined by $s \leq t$ if and only if $\{s\} \cup (S + s) \supset \{t\} \cup (S + t)$.

Let C be a nonempty subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings of C into itself is said to be a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) For each $t \in S$, $T(t)$ is nonexpansive;
(ii) $T(ts) = T(t)T(s)$ for each $t, s \in S$;
(iii) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

We denote by $F(\mathcal{S})$ the set of all common fixed points of a nonexpansive semigroup $\mathcal{S}$, i.e.,

$$F(\mathcal{S}) = \bigcap_{t \in S} F(T(t)).$$

Motivated by Takahashi and Takeuchi [15], we introduce the set $A(\mathcal{S})$ of all common attractive points of the family $\mathcal{S} = \{T(t) : t \in S\}$ of mappings on C, i.e.,

$$A(\mathcal{S}) = \{x \in H : \|T(t)y - x\| \leq \|y - x\|,\ \forall y \in C,\ t \in S\}.$$

Let S be a commutative semigroup and let $B(S)$ be the Banach space of all bounded real-valued functions defined on S with supremum norm. Let $C(S)$ be the Banach space of all continuous bounded real-valued functions defined on S with supremum norm. For each $s \in S$ and $g \in B(S)$, we can define an element $\ell_s g \in B(S)$ by $(\ell_s g)(t) = g(s + t)$ for all $t \in S$. Let X be a subspace of $B(S)$ containing 1 and let X^* be its topological dual. A linear functional μ on X is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(g(t))$ or $\int g(t) d\mu(t)$ instead of $\mu(g)$ for $\mu \in X^*$ and $g \in X$. Furthermore, assume that X is invariant under every $\ell_s, s \in S$, i.e., $\ell_s X \subset X$ for each $s \in S$. Then, a mean μ on X is called *invariant* if $\mu(\ell_s g) = \mu(g)$ for all $s \in S$ and $g \in X$. For $s \in S$, we can define a *point evaluation* δ_s by $\delta_s(g) = g(s)$ for every $g \in B(S)$. A convex combination of point evaluations is called a *finite mean* on S. A finite mean μ on S is also a mean on any subspace X of $B(S)$ containing constants. A net $\{\mu_\alpha\}$ of means on X is said to be *strongly asymptotically invariant* if for each $s \in S$,

$$\|\ell_s^* \mu_\alpha - \mu_\alpha\| \to 0,$$

where ℓ_s^* is the adjoint operator of ℓ_s. The following definition which was introduced by Takahashi [11] is crucial in the fixed point theory for abstract semigroups (see also [4]). Let h be a bounded function of S into H. Let X be a subspace of $B(S)$ containing constants and invariant under every ℓ_s, $s \in S$. Assume that for each $z \in H$, the function $t \mapsto \langle h(t), z\rangle$ is an element of X. Then, for any $\mu \in X^*$ there exists a unique element $h_\mu \in H$ such that

$$\langle h_\mu, z\rangle = (\mu)_t \langle h(t), z\rangle = \int \langle h(t), z\rangle \, d\mu(t), \quad \forall z \in H.$$

If μ is a mean on X, then h_μ is contained in $\overline{\text{co}}\{h(t) : t \in S\}$, where $\overline{\text{co}}A$ is the closure of convex hull of A (for example, see [11, 14]). Sometimes, h_μ will be denoted by $\int h(t) d\mu(t)$. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Assume that for each $x \in C$ and $z \in H$, $\{T(t)x : t \in S\}$ is bounded. Let μ be a mean on $C(S)$. Following [10], we also write $T_\mu x$ instead of $\int T(t)x \, d\mu(t)$. We remark that

T_μ is nonexpansive on C and $T_\mu x = x$ for each $x \in F(\mathcal{S})$. If μ is a finite mean, i.e.,

$$\mu = \sum_{i=1}^{n} a_i \delta_{t_i} \ (t_i \in S, a_i \geq 0, \ \sum_{i=1}^{n} a_i = 1),$$

then we have

$$T_\mu x = \sum_{i=1}^{n} a_i T(t_i) x, \quad \forall x \in C.$$

This fact is important in Section 5 of this paper.

3. Lemmas

In this section, we prove some lemmas which are used in the proof of our main theorem. They are basic properties of common attractive points of nonexpansive semigroups in a Hilbert space.

Lemma 3.1. *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. If $A(\mathcal{S}) \neq \emptyset$, then $F(\mathcal{S}) \neq \emptyset$.*

Proof. Let $u \in A(\mathcal{S})$ and $y = P_C u \in C$. Then, we have $T(t)y \in C$ from $T(t)C \subset C$. Furthermore, we have

$$\|T(t)y - u\| \leqq \|y - u\| = \|P_C u - u\|.$$

By the properties of P_C, we have $T(t)y = P_C u = y$. Therefore $y \in F(\mathcal{S})$. □

Lemma 3.2. *Let H be a Hilbert space, let C be a nonempty subset of H, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Then, $A(\mathcal{S})$ is a closed and convex subset of H.*

Proof. We show that $A(\mathcal{S})$ is closed. Let $\{z_n\} \subset A(\mathcal{S})$ be a sequence which converges strongly to $z \in H$. Take $x \in C$ and $t \in S$. From $z_n \in A(\mathcal{S})$, we have

$$\begin{aligned} \|z - T(t)x\| &\leqq \|z - z_n\| + \|z_n - T(t)x\| \\ &\leqq \|z - z_n\| + \|z_n - x\|. \end{aligned}$$

Since $z_n \to z$, we have

$$\|z - T(t)x\| \leqq \|z - x\|.$$

This implies that $z \in A(\mathcal{S})$. So, $A(\mathcal{S})$ is closed. We prove that $A(\mathcal{S})$ is convex. Let $z_1, z_2 \in A(\mathcal{S}), \alpha \in [0,1]$ and $z = \alpha z_1 + (1-\alpha) z_2$. We prove from (2.1) that for any $x \in C$,

$$\begin{aligned} \|z - T(t)x\|^2 &= \|\alpha z_1 + (1-\alpha) z_2 - T(t)x\|^2 \\ &= \alpha \|z_1 - T(t)x\|^2 + (1-\alpha)\|z_2 - T(t)x\|^2 - \alpha(1-\alpha)\|z_1 - z_2\|^2 \\ &\leqq \alpha \|z_1 - x\|^2 + (1-\alpha)\|z_2 - x\|^2 - \alpha(1-\alpha)\|z_1 - z_2\|^2 \\ &= \|\alpha(z_1 - x) + (1-\alpha)(z_2 - x)\|^2 = \|z - x\|^2. \end{aligned}$$

This implies that $z \in A(\mathcal{S})$. So, $A(\mathcal{S})$ is convex. □

Lemma 3.3. *Let H be a Hilbert space, let C be a nonempty subset of H, and let S be a commutative semigroup. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C. Let $\{u_\alpha\}$ be a net in H such that*

$$\overline{\lim_\alpha} \langle (u_\alpha - y) + (u_\alpha - T(t)y), y - T(t)y \rangle \leqq 0$$

for all $t \in S$ and $y \in C$. If a subnet $\{u_{\alpha_\beta}\}$ of $\{u_\alpha\}$ converges weakly to $u \in H$, then $u \in A(\mathcal{S})$.

Proof. Since $\{u_{\alpha_\beta}\}$ converses weakly to $u \in H$, we have that for any $t \in S$ and $y \in C$,

$$\begin{aligned}\langle (u - y) + (u - T(t)y), y - T(t)y \rangle & \\ &= \lim_\beta \langle (u_{\alpha_\beta} - y) + (u_{\alpha_\beta} - T(t)y), y - T(t)y \rangle \\ &\leq \overline{\lim_\alpha} \langle (u_\alpha - y) + (u_\alpha - T(t)y), y - T(t)y \rangle \\ &\leq 0.\end{aligned}$$

On the other hand, we know from (2.2) that

$$\langle (u - y) + (u - T(t)y), y - T(t)y \rangle = \|u - T(t)y\|^2 - \|u - y\|^2.$$

Thus we have

$$\|u - T(t)y\| \leq \|u - y\|.$$

for all $t \in S$ and $y \in C$. This implies $u \in A(\mathcal{S})$. □

The following was proved by Lau and Takahashi [8] (see also [16]).

Lemma 3.4. *Let H be a Hilbert space, let C be a nonempty subset of H and let D be a nonempty, closed and convex subset of H. Let P be the metric projection from H onto D. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C and $x \in C$. If $\|T(t+s)x - v\| \leqq \|T(t)x - v\|$ for any $v \in D$ and $s, t \in S$, then $\{PT(t)x\}$ converges strongly to $v_0 \in D$.*

4. Nonlinear ergodic theorem

In this section, we prove a nonlinear mean ergodic theorem without convexity for finding a common attractive point of a semigroup of nonexpansive mappings in a Hilbert space by using the ideas of [3, 15] (see also [14]).

Theorem 4.1. *Let H be a Hilbert space, let C be a nonempty subset of H. Let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\{T(t)x : t \in S\}$ is bounded for some $x \in C$. Let $\{\mu_\alpha\}$ be a strongly asymptotically invariant net of means on $C(S)$, i.e., a net of means on $C(S)$ such that*

$$\lim_\alpha \|\mu_\alpha - \ell_s^* \mu_\alpha\| = 0.$$

Then, the following hold:

1. *$A(\mathcal{S})$ is non-empty, closed and convex;*
2. *for any $u \in C$, $\{T_{\mu_\alpha} u\}$ converges weakly to $u_0 \in A(\mathcal{S})$, where $u_0 = \lim_t P_{A(\mathcal{S})} T(t)u$.*

Proof. Since $\{T(t)x : t \in S\}$ is bounded for some $x \in C$, we can define a net $\{T_{\mu_\alpha}x\}$ of elements of H such that

$$\langle T_{\mu_\alpha}x, y\rangle = (\mu_\alpha)_t\langle T(t)x, y\rangle$$

for all $y \in H$. Since $\mathcal{S} = \{T(t) : t \in S\}$ is a nonexpansive semigroup on C, we have that

$$\begin{aligned}\langle (T(s)x-y) + (T(s)x - T(t)y), y - T(t)y\rangle \\ &= \|T(s)x - T(t)y\|^2 - \|T(s)x - y\|^2 \\ &\leq \|T(s)x - T(t)y\|^2 - \|T(s+t)x - T(t)y\|^2\end{aligned}$$

for all $t, s \in S$ and $y \in C$. Applying μ_α to both sides of the inequality, we have that

$$\begin{aligned}&\langle (T_{\mu_\alpha}x - y) + (T_{\mu_\alpha}x - T(t)y), y - T(t)y\rangle \\ &\quad= (\mu_\alpha)_s\langle (T(s)x - y) + (T(s)x - T(t)y), y - T(t)y\rangle \\ &\quad\leq (\mu_\alpha)_s(\|T(s)x - T(t)y\|^2 - \|T(s+t)x - T(t)y\|^2) \\ &\quad= (\mu_\alpha)_s\|T(s)x - T(t)y\|^2 - (\mu_\alpha)_s\|T(s+t)x - T(t)y\|^2 \\ &\quad= (\mu_\alpha - \ell_t^*\mu_\alpha)_s\|T(s)x - T(t)y\|^2 \\ &\quad\leq \|\mu_\alpha - \ell_t^*\mu_\alpha\| \sup_{s\in S}\|T(s)x - T(t)y\|^2\end{aligned}$$

for all $t \in S$ and $y \in C$. Since $\lim_\alpha\|\mu_\alpha - \ell_t^*\mu_\alpha\| = 0$ for each $t \in S$, we have that

$$\overline{\lim_\alpha}\ \langle (T_{\mu_\alpha}x - y) + (T_{\mu_\alpha}x - T(t)y)\,, y - T(t)y\rangle \leqq 0.$$

Since$\{T(t)x\}$ is bounded, so is $\{T_{\mu_\alpha}x\}$. There exists a subnet $\{T_{\mu_{\alpha_\beta}}x\}$ of $\{T_{\mu_\alpha}x\}$ which converges weakly to a point $b \in H$. By Lemma 3.3, we have that $b \in A(\mathcal{S})$. Hence, we have that $A(\mathcal{S})$ is non-empty. By Lemma 3.2, we have that $A(\mathcal{S})$ is closed and convex.

Let us prove (2). Let $u \in C$. Since $A(\mathcal{S})$ is non-empty, we have that

$$\|T(t)u - v\| \leq \|u - v\|$$

for all $t \in S$ and $v \in A(\mathcal{S})$. Then $\{T(t)u\}$ is bounded. Furthermore, we have that

$$\|T(t+s)u - v\| = \|T(s)T(t)u - v\| \leqq \|T(t)u - v\|$$

for each $t, s \in S$ and $v \in A(\mathcal{S})$. By Lemma 3.4, there exists $v_0 \in A(\mathcal{S})$ such that $\lim_t P_{A(\mathcal{S})}T(t)u = v_0 \in A(\mathcal{S})$. Since $P_{A(\mathcal{S})}$ is the metric projection of H onto $A(\mathcal{S})$, we have that

$$\begin{aligned}\|P_{A(\mathcal{S})}T(t+s)u - T(t+s)u\| &\leqq \|P_{A(\mathcal{S})}T(t)u - T(t+s)u\| \\ &= \|P_{A(\mathcal{S})}T(t)u - T(s)T(t)u\| \\ &\leq \|P_{A(\mathcal{S})}T(t)u - T(t)u\|.\end{aligned}$$

Thus, $\{\|T(t)u - P_{A(\mathcal{S})}T(t)u\|\}$ is non-increasing. Let $\{T_{\mu_{\alpha_\beta}}u\}$ be a weakly convergent subnet of $\{T_{\mu_\alpha}u\}$ and let $\{T_{\mu_{\alpha_\beta}}u\}$ converge weakly to a point in $b \in H$. As in the proof of (1), we have $b \in A(\mathcal{S})$. To complete the proof of (2), it is sufficient to show $b = v_0$. We have from the property of $P_{A(\mathcal{S})}$ that

$$\langle T(t)u - P_{A(\mathcal{S})}T(t)u, P_{A(\mathcal{S})}T(t)u - v\rangle \geqq 0 \tag{4.1}$$

for all $t \in S$ and $v \in A(\mathcal{S})$. Since $\{T(t)u\}$ is bounded, so is $\{P_{A(\mathcal{S})}T(t)u\}$. So, there exists $M > 0$ such that $\|T(t)u\| \leqq M$ and $\|P_{A(\mathcal{S})}T(t)u\| \leqq M$ for every $t \in S$. We have that for any $v \in A(\mathcal{S})$ and $t \in S$,

$$\begin{aligned}\langle v - v_0, &T(t)u - P_{A(\mathcal{S})}T(t)u\rangle \\ &\leqq \left\langle P_{A(\mathcal{S})}T(t)u - v_0, T(t)u - P_{A(\mathcal{S})}T(t)u\right\rangle \\ &\leqq \|P_{A(\mathcal{S})}T(t)u - v_0\| \cdot \|T(t)u - P_{A(\mathcal{S})}T(t)u\| \\ &\leqq 2M \cdot \|P_{A(\mathcal{S})}T(t)u - v_0\|.\end{aligned}$$

Then, we have that

$$(\mu_\alpha)_t\langle v - v_0, T(t)u - P_{A(\mathcal{S})}T(t)u\rangle \leqq 2M \cdot (\mu_\alpha)_t\|P_{A(\mathcal{S})}T(t)u - v_0\|. \tag{4.2}$$

Since $\{\mu_\alpha\}$ is a net of means on $C(S)$ such that

$$\lim_\alpha \|\mu_\alpha - \ell_s^*\mu_\alpha\| = 0$$

for each $s \in S$, a cluster point μ of $\{\mu_\alpha\}$ in the weak* topology is an invariant mean on $C(S)$; see the proof of [14, Theorem 3.4.4]. Without loss of generality, we may assume that $\mu_{\alpha_\beta} \rightharpoonup \mu$ in the weak* topology. Replacing α by α_β in (4.2), we have that

$$(\mu_{\alpha_\beta})_t\langle v - v_0, T(t)u - P_{A(\mathcal{S})}T(t)u\rangle \leqq 2M \cdot (\mu_{\alpha_\beta})_t\|P_{A(\mathcal{S})}T(t)u - v_0\| \tag{4.3}$$

and hence

$$(\mu)_t\langle v - v_0, T(t)u - P_{A(\mathcal{S})}T(t)u\rangle \leqq 2M \cdot (\mu)_t\|P_{A(\mathcal{S})}T(t)u - v_0\|. \tag{4.4}$$

Since μ is an invariant mean on $C(S)$, we have from

$$\lim_{t\in S} \|P_{A(\mathcal{S})}T(t)u - v_0\| = 0$$

and $T_{\mu_{\alpha_\beta}}u \rightharpoonup b \in H$,

$$\langle v - v_0, b - v_0\rangle \leqq 0$$

for any $v \in A(\mathcal{S})$. Setting $v = b$, we have $\|b - v_0\| \leqq 0$ and hence $b = v_0$. Thus $\{T_{\mu_\alpha}u\}$ converges weakly to $v_0 \in A(\mathcal{S})$. This completes the proof of (2). □

5. Applications

Throughout this section, let C be a nonempty subset of a Hilbert space H. Using Theorems 4.1, we can prove some nonliear mean ergodic theorems as in [4] and [14].

Theorem 5.1. *Let T be a nonexpansive mapping of C into itself such that $\{T^n x\}$ is bounded for some $x \in C$. Then, the following hold:*

1. *$A(T)$ is non-empty, closed and convex:*
2. *for any $u \in C$, $\{\frac{1}{n}\sum_{i=0}^{n-1} T^i u\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)}T^n u$.*

Proof. Put $S = \mathbb{Z}^+$ in Theorem 4.1 and define

$$\mu_n(f) = \frac{1}{n} \sum_{i=0}^{n-1} f(i)$$

for all $n \in \mathbb{N}$ and $f \in B(S)$. We have that $\{\mu_n : n \in \mathbb{N}\}$ is a strongly asymptotically invariant sequence of means on $B(S)$. Furthermore, we have that for any $u \in C$ and $n \in \mathbb{N}$,

$$T_{\mu_n} u = \frac{1}{n} \sum_{i=0}^{n-1} T^i u.$$

Therefore, we obtain Theorem 5.1 by using Theorem 4.1. □

Theorem 5.2. *Let T be a nonexpansive mapping of C into itself such that $\{T^n x\}$ is bounded for some $x \in C$. Let $\{q_{n,m} : n, m \in \mathbb{Z}^+\}$ be a sequence of real numbers such that $q_{n,m} \geq 0$, $\sum_{m=0}^{\infty} q_{n,m} = 1$ for each $n \in \mathbb{Z}^+$ and $\lim_{n\to\infty} \sum_{m=0}^{\infty} |q_{n,m+1} - q_{n,m}| = 0$. Then, the following hold:*

1. *$A(T)$ is non-empty, closed and convex:*
2. *for any $u \in C$, $\{\sum_{m=0}^{\infty} q_{n,m} T^m u\}$ converges weakly to $u_0 \in A(T)$, where $u_0 = \lim_{n\to\infty} P_{A(T)} T^n u$.*

Proof. Put $S = \mathbb{Z}^+$ in Theorem 4.1 and define

$$\mu_n(f) = \sum_{m=0}^{\infty} q_{n,m} f(m)$$

for all $n \in \mathbb{Z}^+$ and $f \in B(S)$. We have that $\{\mu_n : n \in \mathbb{Z}^+\}$ is a strongly asymptotically invariant sequence of means on $B(S)$. Furthermore, we have that for any $u \in C$ and $n \in \mathbb{Z}^+$,

$$T_{\mu_n} u = \sum_{m=0}^{\infty} q_{n,m} T^m u.$$

Therefore, we obtain Theorem 5.2 by using Theorem 4.1. □

Theorem 5.3. *Let T and U be nonexpansive mappings of C into itself such that $\{T^i U^j x : i, j \in \mathbb{Z}^+\}$ is bounded for some $x \in C$. Then, the following hold:*

1. *$A(T) \cap A(U)$ is non-empty, closed and convex;*
2. *for any $u \in C$, $\{\frac{1}{(n)^2} \sum_{i,j=0}^{n-1} T^i U^j u\}$ converges weakly to $u_0 \in A(T) \cap A(S)$.*

Proof. Put $S = \mathbb{Z}^+ \times \mathbb{Z}^+$ in Theorem 4.1 and define

$$\mu_n(f) = \frac{1}{n^2} \sum_{i,j=0}^{n-1} f(i, j)$$

for all $n \in \mathbb{N}$ and $f \in B(S)$. We have that $\{\mu_n : n \in \mathbb{N}\}$ is a strongly asymptotically invariant sequence of means on $B(S)$. Furthermore, we have that for any $u \in C$ and $n \in \mathbb{N}$,

$$T_{\mu_n} u = \frac{1}{(n)^2} \sum_{i,j=0}^{n-1} T^i U^j u.$$

Therefore, we obtain Theorem 5.3 by using Theorem 4.1. □

Let C be a nonempty subset of a Hilbert space H. A family $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ of mappings of C into itself satisfying the following conditions is said to be a one-parameter nonexpansive semigroup on C:

(i) For each $t \in \mathbb{R}^+$, $T(t)$ is nonexpansive;
(ii) $T(0) = I$;
(iii) $T(t+s) = T(t)T(s)$ for every $t, s \in \mathbb{R}^+$;
(iv) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

Theorem 5.4. *Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C such that $\{T(t)x : t \in \mathbb{R}^+\}$ is bounded for some $x \in C$. Then, the following hold:*

(1) *$A(\mathcal{S})$ is non-empty, closed and convex:*
(2) *for any $u \in C$, $\{\frac{1}{\lambda}\int_0^\lambda T(s)u\,ds\}$ converges weakly to $u_0 \in A(\mathcal{S})$, as $\lambda \to \infty$, where $u_0 = \lim_{t\to\infty} P_{A(\mathcal{S})}T(t)u$.*

Proof. Put $S = \mathbb{R}^+$ in Theorem 4.1. Define

$$\mu_\lambda(f) = \frac{1}{\lambda}\int_0^\lambda f(t)dt$$

for all $\lambda > 0$ and $f \in C(S)$. We have that $\{\mu_\lambda : 0 < \lambda < \infty\}$ is a strongly asymptotically invariant net of means on X. Furthermore, we have that for any $u \in C$ and $\lambda > 0$,

$$T_{\mu_\lambda}u = \frac{1}{\lambda}\int_0^\lambda T(s)u\,ds.$$

Therefore, we obtain Theorem 5.4 by using Theorem 4.1. □

Theorem 5.5. *Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C such that $\{T(t)x : t \in \mathbb{R}^+\}$ is bounded for some $x \in C$. Then, the following hold:*

(1) *$A(\mathcal{S})$ is non-empty, closed and convex:*
(2) *for any $u \in C$, $\{r\int_0^\infty e^{-rt}T(t)u\,dt\}$ converges weakly to $u_0 \in A(\mathcal{S})$,as $r \to 0$. where $u_0 = \lim_{t\to\infty} P_{A(\mathcal{S})}T(t)u$.*

Proof. Put $S = \mathbb{R}^+$ in Theorem 4.1. Define

$$\mu_r(f) = r\int_0^\infty e^{-rt}f(t)dt$$

for all $r > 0$ and $f \in C(S)$. We have that $\{\mu_r : 0 < r < \infty\}$ is a strongly asymptotically invariant net of means on X. Furthermore, we have that for any $u \in C$ and $r > 0$,

$$T_{\mu_r}u = r\int_0^\infty e^{-rt}T(t)u\,dt.$$

Therefore, we obtain Theorem 5.5 by using Theorem 4.1. □

Theorem 5.6. *Let $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ be a one-parameter nonexpansive semigroup on C such that $\{T(t)x : t \in \mathbb{R}^+\}$ is bounded for some $x \in C$. Let q be a continuous function from $\mathbb{R}^+ \times \mathbb{R}^+$ into $\mathbb{R}$ such that $\sup_{h\geq 0}\int_0^\infty |q(h,t)|\,dt < \infty$, $\lim_{h\to\infty}\int_0^\infty q(h,t)\,dt = 1$, $\lim_{h\to\infty}\int_0^\infty |q(h,t+s) - q(h,t)|\,dt = 0$ for all $s \in \mathbb{R}^+$. Then, the following hold:*

(1) *$A(\mathcal{S})$ is non-empty, closed and convex:*
(2) *for any $u \in C$, $\{\int_0^\infty q(h,t)T(t)udt\}$ converges weakly to $u_0 \in A(\mathcal{S})$, as $h \to \infty$, where $u_0 = \lim_{t\to\infty} P_{A(\mathcal{S})}T(t)u$.*

Proof. Put $S = \mathbb{R}^+$ in Theorem 4.1. Define

$$\mu_h(f) = \int_0^\infty q(h,t)f(t)dt$$

for all $h > 0$ and $f \in C(S)$. As in the proof of [4, Theorem 7], we have that $\{\mu_h : 0 < h < \infty\}$ is a strongly asymptotically invariant net of means on $C(S)$ (see also [1, Theorem 5.7], [14]). Furthermore, we have that for any $u \in C$ and $h > 0$,

$$T_{\mu_h}u = \int_0^\infty q(h,t)T(t)udt.$$

Therefore, we obtain Theorem 5.6 by using Theorems 4.1. □

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