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EXISTENCE AND MEAN APPROXIMATION OF FIXED POINTS OF GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce a broad class of nonlinear mappings in a Hilbert space which covers the class of super generalized hybrid mappings and the class of widely generalized hybrid mappings defined by Kocourek, Takahashi and Yao [11] and the authors [10], respectively. Then we prove fixed point theorems for such new mappings. Furthermore, we prove nonlinear ergodic theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly prove Browder and Petryshyn's fixed point theorem [5] for strict pseudo-contractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [11] for super generalized hybrid mappings.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be nonexpansive [16], nonspreading [13], hybrid [17] if

$$\|Tx - Ty\| \leq \|x - y\|,$$
$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2,$$
$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$, respectively; see also [8] and [19]. These mappings are independent and they are deduced from a firmly nonexpansive mapping in a Hilbert space; see [17]. A mapping $F : C \to H$ is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Goebel and Kirk [7]. Motivated by these mappings, Kocourek, Takahashi and Yao [11] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for any $x, y \in C$. We call such a mapping an (α, β)-generalized hybrid mapping. We observe that the class of the mappings covers the classes of well-known mappings. For example, an (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi

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[12] and Iemoto and Takahashi [8]. Moreover, they proved the following nonlinear ergodic theorem.

Theorem 1.1 ([11]). *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H, let T be a generalized hybrid mapping from C into itself which has a fixed point, and let P be the metric projection of H onto the set of fixed points of T. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T, where $p = \lim_{n\to\infty} PT^n x$.

Furthermore, they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping $T : C \to H$ is called super generalized hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned}
&\alpha\|Tx - Ty\|^2 + (1 - \alpha + \gamma)\|x - Ty\|^2 \\
&\quad \le (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\
&\qquad + (\alpha - \beta)\gamma\|x - Tx\|^2 + \gamma\|y - Ty\|^2
\end{aligned}$$

for any $x, y \in C$. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. Very recently, the authors [10] also defined a class of nonlinear mappings in a Hilbert space which covers the class of contractive mappings and the class of generalized hybrid mappings defined by Kocourek, Takahashi and Yao [11]. A mapping T from C into H is said to be widely generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ such that

$$\begin{aligned}
&\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\
&\qquad\qquad + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \le 0
\end{aligned}$$

for any $x, y \in C$.

In this paper, motivated by these classes of nonlinear mappings, we introduce a broad class of nonlinear mappings in a Hilbert space which covers the class of super generalized hybrid mappings and the class of widely generalized hybrid mappings. Then we prove fixed point theorems for such new mappings in a Hilbert space. Furthermore, we prove nonlinear ergodic theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly prove Browder and Petryshyn's fixed point theorem [5] for strict pseudo-contractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [11] for super generalized hybrid mappings.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let A be a nonempty subset of H. We denote by $\overline{co}A$ the closure of the convex hull of A. In a Hilbert space, it is known that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \tag{2.1}$$

for any $x, y \in H$ and for any $\alpha \in \mathbb{R}$; see [16]. Furthermore, we have that

$$2\langle x-y, z-w\rangle = \|x-w\|^2 + \|y-z\|^2 - \|x-z\|^2 - \|y-w\|^2 \tag{2.2}$$

for any $x, y, z, w \in H$. Let C be a nonempty subset of H and let T be a mapping from C into H. We denote by $F(T)$ the set of fixed points of T. A mapping T from C into H with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for any $x \in F(T)$ and for any $y \in C$. It is well-known that if C is closed and convex and $T : C \to C$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Ito and Takahashi [9]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving that $F(T)$ is closed, take a sequence $\{z_n\} \subset F(T)$ with $z_n \to z$. Since C is weakly closed, we have $z \in C$. Furthermore, from

$$\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,$$

z is a fixed point of T and so $F(T)$ is closed. Let us show that $F(T)$ is convex. For $x, y \in F(T)$ and $\alpha \in [0,1]$, put $z = \alpha x + (1-\alpha)y$. Then we have from (2.1) that

$$\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)^2\|x-y\|^2 + (1-\alpha)\alpha^2\|x-y\|^2 - \alpha(1-\alpha)\|x-y\|^2 \\
&= \alpha(1-\alpha)(1-\alpha+\alpha-1)\|x-y\|^2 \\
&= 0
\end{aligned}$$

and hence $Tz = z$. This implies that $F(T)$ is convex. Let D be a nonempty closed convex subset of H and $x \in H$. Then we know that there exists a unique nearest point $z \in D$ such that $\|x - z\| = \inf_{y \in D} \|x - y\|$. We denote such a correspondence by $z = P_D x$. The mapping P_D is called the metric projection of H onto D. It is known that P_D is nonexpansive and

$$\langle x - P_D x, P_D x - u\rangle \geq 0$$

for any $x \in H$ and for any $u \in D$; see [16] for more details. For proving a nonlinear ergodic theorem in this paper, we also need the following lemma proved by Takahashi and Toyoda [18].

Lemma 2.1. *Let D be a non-empty closed convex subset of H. Let P be the metric projection from H onto D. Let $\{u_n\}$ be a sequence in H. If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in D$ and for any $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in D$.*

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean μ is called a Banach limit on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$.

We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. See [15] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we can obtain the following result; see [14] and [15].

Lemma 2.2. *Let H be a real Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on l^∞. Then there exists a unique point $z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle$$

for any $y \in H$.

3. Fixed point theorems

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is said to be widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \le 0 \end{aligned} \tag{3.1}$$

for any $x, y \in C$. Such a mapping T is called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [11] if $\alpha+\beta = -\gamma-\delta = 1$ and $\varepsilon = \zeta = \eta = 0$. We first prove fixed point theorems for widely more generalized hybrid mappings in a Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ *and* $\zeta + \eta \ge 0$*;*
(2) $\alpha + \beta + \gamma + \delta \ge 0$, $\alpha + \beta + \zeta + \eta > 0$ *and* $\varepsilon + \eta \ge 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2).

Proof. Suppose that T has a fixed point z. Then $\{T^n z \mid n = 0, 1, \dots\} = \{z\}$. Therefore $\{T^n z \mid n = 0, 1, \dots\}$ is bounded.

Conversely suppose that there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \dots\}$ is bounded. Since T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself, we obtain that

$$\begin{aligned} \alpha\|Tx - T^{n+1}z\|^2 + \beta\|x - T^{n+1}z\|^2 + \gamma\|Tx - T^n z\|^2 + \delta\|x - T^n z\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|T^n z - T^{n+1}z\|^2 + \eta\|(x - Tx) - (T^n z - T^{n+1}z)\|^2 \le 0 \end{aligned}$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $x \in C$. By (2.2) we obtain that

$$\begin{aligned} &\|(x - Tx) - (T^n z - T^{n+1} z)\|^2 \\ &\quad = \|x - Tx\|^2 + \|T^n z - T^{n+1} z\|^2 - 2\langle x - Tx, T^n z - T^{n+1} z\rangle \\ &\quad = \|x - Tx\|^2 + \|T^n z - T^{n+1} z\|^2 + \|x - T^n z\|^2 + \|Tx - T^{n+1} z\|^2 \\ &\qquad - \|x - T^{n+1} z\|^2 - \|Tx - T^n z\|^2. \end{aligned}$$

Thus we have that

$$\begin{aligned} &(\alpha + \eta)\|Tx - T^{n+1} z\|^2 + (\beta - \eta)\|x - T^{n+1} z\|^2 + (\gamma - \eta)\|Tx - T^n z\|^2 \\ &\quad + (\delta + \eta)\|x - T^n z\|^2 + (\varepsilon + \eta)\|x - Tx\|^2 + (\zeta + \eta)\|T^n z - T^{n+1} z\|^2 \leq 0. \end{aligned}$$

From

$$\begin{aligned} &(\gamma - \eta)\|Tx - T^n z\|^2 \\ &\quad = (\alpha + \gamma)(\|x - Tx\|^2 + \|x - T^n z\|^2 - 2\langle x - Tx, x - T^n z\rangle) \\ &\qquad - (\alpha + \eta)\|Tx - T^n z\|^2, \end{aligned}$$

we have that

$$\begin{aligned} &(\alpha + \eta)\|Tx - T^{n+1} z\|^2 + (\beta - \eta)\|x - T^{n+1} z\|^2 \\ &\quad + (\alpha + \gamma)(\|x - Tx\|^2 + \|x - T^n z\|^2 - 2\langle x - Tx, x - T^n z\rangle) \\ &\quad - (\alpha + \eta)\|Tx - T^n z\|^2 + (\delta + \eta)\|x - T^n z\|^2 \\ &\quad + (\varepsilon + \eta)\|x - Tx\|^2 + (\zeta + \eta)\|T^n z - T^{n+1} z\|^2 \leq 0 \end{aligned}$$

and hence

$$\begin{aligned} &(\alpha + \eta)(\|Tx - T^{n+1} z\|^2 - \|Tx - T^n z\|^2) + (\beta - \eta)\|x - T^{n+1} z\|^2 \\ &\quad - 2(\alpha + \gamma)\langle x - Tx, x - T^n z\rangle + (\alpha + \gamma + \delta + \eta)\|x - T^n z\|^2 \\ &\quad + (\alpha + \gamma + \varepsilon + \eta)\|x - Tx\|^2 + (\zeta + \eta)\|T^n z - T^{n+1} z\|^2 \leq 0. \end{aligned}$$

By $\alpha + \beta + \gamma + \delta \geq 0$, we have that

$$-(\beta - \eta) = -(\beta + \delta) + \delta + \eta \leq \alpha + \gamma + \delta + \eta.$$

From this inequality and $\zeta + \eta \geq 0$ we obtain that

$$\begin{aligned} &(\alpha + \eta)(\|Tx - T^{n+1} z\|^2 - \|Tx - T^n z\|^2) \\ &\quad + (\beta - \eta)(\|x - T^{n+1} z\|^2 - \|x - T^n z\|^2) \\ &\quad - 2(\alpha + \gamma)\langle x - Tx, x - T^n z\rangle + (\alpha + \gamma + \varepsilon + \eta)\|x - Tx\|^2 \leq 0. \end{aligned}$$

Applying a Banach limit μ to both sides of this inequality, we obtain that

$$\begin{aligned} &(\alpha + \eta)(\mu_n\|Tx - T^{n+1} z\|^2 - \mu_n\|Tx - T^n z\|^2) \\ &\quad + (\beta - \eta)(\mu_n\|x - T^{n+1} z\|^2 - \mu_n\|x - T^n z\|^2) \\ &\quad - 2(\alpha + \gamma)\mu_n\langle x - Tx, x - T^n z\rangle + (\alpha + \gamma + \varepsilon + \eta)\mu_n\|x - Tx\|^2 \leq 0 \end{aligned}$$

and hence

$$-2(\alpha + \gamma)\mu_n\langle x - Tx, x - T^n z\rangle + (\alpha + \gamma + \varepsilon + \eta)\|x - Tx\|^2 \leq 0. \tag{3.2}$$

Since there exists $p \in C$ by Lemma 2.2 such that

$$\mu_n\langle y, T^n z\rangle = \langle y, p\rangle$$

for any $y \in H$, we obtain by (3.2) that

$$-2(\alpha+\gamma)\langle x - Tx, x - p\rangle + (\alpha+\gamma+\varepsilon+\eta)\|x - Tx\|^2 \leq 0.$$

Putting $x = p$, we obtain that

$$(\alpha+\gamma+\varepsilon+\eta)\|p - Tp\|^2 \leq 0.$$

Since $\alpha+\gamma+\varepsilon+\eta > 0$, we obtain that $\|p - Tp\|^2 \leq 0$ and hence $Tp = p$.

Next suppose that $\alpha+\beta+\gamma+\delta > 0$. Let p_1 and p_2 be fixed points of T. Then

$$\begin{aligned}
&\alpha\|Tp_1 - Tp_2\|^2 + \beta\|p_1 - Tp_2\|^2 + \gamma\|Tp_1 - p_2\|^2 + \delta\|p_1 - p_2\|^2 \\
&\qquad + \varepsilon\|p_1 - Tp_1\|^2 + \zeta\|p_2 - Tp_2\|^2 + \eta\|(p_1 - Tp_1) - (p_2 - Tp_2)\|^2 \\
&= (\alpha+\beta+\gamma+\delta)\|p_1 - p_2\|^2 \leq 0
\end{aligned}$$

and hence $p_1 = p_2$. Therefore a fixed point of T is unique.

In the case of $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$ and $\varepsilon+\eta \geq 0$, we can obtain the result by replacing the variables x and y. $\square$

As a direct consequence of Theorem 3.1, we obtain the following.

Theorem 3.2. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$ *and* $\zeta+\eta \geq 0$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$ *and* $\varepsilon+\eta \geq 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

The following theorem is an extension of Theorem 3.2.

Theorem 3.3. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta > 0$ *and* $[0,1)\cap\{\lambda \mid (\alpha+\beta)\lambda+\zeta+\eta \geq 0\} \neq \emptyset$;
(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$ *and* $[0,1)\cap\{\lambda \mid (\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0\} \neq \emptyset$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha+\beta+\gamma+\delta > 0$ on the conditions (1) *and* (2).

Proof. Let $\lambda \in [0,1)\cap\{\lambda \mid (\alpha+\beta)\lambda+\zeta+\eta \geq 0\}$ and define $S = (1-\lambda)T + \lambda I$. Since C is convex, S is a mapping from C into itself. Since C is bounded, $\{S_n z \mid n = 0,1,\dots\}$ is bounded for any $z \in C$. Since $\lambda \neq 1$, we obtain that $F(S) = F(T)$. Moreover, from $T = \frac{1}{1-\lambda}S - \frac{\lambda}{1-\lambda}I$ and (2.1), we have that

$$\begin{aligned}
&\alpha\left\|\left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\|^2 \\
&\qquad + \beta\left\|x - \left(\frac{1}{1-\lambda}Sy - \frac{\lambda}{1-\lambda}y\right)\right\|^2 + \gamma\left\|\left(\frac{1}{1-\lambda}Sx - \frac{\lambda}{1-\lambda}x\right) - y\right\|^2 \\
&\qquad + \delta\|x - y\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\left\|x-\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)\right\|^2+\zeta\left\|y-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right\|^2\\
&+\eta\left\|\left(x-\left(\frac{1}{1-\lambda}Sx-\frac{\lambda}{1-\lambda}x\right)\right)-\left(y-\left(\frac{1}{1-\lambda}Sy-\frac{\lambda}{1-\lambda}y\right)\right)\right\|^2\\
=\ &\alpha\left\|\frac{1}{1-\lambda}(Sx-Sy)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2\\
&+\beta\left\|\frac{1}{1-\lambda}(x-Sy)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2\\
&+\gamma\left\|\frac{1}{1-\lambda}(Sx-y)-\frac{\lambda}{1-\lambda}(x-y)\right\|^2+\delta\|x-y\|^2\\
&+\varepsilon\left\|\frac{1}{1-\lambda}(x-Sx)\right\|^2+\zeta\left\|\frac{1}{1-\lambda}(y-Sy)\right\|^2\\
&+\eta\left\|\frac{1}{1-\lambda}(x-Sx)-\frac{1}{1-\lambda}(y-Sy)\right\|^2\\
=\ &\frac{\alpha}{1-\lambda}\|Sx-Sy\|^2+\frac{\beta}{1-\lambda}\|x-Sy\|^2\\
&+\frac{\gamma}{1-\lambda}\|Sx-y\|^2+\left(-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta\right)\|x-y\|^2\\
&+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}\|x-Sx\|^2+\frac{\zeta+\beta\lambda}{(1-\lambda)^2}\|y-Sy\|^2\\
&+\frac{\eta+\alpha\lambda}{(1-\lambda)^2}\|(x-Sx)-(y-Sy)\|^2\leq 0.
\end{aligned}
$$

Therefore S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, \frac{\gamma}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta, \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}, \frac{\zeta+\beta\lambda}{(1-\lambda)^2}, \frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping. Furthermore, we obtain that

$$
\begin{aligned}
\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta &= \alpha+\beta+\gamma+\delta\geq 0,\\
\frac{\alpha}{1-\lambda}+\frac{\gamma}{1-\lambda}+\frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{\alpha+\gamma+\varepsilon+\eta}{(1-\lambda)^2}>0,\\
\frac{\zeta+\beta\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\beta)\lambda+\zeta+\eta}{(1-\lambda)^2}\geq 0.
\end{aligned}
$$

Therefore by Theorem 3.1 we obtain $F(S)\neq\emptyset$.

Next suppose that $\alpha+\beta+\gamma+\delta>0$. Let p_1 and p_2 be fixed points of T. Then

$$
\begin{aligned}
&\alpha\|Tp_1-Tp_2\|^2+\beta\|p_1-Tp_2\|^2+\gamma\|Tp_1-p_2\|^2+\delta\|p_1-p_2\|^2\\
&\quad+\varepsilon\|p_1-Tp_1\|^2+\zeta\|p_2-Tp_2\|^2+\eta\|(p_1-Tp_1)-(p_2-Tp_2)\|^2\\
&=(\alpha+\beta+\gamma+\delta)\|p_1-p_2\|^2\leq 0
\end{aligned}
$$

and hence $p_1=p_2$. Therefore a fixed point of T is unique.

In the case of $\alpha+\beta+\gamma+\delta\geq 0$, $\alpha+\beta+\zeta+\eta>0$ and $[0,1)\cap\{\lambda\mid(\alpha+\gamma)\lambda+\varepsilon+\eta\geq 0\}\neq\emptyset$, we can obtain the result by replacing the variables x and y. $\square$

Remark 3.4. We can also prove Theorems 3.1 and 3.3 by using the condition

$$-\beta-\delta+\varepsilon+\eta>0, \quad or \quad -\gamma-\delta+\varepsilon+\eta>0$$

instead of the condition

$$\alpha+\gamma+\varepsilon+\eta>0, \quad or \quad \alpha+\beta+\zeta+\eta>0,$$

respectly. In fact, in the case of the condition $-\beta-\delta+\varepsilon+\eta>0$, we obtain from $\alpha+\beta+\gamma+\delta\geq 0$ that

$$0<-\beta-\delta+\varepsilon+\eta\leq\alpha+\gamma+\varepsilon+\eta.$$

Thus we obtain the desired result by Theorems 3.1 and 3.3. Similary, in the case of $-\gamma-\delta+\varepsilon+\eta>0$, we can obtain the result by using the case of $\alpha+\beta+\zeta+\eta>0$.

4. Fixed point theorems for well-known mappings

An $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping T above with $\alpha=1$, $\beta=\gamma=\varepsilon=\zeta=\eta=0$ and $-1<\delta<0$ is a contractive mapping. Using Theorem 3.1, we can show the Banach fixed point theorem in a Hilbert space.

Theorem 4.1 (The Banach fixed point theorem)**.** *Let H be a real Hilbert space and let T be a contractive mapping from H into H, that is, there exists a real number α with $0<\alpha<1$ such that*

$$\|Tx-Ty\|\leq\alpha\|x-y\|$$

for any $x,y\in H$. Then T has a unique fixed point.

Proof. Since

$$\begin{aligned}\|T^nx-x\| &\leq \|T^nx-T^{n-1}x\|+\|T^{n-1}x-T^{n-2}x\|+\cdots+\|Tx-x\|\\ &\leq (\alpha^{n-1}+\alpha^{n-2}+\cdots+1)\|Tx-x\|\\ &\leq \frac{1}{1-\alpha}\|Tx-x\|\end{aligned}$$

for any $x\in H$, $\{T^nx \mid n=0,1,\ldots\}$ is bounded. By Theorem 3.1 T has a unique fixed point. □

Using Theorem 3.1, we can show Kocourek, Takahashi and Yao's fixed point theorem [11] for generalized hybrid mappings in a Hilbert space.

Theorem 4.2 ([11])**.** *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be a generalized hybrid mapping from C into itself, that is, there exist real numbers α and β such that*

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2\leq\beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

for any $x,y\in C$. Then T has a fixed point if and only if there exists $z\in C$ such that $\{T^nz \mid n=0,1,\ldots\}$ is bounded.

Proof. An (α,β)-generalized hybrid mapping T from C into itself is an $(\alpha,1-\alpha,-\beta,-(1-\beta),0,0,0)$-widely more generalized hybrid mapping. Furthermore, $\alpha+(1-\alpha)-\beta-(1-\beta)=0\geq 0$, $\alpha+(1-\alpha)+0+0=1>0$ and $0+0=0\geq 0$, that is, it satisfies the condition (2) in Theorem 3.1. Then we obtain the desired result from Theorem 3.1. □

Using Theorem 3.1, we can show Kawasaki and Takahashi's fixed point theorem [10] for widely generalized hybrid mappings in a Hilbert space.

Theorem 4.3 ([10]). *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping from C into itself which satisfies the following condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \gamma + \varepsilon > 0$*;*
(2) $\alpha + \beta + \gamma + \delta \geq 0$ *and* $\alpha + \beta + \zeta > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \dots\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) *and* (2).

Proof. Since T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-widely generalized hybrid mapping, we obtain that

$$\begin{aligned} &\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0 \end{aligned}$$

for any $x, y \in C$. In the case of $\alpha + \gamma + \varepsilon > 0$, from

$$\varepsilon\|x - Tx\|^2 \leq \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\},$$

we obtain that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \varepsilon\|x - Tx\|^2 \leq 0,$$

that is, it is an $(\alpha, \beta, \gamma, \delta, \varepsilon, 0, 0)$-widely more generalized hybrid mapping. Furthermore, we have that $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+0 = \alpha+\gamma+\varepsilon > 0$ and $0+0 = 0 \geq 0$, that is, it satisfies the condition (1) in Theorem 3.1. Then we obtain the desired result from Theorem 3.1. In the case of $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta + \zeta > 0$, we can obtain the result by replacing the variables x and y. $\square$

Note that an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping T with $\alpha = 1$, $\beta = \gamma = \varepsilon = \zeta = 0$, $\delta = -1$ and $\eta = -k \in (-1, 0]$ is a strict pseudo-contractive mapping in the case of Browder and Petryshyn [5]. Using Theorem 3.3, we can show the following fixed point theorem in a Hilbert space.

Theorem 4.4. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be a strict pseudo-contractive mapping from C into itself, that is, there exists a real number k with $0 \leq k < 1$ such that*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2$$

for any $x, y \in C$. Then T has a fixed point.

Proof. A strict pseudo-contractive mapping T from C into itself is an $(1, 0, 0, -1, 0, 0, -k)$-widely more generalized hybrid mapping. Furthermore, $1+0+0+(-1) = 0 \geq 0$, $1+0+0+(-k) = 1-k > 0$ and $[0, 1) \cap \{\lambda \mid (1+0)\lambda + 0 - k \geq 0\} = [k, 1) \neq \emptyset$, that is, it satisfies the condition (1) in Theorem 3.3. Then we obtain the desired result from Theorem 3.3. $\square$

Using Theorem 3.3, we can show the following fixed point theorem for super generalized hybrid mappings in a Hilbert space.

Theorem 4.5. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be a super generalized hybrid mapping from C into itself, that is, there exist real numbers α, β and γ such that*

$$\begin{aligned}\alpha\|Tx - Ty\|^2 &+ (1-\alpha+\gamma)\|x - Ty\|^2 \\ &\le (\beta + (\beta-\alpha)\gamma)\|Tx - y\|^2 + (1-\beta-(\beta-\alpha-1)\gamma)\|x-y\|^2 \\ &\quad + (\alpha-\beta)\gamma\|x - Tx\|^2 + \gamma\|y - Ty\|^2\end{aligned}$$

for any $x, y \in C$. Suppose that $\alpha - \beta \ge 0$ or $\gamma \ge 0$. Then T has a fixed point.

Proof. An (α,β,γ)-super generalized hybrid mapping T from C into itself is an $(\alpha, 1-\alpha+\gamma, -\beta-(\beta-\alpha)\gamma, -1+\beta+(\beta-\alpha-1)\gamma, -(\alpha-\beta)\gamma, -\gamma, 0)$-widely more generalized hybrid mapping. Furthermore, $\alpha + (1-\alpha+\gamma) + (-\beta-(\beta-\alpha)\gamma) + (-1+\beta+(\beta-\alpha-1)\gamma) = 0 \ge 0$ and $\alpha + (1-\alpha+\gamma) + (-\gamma) + 0 = 1 > 0$, that is, it satisfies the first and second conditions $\alpha+\beta+\gamma+\delta \ge 0$ and $\alpha+\beta+\zeta+\eta > 0$ in (2) of Theorem 3.3. Moreover, we have that

$$\begin{aligned}[0,1) &\cap \{\lambda \mid (\alpha + (-\beta-(\beta-\alpha)\gamma))\lambda + (-(\alpha-\beta)\gamma) + 0 \ge 0\} \\ &= [0,1) \cap \{\lambda \mid (\alpha-\beta)((1+\gamma)\lambda - \gamma) \ge 0\}.\end{aligned}$$

If $\alpha - \beta > 0$, then

$$\begin{aligned}[0,1) \cap \{\lambda \mid (\alpha-\beta)((1+\gamma)\lambda-\gamma) \ge 0\} &= [0,1) \cap \{\lambda \mid (1+\gamma)\lambda - \gamma \ge 0\} \\ &= \begin{cases} [0,1) & \text{if } \gamma < 0, \\ \left[\frac{\gamma}{1+\gamma}, 1\right) & \text{if } \gamma \ge 0, \end{cases} \\ &\ne \emptyset,\end{aligned}$$

that is, it satisfies the third condition $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda + \varepsilon + \eta \ge 0\} \ne \emptyset$ in (2) of Theorem 3.3. If $\alpha - \beta = 0$, then

$$[0,1) \cap \{\lambda \mid (\alpha-\beta)((1+\gamma)\lambda-\gamma) \ge 0\} = [0,1) \ne \emptyset,$$

that is, it satisfies the third condition $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda + \varepsilon + \eta \ge 0\} \ne \emptyset$ in (2) of Theorem 3.3. If $\alpha - \beta < 0$ and $\gamma \ge 0$, then

$$\begin{aligned}[0,1) \cap \{\lambda \mid (\alpha-\beta)((1+\gamma)\lambda-\gamma) \ge 0\} &= [0,1) \cap \{\lambda \mid (1+\gamma)\lambda - \gamma \le 0\} \\ &= \left[0, \frac{\gamma}{1+\gamma}\right] \ne \emptyset,\end{aligned}$$

that is, it satisfies the third condition $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda + \varepsilon + \eta \ge 0\} \ne \emptyset$ in (2) of Theorem 3.3. Then we obtain the desired result from Theorem 3.3. □

Compare Theorem 4.5 with Kocourek, Takahashi and Yao's theorem [11]. The case of $\alpha - \beta \ge 0$ is new.

5. Nonlinear ergodic theorems

In this section, using the technique developed by Takahashi [14], we prove a nonlinear ergodic theorem of Baillon's type for widely more generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following lemmas.

Lemma 5.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself which has a fixed point and satisfies the condition:*

$$\alpha+\gamma+\varepsilon+\eta>0, \text{ or } \alpha+\beta+\zeta+\eta>0.$$

Then $F(T)$ is closed.

Proof. Suppose that $\{x_n \mid n=1,2,\dots\}\subset F(T)$ is convergent to $x\in H$. We show $x\in F(T)$. Putting $y=x_n$ in (3.1), we have that

$$\begin{aligned}\alpha\|Tx-Tx_n\|^2&+\beta\|x-Tx_n\|^2+\gamma\|Tx-x_n\|^2+\delta\|x-x_n\|^2\\&+\varepsilon\|x-Tx\|^2+\zeta\|x_n-Tx_n\|^2+\eta\|(x-Tx)-(x_n-Tx_n)\}^2\le 0\end{aligned}$$

and hence

$$(\alpha+\gamma)\|Tx-x_n\|^2+(\beta+\delta)\|x-x_n\|^2+(\varepsilon+\eta)\|x-Tx\|^2\le 0. \tag{5.1}$$

Letting $n\to\infty$, we obtain that

$$(\alpha+\gamma+\varepsilon+\eta)\|Tx-x\|^2\le 0. \tag{5.2}$$

Since $\alpha+\gamma+\varepsilon+\eta>0$, we have from (5.2) that $x\in F(T)$. Therefore $F(T)$ is closed. Similarly, we can obtain the desired result for the case of $\alpha+\beta+\zeta+\eta>0$. □

Lemma 5.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself such that $F(T)\neq\emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta\ge 0$ *and* $\alpha+\gamma+\varepsilon+\eta>0$*;*
(2) $\alpha+\beta+\gamma+\delta\ge 0$ *and* $\alpha+\beta+\zeta+\eta>0$.

Then $F(T)$ is convex.

Proof. For $x_1,x_2\in F(T)$ and $\lambda\in\mathbb{R}$ with $0\le\lambda\le 1$, put $x=(1-\lambda)x_1+\lambda x_2$. We show that $x\in F(T)$. Putting $y=x_1$ in (3.1), we have that

$$\begin{aligned}\alpha\|Tx-Tx_1\|^2&+\beta\|x-Tx_1\|^2+\gamma\|Tx-x_1\|^2+\delta\|x-x_1\|^2\\&+\varepsilon\|x-Tx\|^2+\zeta\|x_1-Tx_1\|^2+\eta\|(x-Tx)-(x_1-Tx_1)\|^2\le 0\end{aligned}$$

and hence

$$(\alpha+\gamma)\|Tx-x_1\|^2+(\beta+\delta)\lambda^2\|x_1-x_2\|^2+(\varepsilon+\eta)\|x-Tx\|^2\le 0. \tag{5.3}$$

Similarly, putting $y=x_2$ in (3.1), we have that

$$(\alpha+\gamma)\|Tx-x_2\|^2+(\beta+\delta)(1-\lambda)^2\|x_1-x_2\|^2+(\varepsilon+\eta)\|x-Tx\|^2\le 0. \tag{5.4}$$

Therefore we obtain from (5.3) that

$$\begin{aligned}(\alpha+\gamma)\|Tx-x_1\|^2&+(\beta+\delta)\lambda^2\|x_1-x_2\|^2\\&+(\varepsilon+\eta)(\|Tx-x_1\|^2+\lambda^2\|x_1-x_2\|^2+2\lambda\langle Tx-x_1,x_1-x_2\rangle)\le 0.\end{aligned}$$

Thus we have that

$$\begin{aligned}(\alpha+\gamma+\varepsilon+\eta)\|Tx-x_1\|^2&+(\beta+\delta+\varepsilon+\eta)\lambda^2\|x_1-x_2\|^2\\&+2(\varepsilon+\eta)\lambda\langle Tx-x_1,x_1-x_2\rangle)\le 0.\end{aligned} \tag{5.5}$$

Similarly, we have from (5.4) that

$$(\alpha+\gamma+\varepsilon+\eta)\|Tx-x_2\|^2+(\beta+\delta+\varepsilon+\eta)(1-\lambda)^2\|x_1-x_2\|^2 \\ -2(\varepsilon+\eta)(1-\lambda)\langle Tx-x_2, x_1-x_2\rangle) \leq 0. \tag{5.6}$$

Using (2.1), (5.5), (5.6), $\alpha+\gamma+\varepsilon+\eta>0$ and $\alpha+\beta+\gamma+\delta\geq 0$, we obtain that

$$\begin{aligned}
&\|Tx-x\|^2\\
&\quad=\|Tx-((1-\lambda)x_1+\lambda x_2)\|^2\\
&\quad=(1-\lambda)\|Tx-x_1\|^2+\lambda\|Tx-x_2\|^2-\lambda(1-\lambda)\|x_1-x_2\|^2\\
&\quad\leq(1-\lambda)\left(-\frac{(\beta+\delta+\varepsilon+\eta)\lambda^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\right.\\
&\qquad\qquad\left.-\frac{2(\varepsilon+\eta)\lambda}{\alpha+\gamma+\varepsilon+\eta}\langle Tx-x_1, x_1-x_2\rangle\right)\\
&\qquad+\lambda\left(-\frac{(\beta+\delta+\varepsilon+\eta)(1-\lambda)^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\right.\\
&\qquad\qquad\left.+\frac{2(\varepsilon+\eta)(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\langle Tx-x_2, x_1-x_2\rangle\right)\\
&\qquad-\lambda(1-\lambda)\|x_1-x_2\|^2\\
&\quad=-\frac{(\beta+\delta+\varepsilon+\eta)\lambda^2(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\\
&\qquad-\frac{(\beta+\delta+\varepsilon+\eta)\lambda(1-\lambda)^2}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\\
&\qquad+\frac{2(\varepsilon+\eta)\lambda(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2-\lambda(1-\lambda)\|x_1-x_2\|^2\\
&\quad=-\frac{(\alpha+\beta+\gamma+\delta)\lambda(1-\lambda)}{\alpha+\gamma+\varepsilon+\eta}\|x_1-x_2\|^2\leq 0
\end{aligned}$$

and hence $x\in F(T)$. Thus $F(T)$ is convex. Similarly, we can obtain the desired result in the case of $\alpha+\beta+\zeta+\eta>0$. □

Lemma 5.3. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself such that $F(T)\neq\emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta\geq 0$, $\zeta+\eta\geq 0$ *and* $\alpha+\beta>0$*;*
(2) $\alpha+\beta+\gamma+\delta\geq 0$, $\varepsilon+\eta\geq 0$ *and* $\alpha+\gamma>0$*.*

Then T is quasi-nonexpansive.

Proof. Suppose that the condition (2) holds. We have from (3.1) that for any $x\in C$ and for any $y\in F(T)$,

$$\begin{aligned}
&\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2\\
&\qquad+\varepsilon\|x-Tx\|^2+\zeta\|y-Ty\|^2+\eta\|(x-Tx)-(y-Ty)\|^2\\
&\quad=(\alpha+\gamma)\|Tx-y\|^2+(\beta+\delta)\|x-y\|^2+(\varepsilon+\eta)\|x-Tx\|^2\leq 0.
\end{aligned}$$

We obtain from $\alpha+\gamma>0$ that

$$\|Tx-y\|^2 \leq -\frac{\beta+\delta}{\alpha+\gamma}\|x-y\|^2 - \frac{\varepsilon+\eta}{\alpha+\gamma}\|x-Tx\|^2.$$

Since $-\frac{\beta+\delta}{\alpha+\gamma} \leq 1$ from $\alpha+\beta+\gamma+\delta \geq 0$ and $-\frac{\varepsilon+\eta}{\alpha+\gamma} \leq 0$ from $\varepsilon+\eta \geq 0$, we obtain that $\|Tx-y\|^2 \leq \|x-y\|^2$ and hence $\|Tx-y\| \leq \|x-y\|$. Thus T is quasi-nonexpansive. Similarly, we can obtain the desired result for the case of the condition (1). $\square$

We also have the following lemma.

Lemma 5.4. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $[0,1) \cap \{\lambda \mid (\alpha+\beta)\lambda+\zeta+\eta \geq 0\} \neq \emptyset$ *and* $\alpha+\beta>0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0\} \neq \emptyset$ *and* $\alpha+\gamma>0$.

Take $\lambda \in [0,1) \cap \{\lambda \mid (\alpha+\beta)\lambda+\zeta+\eta \geq 0\}$ or $\lambda \in [0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0\}$. Then $(1-\lambda)T+\lambda I$ is quasi-nonexpansive.

Proof. Let $\lambda \in [0,1) \cap \{\lambda \mid (\alpha+\beta)\lambda+\zeta+\eta \geq 0\}$ and define $S=(1-\lambda)T+\lambda I$. As in the proof of Theorem 3.3, S is a mapping from C into itself and $F(S)=F(T)$. Furthermore, S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, \frac{\gamma}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta, \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}, \frac{\zeta+\beta\lambda}{(1-\lambda)^2}, \frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping. We also obtain that

$$\begin{aligned}
\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda}+\frac{\gamma}{1-\lambda}-\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta &= \alpha+\beta+\gamma+\delta \geq 0, \\
\frac{\zeta+\beta\lambda}{(1-\lambda)^2}+\frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\beta)\lambda+\zeta+\eta}{(1-\lambda)^2} \geq 0, \\
\frac{\alpha}{1-\lambda}+\frac{\beta}{1-\lambda} &= \frac{\alpha+\beta}{1-\lambda} > 0.
\end{aligned}$$

By Lemma 5.3, S is quasi-nonexpansive. Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta \geq 0$, $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda+\varepsilon+\eta \geq 0\} \neq \emptyset$ and $\alpha+\gamma>0$. $\square$

Now we have the following nonlinear ergodic theorem for widely more generalized hybrid mappings in a Hilbert space.

Theorem 5.5. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha,\beta,\gamma,\delta,\varepsilon,\zeta,\eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\gamma+\varepsilon+\eta>0$, $\zeta+\eta \geq 0$ *and* $\alpha+\beta>0$*;*

(2) $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta>0$, $\varepsilon+\eta \geq 0$ *and* $\alpha+\gamma>0$.

Then for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection of H onto $F(T)$ and $p=\lim_{n\to\infty} PT^n x$.

Proof. Since $F(T)$ is nonempty and T is quasi-nonexpansive from Lemma 5.3, we obtain that

$$\|T^{n+1}x - y\| \leq \|T^n x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$, we have that $\{T^n x\}$ is bounded for any $x \in C$. Since

$$\|S_n x - y\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - y\| \leq \|x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$, $\{S_n x \mid n = 0, 1, \dots\}$ is also bounded. Therefore there exists a strictly increasing sequence $\{n_i\}$ and $p \in H$ such that $\{S_{n_i} x \mid i = 0, 1, \dots\}$ is weakly convergent to p. Since C is closed and convex, C is weakly closed. Thus $p \in C$. We first show that $p \in F(T)$. Indeed, using $\alpha + \beta + \gamma + \delta \geq 0$ and $\zeta + \eta \geq 0$, as in the proof of Theorem 3.1 we have that

$$\begin{aligned}
&(\alpha + \eta)(\|Tz - T^{k+1}x\|^2 - \|Tz - T^k x\|^2) \\
&\quad + (\beta - \eta)(\|z - T^{k+1}x\|^2 - \|z - T^k x\|^2) \\
&\quad - 2(\alpha + \gamma)\langle z - Tz, z - T^k x\rangle + (\alpha + \gamma + \varepsilon + \eta)\|z - Tz\|^2 \leq 0
\end{aligned}$$

for any $k \in \mathbb{N} \cup \{0\}$ and for any $z \in C$. Summing up these inequalities with respect to $k = 0, 1, \dots, n-1$ and dividing by n, we obtain that

$$\begin{aligned}
&\frac{\alpha + \eta}{n}(\|Tz - T^n x\|^2 - \|Tz - x\|^2) + \frac{\beta - \eta}{n}(\|z - T^n x\|^2 - \|z - x\|^2) \\
&\quad - 2(\alpha + \gamma)\langle z - Tz, z - S_n x\rangle + (\alpha + \gamma + \varepsilon + \eta)\|z - Tz\|^2 \leq 0.
\end{aligned}$$

Replacing n by n_i, we obtain that

$$\begin{aligned}
&\frac{\alpha + \eta}{n_i}(\|Tz - T^{n_i} x\|^2 - \|Tz - x\|^2) + \frac{\beta - \eta}{n_i}(\|z - T^{n_i} x\|^2 - \|z - x\|^2) \\
&\quad - 2(\alpha + \gamma)\langle z - Tz, z - S_{n_i} x\rangle + (\alpha + \gamma + \varepsilon + \eta)\|z - Tz\|^2 \leq 0.
\end{aligned}$$

Letting $i \to \infty$, we obtain that

$$-2(\alpha + \gamma)\langle z - Tz, z - p\rangle + (\alpha + \gamma + \varepsilon + \eta)\|z - Tz\|^2 \leq 0.$$

Putting $z = p$, we obtain that

$$(\alpha + \gamma + \varepsilon + \eta)\|p - Tp\|^2 \leq 0.$$

Since $\alpha + \gamma + \varepsilon + \eta > 0$, we obtain that $Tp = p$.

Since $F(T)$ is closed and convex from Lemmas 5.1 and 5.2, the metric projection P from H onto $F(T)$ is well-defined. By Lemma 2.1, there exists $q \in F(T)$ such that $\{PT^n x \mid n = 0, 1, \dots\}$ is convergent to q. To complete the proof, we show that $q = p$. Note that the metric projection P satisfies

$$\langle z - Pz, Pz - u\rangle \geq 0$$

for any $z \in H$ and for any $u \in F(T)$; see [15]. Therefore

$$\langle T^k x - PT^k x, PT^k x - y\rangle \geq 0$$

for any $k \in \mathbb{N} \cup \{0\}$ and for any $y \in F(T)$. Since P is the metric projection and T is quasi-nonexpansive, we obtain that

$$\begin{aligned} \|T^n x - PT^n x\| &\leq \|T^n x - PT^{n-1} x\| \\ &\leq \|T^{n-1} x - PT^{n-1} x\|, \end{aligned}$$

that is, $\{\|T^n x - PT^n x\| \mid n = 0, 1, \dots\}$ is non-increasing. Therefore we obtain

$$\begin{aligned} \langle T^k x - PT^k x, y - q\rangle &\leq \langle T^k x - PT^k x, PT^k x - q\rangle \\ &\leq \|T^k x - PT^k x\| \cdot \|PT^k x - q\| \\ &\leq \|x - Px\| \cdot \|PT^k x - q\|. \end{aligned}$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n-1$ and dividing by n, we obtain

$$\left\langle S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x, y - q \right\rangle \leq \frac{\|x - Px\|}{n} \sum_{k=0}^{n-1} \|PT^k x - q\|.$$

Since $\{S_{n_i} x \mid i = 0, 1, \dots\}$ is weakly convergent to p and $\{PT^n x \mid n = 0, 1, \dots\}$ is convergent to q, we obtain that

$$\langle p - q, y - q\rangle \leq 0.$$

Putting $y = p$, we obtain

$$\|p - q\|^2 \leq 0$$

and hence $q = p$. This completes the proof.

Similarly, we can obtain the desired result for the case of $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, $\varepsilon + \eta \geq 0$ and $\alpha + \gamma > 0$. $\square$

We also have the following nonlinear ergodic theorem.

Theorem 5.6. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself such that $F(T) \neq \emptyset$ and it satisfies the condition* (1) *or* (2)*:*

(1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$,
$[0, 1) \cap \{\lambda \mid (\alpha + \beta)\lambda + \zeta + \eta \geq 0\} \neq \emptyset$ *and* $\alpha + \beta > 0$*;*

(2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$,
$[0, 1) \cap \{\lambda \mid (\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$ *and* $\alpha + \gamma > 0$.

Take $\lambda \in [0, 1) \cap \{\lambda \mid (\alpha + \beta)\lambda + \zeta + \eta \geq 0\}$ or $\lambda \in [0, 1) \cap \{\lambda \mid (\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0\}$. Then for any $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} ((1 - \lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T, where P is the metric projection of H onto $F(T)$ and $p = \lim_{n\to\infty} P((1 - \lambda)T + \lambda I)^n x$.

Proof. Let $\lambda \in [0,1) \cap \{\lambda \mid (\alpha+\beta)\lambda + \zeta + \eta \geq 0\}$ and $S = (1-\lambda)T + \lambda I$. Since S is an $\left(\frac{\alpha}{1-\lambda}, \frac{\beta}{1-\lambda}, \frac{\gamma}{1-\lambda}, -\frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma)+\delta, \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2}, \frac{\zeta+\beta\lambda}{(1-\lambda)^2}, \frac{\eta+\alpha\lambda}{(1-\lambda)^2}\right)$-widely more generalized hybrid mapping from C into itself and

$$\begin{aligned}
\frac{\alpha}{1-\lambda} + \frac{\beta}{1-\lambda} + \frac{\gamma}{1-\lambda} - \frac{\lambda}{1-\lambda}(\alpha+\beta+\gamma) + \delta &= \alpha+\beta+\gamma+\delta \geq 0, \\
\frac{\alpha}{1-\lambda} + \frac{\gamma}{1-\lambda} + \frac{\varepsilon+\gamma\lambda}{(1-\lambda)^2} + \frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{\alpha+\gamma+\varepsilon+\eta}{(1-\lambda)^2} > 0, \\
\frac{\zeta+\beta\lambda}{(1-\lambda)^2} + \frac{\eta+\alpha\lambda}{(1-\lambda)^2} &= \frac{(\alpha+\beta)\lambda+\zeta+\eta}{(1-\lambda)^2} \geq 0, \\
\frac{\alpha}{1-\lambda} + \frac{\beta}{1-\lambda} &= \frac{\alpha+\beta}{1-\lambda} > 0,
\end{aligned}$$

by Theorem 5.5 $S_n x$ is weakly convergent to $p \in F(S) = F(T)$.

Since $F(S)$ is closed and convex from Lemmas 5.1 and 5.2, the metric projection P from H onto $F(S)$ is well-defined. Since S is quasi-nonexpansive from Lemma 5.4, we obtain that

$$\|S^{n+1}x - y\| \leq \|S^n x - y\|$$

for any $n \in \mathbb{N} \cup \{0\}$ and for any $y \in F(S)$. Therefore we can obtain the desired result similarly to the proof of Theorem 5.5.

Similarly, we can obtain the desired result for the case of $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta+\zeta+\eta > 0$, $[0,1) \cap \{\lambda \mid (\alpha+\gamma)\lambda + \varepsilon + \eta \geq 0\} \neq \emptyset$ and $\alpha+\gamma > 0$. $\square$

References

[1] J.-B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Sér. A–B, **280** (1975), 1511–1514.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, *Fund. Math.* **3** (1922), 133–181.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), 123–145.

[4] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[5] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl., **20** (1967), 197–228.

[6] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[7] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990.

[8] S. Iemoto and W. Takahashi, *Approximating common fixed points of nonexpansive mappings and nonspreading mappings in a Hilbert space*, Nonlinear Anal. **71** (2009), 2082–2089.

[9] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), 493–508.

[10] T. Kawasaki and W. Takahashi, *Fixed point and nonlinear ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **13** (2012), 529–540.

[11] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math., **14** (2010), 2497–2511.

[12] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.

[13] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.
[14] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.
[15] W. Takahash, *Nonlinear Functional Analysis. Fixed Points Theory and its Applications*, Yokohama Publishers, Yokohama, 2000.
[16] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.
[17] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[18] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.
[19] W. Takahashi and J.-C. Yao, *Fixed point theorems and ergodic theorems for nonlinear mappings in Hilbert spaces*, Taiwanese J. Math. **15** (2011), 457–472.

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