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EQUILIBRIA OF ABSTRACT ECONOMIES WITH APPLICATIONS

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Abstract. In a recent paper, we established a common existence theorem of maximal elements for majorized Q_α-condensing mappings in *l.c.*-spaces. As an application, we derive a new existence theorem of equilibria for noncompact abstract economies. Finally, the existence of solutions for a system of generalized quasi-variational inequalities in *l.c.*-spaces are also derived.

1. Introduction and preliminary

Since Arrow and Debreu [1] established the existence theorem of Walrasian equilibria, the result has been extended in many directions. For more details, the reader might consult the references [4, 5, 9, 10, 11] and the references therein. Let I be any (finite or infinite) set of agents. An ***abstract economy*** is defined as a family of order quadruples $\mathbf{\Omega} := (X_\alpha, A_\alpha, B_\alpha, P_\alpha)_{\alpha\in I}$ such that for each $\alpha \in I$, X_α is a topological space, $A_\alpha, B_\alpha : X \longrightarrow 2^{X_\alpha}$ are constraint correspondences, where $X := \prod_{\alpha\in I} X_\alpha$, and $P_\alpha : X \longrightarrow 2^{X_\alpha}$ is a preference correspondence. An ***equilibrium point*** of $\mathbf{\Omega}$ is a point $\hat{x} \in X$ such that for each $\alpha \in I$, $\hat{x}_\alpha \in \mathrm{cl}B_\alpha(\hat{x})$ and $A_\alpha(\hat{x}) \cap P_\alpha(\hat{x}) = \emptyset$, where $\hat{x}_\alpha = \pi_\alpha(\hat{x})$ denotes the projection of $\hat{x}$ onto X_α. When $A_\alpha = B_\alpha$ and X_α is a topological vector space for each $\alpha \in I$, our definitions of an abstract economy and an equilibrium point coincide with the standard definition of Shafer and Sonnenschein [8].

Throughout this paper, all topological spaces are assumed to be Hausdorff. First we recall some definitions and notations. For a nonempty set X, we denote the set of all subsets of X and the set of all nonempty finite subsets of X by 2^X and $\langle X\rangle$, respectively. In addition, for any subset C of a topological space X, the closure of C is denoted by $\mathrm{cl}_X C$.

An ***H-space*** is a topological space X, equipped with a family $\{\Gamma_D\}$ of some nonempty contractible subsets of X, indexed by $D \in \langle X\rangle$ such that $\Gamma_D \subset \Gamma_{D'}$ whenever $D \subset D'$. A nonempty subset C of X is said to be ***H-convex***, if $\Gamma_D \subseteq C$ for all $D \in \langle C\rangle$. For a nonempty subset C of X, the ***H-convex hull*** of C is defined as

$$H\text{-}coC := \bigcap\{K \mid K \text{ is } H\text{-convex in } X \text{ and } C \subseteq K\}.$$

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In fact, C is H-convex if and only if $C = H\text{-}coC$. Moreover, for any finite set $D \in \langle X \rangle$, $H\text{-}coD$ is called a ***polytope***. We shall say that an H-space X is an ***H-space with precompact polytopes***, if any polytope of X is precompact. For example, a locally convex topological vector space X is an H-space with precompact polytopes, by setting $\Gamma_D = coD$ for all $D \in \langle X \rangle$.

An H-space $(X, \{\Gamma_D\})$ is called an ***l.c.-space***, if X is an uniform space whose topology is induced by its uniformity $\mathcal{U}$, and there is a base $\mathcal{B}$ consisting of symmetric entourages in $\mathcal{U}$ such that for each $V \in \mathcal{B}$, the set $V(E) := \{y \in X \mid (x, y) \in V \text{ for some } x \in E\}$ is H-convex whenever E is H-convex. For more details about uniform spaces, we refer to [7, 11, 12]. We shall use the notation $(X, \mathcal{U}, \mathcal{B})$ to stand for an $l.c.$-space, and adopt the following ***measure of precompactness*** of a subset A in X, defined by

$$Q(A) := \{V \in \mathcal{B} \mid A \subseteq \mathrm{cl}_X V(K) \text{ for some precompact set } K \text{ of } X\}.$$

Let $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)_{\alpha \in I}$ be a family of $l.c.$-spaces with precompact polytope, where I is a finite or infinite index set, and let $X := \prod_{\alpha \in I} X_\alpha$ be the product H-space. For each $\alpha \in I$, let π_α be the projection of X onto X_α, and Q_α be a measure of precompactness in X_α. We say that a set-valued mapping $T_\alpha : X \longrightarrow 2^{X_\alpha}$ is ***Q_α-condensing***, if $Q_\alpha(\pi_\alpha(C)) \subsetneq Q_\alpha(T_\alpha(C))$ for each $C \in X$ satisfying $\pi_\alpha(C)$ is a nonprecompact subset of X_α. It is clear that $T_\alpha : X \longrightarrow 2^{X_\alpha}$ is Q_α-condensing whenever X_α is compact. In case the index set I is singleton and π_α is the identity mapping, the Q_α-condensing mapping reduces to an usual Q-condensing mapping.

2. Equilibria of abstract economies

In order to establish our main result, we first list some fundamental notions. Let X be a topological space, Y an H-space, $S, T : X \longrightarrow 2^Y$ two set-valued mappings, and $\theta : X \longrightarrow Y$ be a single-valued map.

(1) T is said to be ***upper semicontinuous***, if for each $x \in X$ and each open subset V of Y with $T(x) \subseteq V$, there exists a neighborhood N_x of x such that $T(z) \subseteq V$ for all $z \in N_x$.
(2) T is said to be ***almost upper semicontinuous***, if for each $x \in X$ and each open subset V of Y with $T(x) \subseteq V$, there exists a neighborhood N_x of x such that $T(z) \subseteq \mathrm{cl}_Y V$ for all $z \in N_x$.
(3) The set-valued mappings $S \cap T : X \longrightarrow 2^Y$ and $\mathrm{cl}T : X \longrightarrow 2^Y$ are defined by

$$(S \cap T)(x) := S(x) \cap T(x) \quad \text{and} \quad \mathrm{cl}T(x) := \mathrm{cl}_Y T(x) \text{ for each } x \in X.$$

(4) T is said to be ***of class $\mathcal{L}_\theta$***, if
   (a) for each $x \in X$, $\theta(x) \notin H\text{-}coT(x)$,
   (b) for each $y \in Y$, $T^{-1}(y)$ is compactly open in X.
(5) A set-valued mapping $T_x : X \longrightarrow 2^Y$ is said to be an ***$\mathcal{L}_\theta$-majorant of T at x***, if there exists an open neighborhood N_x of x in X such that
   (a) for each $z \in N_x$, $T(z) \subseteq T_x(z)$ and $\theta(z) \notin H\text{-}coT_x(z)$,
   (b) for each $y \in Y$, $T_x^{-1}(y)$ is compactly open in X.
(6) T is said to be ***$\mathcal{L}_\theta$-majorized***, if for each $x \in X$ with $T(x) \neq \emptyset$, there exists an $\mathcal{L}_\theta$-majorant of T at x.

The following Theorem A [12, Theorem 2.1] and Theorem B [6, Theorem 2.6] shall paly an important role.

Theorem A. *Let* $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)_{\alpha\in I}$ *be a family of l.c.-spaces with precompact polytopes,* $X := \prod_{\alpha\in I} X_\alpha$*, and* $T_\alpha : X \longrightarrow 2^{X_\alpha}$ *be* Q_α*-condensing for each* $\alpha \in I$*. Then there exists a nonempty compact* H*-convex subset* $K := \prod_{\alpha\in I} K_\alpha$ *of* X *such that* $T_\alpha(K) \subseteq K_\alpha$*.*

Theorem B. *Let* $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)_{\alpha\in I}$ *be a family of l.c.-spaces with precompact polytopes,* $X := \prod_{\alpha\in I} X_\alpha$*, and* $T_\alpha : X \longrightarrow 2^{X_\alpha}$ *be* $\mathcal{L}_{\pi_\alpha}$*-majorized* Q_α*-condensing. If for each* $\alpha \in I$*, the set* $\{x \in X \mid T_\alpha(x) \neq \emptyset\}$ *is compactly open in* X*, then there exists* $\hat{x} \in X$ *such that* $T_\alpha(\hat{x}) = \emptyset$ *for all* $\alpha \in I$*; that is,* $\hat{x}$ *is a common maximal element of* T_α*.*

Now, we are ready to deduce an existence theorem of equilibria for noncompact abstract economies.

Theorem 2.1. *Let* $\mathbf{\Omega} := (X_\alpha, A_\alpha, B_\alpha, P_\alpha)_{\alpha\in I}$ *be an abstract economy, and* $X := \prod_{\alpha\in I} X_\alpha$*, where* I *is a set of agents such that for each* $\alpha \in I$*,*

(1) X_α *is an l.c.-space with precompact polytopes,*
(2) *for each* $x \in X$*,* $A_\alpha(x) \neq \emptyset$ *and* H*-co*$A_\alpha(x) \subseteq \mathrm{cl}B_\alpha(x)$*,*
(3) *for each* $y_\alpha \in X_\alpha$*,* $A_\alpha^{-1}(y_\alpha)$ *is compactly open in* X*,*
(4) $\mathrm{cl}B_\alpha : X \longrightarrow 2^{X_\alpha}$ *is upper semicontinuous and* Q_α*-condensing,*
(5) $A_\alpha \cap P_\alpha$ *is* $\mathcal{L}_{\pi_\alpha}$*-majorized,*
(6) *the set* $\{x \in X \mid (A_\alpha \cap P_\alpha)(x) \neq \emptyset\}$ *is compactly open in* X*.*

Then $\mathbf{\Omega}$ *has an equilibrium point.*

Proof. Since each $\mathrm{cl}B_\alpha$ is Q_α-condensing, by Theorem A, there exists a nonempty compact H-convex subset $K := \prod_{\alpha\in I} K_\alpha$ of X such that $\mathrm{cl}B_\alpha(K) \subseteq K_\alpha$. By (2), it follows that $A_\alpha(x) \subseteq K_\alpha$ for each $x \in K$. Notice that the set $F_\alpha := \{x \in K \mid x_\alpha \in \mathrm{cl}B_\alpha(x)\}$ is closed in K for each $\alpha \in I$, since $\mathrm{cl}B_\alpha$ is upper semicontinuous. Now, we define $T_\alpha : K \longrightarrow 2^{K_\alpha}$ by

$$T_\alpha(x) := \begin{cases} (A_\alpha \cap P_\alpha)(x) & \text{, if } x \in F_\alpha, \\ H\text{-}coA_\alpha(x) & \text{, if } x \in K \setminus F_\alpha. \end{cases}$$

We shall show that T_α satisfies the hypotheses of Theorem B. First, for each $\alpha \in I$, we have

$$\begin{aligned} \{x \in K \mid T_\alpha(x) \neq \emptyset\} &= \{x \in F_\alpha \mid T_\alpha(x) \neq \emptyset\} \cup \{x \in K \setminus F_\alpha \mid T_\alpha(x) \neq \emptyset\} \\ &= (F_\alpha \cap \{x \in K \mid (A_\alpha \cap P_\alpha)(x) \neq \emptyset\}) \cup (K \setminus F_\alpha) \\ &= \{x \in K \mid (A_\alpha \cap P_\alpha)(x) \neq \emptyset\} \cup (K \setminus F_\alpha), \end{aligned}$$

which is open in K by (6), since K is compact and $K \setminus F_\alpha$ is open in K.

Next, for each $x \in K$ with $T_\alpha(x) \neq \emptyset$, we will find an $\mathcal{L}_{\pi_\alpha}$-majorant of T_α at x. To see this, we consider the following two cases:

Case 1. $x \in K \setminus F_\alpha$.

Let $S_x := H\text{-}coA_\alpha$, and $N_x := K \setminus F_\alpha$. Then N_x is an open neighborhood of x in K, satisfying the following facts:

(a) $T_\alpha(z) = S_x(z)$ for each $z \in N_x$, and $z_\alpha = \pi_\alpha(z) \notin H\text{-}coA_\alpha(z) = H\text{-}coS_x(z)$ by (2).

(b) $S_x^{-1}(y_\alpha)$ is open in K for all $y_\alpha \in K_\alpha$, by (3) and [5, Lemma 3.1].

Therefore, S_x is an $\mathcal{L}_{\pi_\alpha}$-majorant of T_α at x.

Case 2. $x \in F_\alpha$.

Since $A_\alpha \cap P_\alpha$ is $\mathcal{L}_{\pi_\alpha}$-majorized by (5), $A_\alpha \cap P_\alpha$ admits a $\mathcal{L}_{\pi_\alpha}$-majorant $T_x : K \longrightarrow 2^{K_\alpha}$ at x; that is, there exists an open neighborhood N_x of x in K such that

(a') for each $z \in N_x$, $T_\alpha(z) = (A_\alpha \cap P_\alpha)(z) \subseteq T_x(z)$ and $z_\alpha \notin H\text{-}coT_x(z)$.

(b') for each $y_\alpha \in K_\alpha$, $T_x^{-1}(y_\alpha)$ is (compactly) open in K.

Define $S_x : K \longrightarrow 2^{K_\alpha}$ by

$$S_x(z) := \begin{cases} (H\text{-}coA_\alpha(z)) \cap T_x(z) & , \text{if } z \in F_\alpha, \\ H\text{-}coA_\alpha(z) & , \text{if } z \notin F_\alpha. \end{cases}$$

Note that for each $z \in N_x$, $z \notin F_\alpha$. It follows that $z_\alpha \notin H\text{-}coS_x(z)$ by (a'). Moreover, for any $y_\alpha \in K_\alpha$, the set

$$\begin{aligned} S_x^{-1}(y_\alpha) &= \{z \in F_\alpha \mid y_\alpha \in S_x(z)\} \cup \{z \in K \setminus F_\alpha \mid y_\alpha \in S_x(z)\} \\ &= \{z \in F_\alpha \mid y_\alpha \in (H\text{-}coA_\alpha(z)) \cap T_x(z)\} \cup \{z \in K \setminus F_\alpha \mid y_\alpha \in H\text{-}coA_\alpha(z)\} \\ &= \left[F_\alpha \cap (H\text{-}coA_\alpha)^{-1}(y_\alpha) T_x^{-1}(y_\alpha)\right] \cup \left[(K \setminus F_\alpha) \cap (H\text{-}coA_\alpha)^{-1}(y_\alpha)\right] \\ &= \left[T_x^{-1}(y_\alpha) \cup (K \setminus F_\alpha)\right] \cap (H\text{-}coA_\alpha)^{-1}(y_\alpha) \end{aligned}$$

is open in K by (3) and [5, Lemma 3.1]. Therefore, S_x is an $\mathcal{L}_{\pi_\alpha}$-majorant of T_α at x.

Consequently, T_α is $\mathcal{L}_{\pi_\alpha}$-majorized, and hence by Theorem B, there exists $\hat{x} \in K$ such that $T_\alpha(\hat{x}) = \emptyset$ for all $\alpha \in I$. By (2), it follows that $\hat{x} \in F_\alpha$ and $T_\alpha(\hat{x}) = A_\alpha(\hat{x}) \cap P_\alpha(\hat{x})$ for each $\alpha \in I$. Hence $\hat{x}_\alpha \in \mathrm{cl}B_\alpha(\hat{x})$ and $A_\alpha(\hat{x}) \cap P_\alpha(\hat{x}) = \emptyset$. Thus, $\hat{x}$ is an equilibrium point of $\mathbf{\Omega}$. □

We remark that Theorem 2.1 improves and generalizes [4, Theorem 4] and [10, Theorem 7] to general noncompact *l.c.*-spaces. As a consequence, we have an extension of Ding [4] as follows.

Corollary 2.2. *Let $\mathbf{\Omega} := (X_\alpha, A_\alpha, P_\alpha)_{\alpha \in I}$ be an abstract economy, where I is a set of agents such that for each $\alpha \in I$,*

(1) *X_α is a compact H-space,*

(2) *$A_\alpha : X \longrightarrow 2^{X_\alpha}$ is almost upper semicontinuous with nonempty H-convex values,*

(3) *$A_\alpha^{-1}(x_\alpha)$ is open for each $x_\alpha \in X_\alpha$,*

(4) *$A_\alpha \cap P_\alpha$ is $\mathcal{L}_{\pi_\alpha}$-majorized,*

(5) *the set $\{x \in X \mid (A_\alpha \cap P_\alpha)(x) \neq \emptyset\}$ is open in X.*

Then $\mathbf{\Omega}$ has an equilibrium point.

Proof. First, we note that the set-valued mapping $\mathrm{cl}A_\alpha$ is upper semicontinuous and Q_α-condensing, since A_α is almost upper semicontinuous and X_α is compact. Next, although the existence of K in proving Theorem 2.1 requires the conditions (1) and (4) of Theorem 2.1, we can directly take $K_\alpha := X_\alpha$ and $K := \prod_{\alpha\in I} X_\alpha$, instead of using those conditions. Thus, following the proof of Theorem 2.1 with $A_\alpha = B_\alpha$, the abstract economy $(X_\alpha, A_\alpha, A_\alpha, P_\alpha)_{\alpha\in I}$ has an equilibrium point; that is, $\mathbf{\Omega}$ has an equilibrium point. □

3. System of quasi-variational inequalities

Let I be an index set, $(X_\alpha)_{\alpha\in I}$, $(Y_\alpha)_{\alpha\in I}$ two families of topological spaces, and $X := \prod_{\alpha\in I} X_\alpha$, $Y := \prod_{\alpha\in I} Y_\alpha$. For each $\alpha \in I$, let $T_\alpha : X \longrightarrow 2^{Y_\alpha}$, $A_\alpha : X \longrightarrow 2^{X_\alpha}$ be two set-valued mappings, and $\phi_\alpha : X \times Y_\alpha \times X_\alpha \longrightarrow \mathrm{R}$ be a real-valued function. The ***system of generalized quasi-variational inequalities*** (in short, **SGQVI**) is defined as follow:

$$(\mathbf{SGQVI}) : \begin{cases} \text{Find } (\hat{x}, \hat{y}) \in X \times Y \text{ such that for each } \alpha \in I, \\ \hat{x}_\alpha \in \mathrm{cl}A_\alpha(\hat{x}),\ \hat{y}_\alpha \in T_\alpha(\hat{x}), \text{ and } \phi_\alpha(\hat{x}, \hat{y}_\alpha, z_\alpha) \geq 0 \text{ for all } z_\alpha \in A_\alpha(\hat{x}). \end{cases}$$

If the index set $I = \{1\}$, then SGQVI reduces to the quasi-variational inequality: Find $(\hat{x}, \hat{y}) \in X \times Y$ such that $\hat{x} \in \mathrm{cl}A(\hat{x}), \hat{y} \in T(\hat{x})$, and $\phi(\hat{x}, \hat{y}, z) \geq 0$ for all $z \in A(\hat{x})$.

As an application to SGQVI, we recall that a topological space X is called ***acyclic***, if all of its reduced Čech homology groups over rationals vanish. In particular, any contractible space is acyclic, and thus any convex or star-shaped set is acyclic. In an H-space X, a function $f : X \longrightarrow \mathrm{R}\cup\{\pm\infty\}$ is called ***H-quasiconvex***, provided that for each $r \in \mathrm{R}$, the set $\{x \in X \mid f(x) < r\}$ is H-convex. Next result provides an existence theorem of solutions to SGQVI.

Theorem 3.1. *Let $(X_\alpha, \mathcal{U}_\alpha, \mathcal{B}_\alpha)_{\alpha\in I}$ be a family of l.c.-spaces with precompact polytopes, $(Y_\alpha)_{\alpha\in I}$ a family of topological space, and $X := \prod_{\alpha\in I} X_\alpha$, $Y := \prod_{\alpha\in I} Y_\alpha$. Suppose that for each $\alpha \in I$, $A_\alpha : X \longrightarrow 2^{X_\alpha}$ is an almost upper semicontinuous Q_α-condensing set-valued mapping with nonempty H-convex values, satisfying $A_\alpha^{-1}(x_\alpha)$ is compactly open for each $x_\alpha \in X_\alpha$, and $T_\alpha : X \longrightarrow 2^{Y_\alpha}$ is upper semicontinuous with nonempty compact values. If $\phi_\alpha : X \times Y_\alpha \times X_\alpha \longrightarrow \mathrm{R}$ is an upper semicontinuous function such that*

1. *for each $(x, y_\alpha) \in X \times Y_\alpha$, $z_\alpha \mapsto \phi_\alpha(x, y_\alpha, z_\alpha)$ is H-quasiconvex,*
2. *for each $x \in X$, there exists $y_\alpha \in T_\alpha(x)$ such that $\phi_\alpha(x, y_\alpha, x_\alpha) \geq 0$,*
3. *for each $(x, z_\alpha) \in X \times X_\alpha$, the set $\{y_\alpha \in T_\alpha(x) \mid \phi_\alpha(x, y_\alpha, z_\alpha) \geq 0\}$ is acyclic,*

then there is a solution to **SGQVI**.

Proof. For each $\alpha \in I$, we define a set-valued mapping $P_\alpha : X \longrightarrow 2^{X_\alpha}$ by

$$P_\alpha(x) := \left\{ z_\alpha \in X_\alpha \mid \sup_{y_\alpha \in T_\alpha(x)} \phi_\alpha(x, y_\alpha, z_\alpha) < 0 \right\}, \quad \forall\, x \in X.$$

Then each $P_\alpha(x)$ is an H-convex set, since ϕ_α is H-quasiconvex in z_α. Since ϕ_α is upper semicontinuous and $T_\alpha : X \longrightarrow 2^{Y_\alpha}$ is upper semicontinuous with nonempty compact values, the function $x \mapsto \sup_{y_\alpha \in T_\alpha(x)} \phi_\alpha(x, y_\alpha, z_\alpha)$ is upper semicontinuous, by Proposition 21 of [2, p.119]. Hence for each $z_\alpha \in X_\alpha$,

$$P_\alpha^{-1}(z_\alpha) = \{x \in X \mid z_\alpha \in P_\alpha(x)\} = \left\{ x \in X \mid \sup_{y_\alpha \in T_\alpha(x)} \phi_\alpha(x, y_\alpha, z_\alpha) < 0 \right\}$$

is open in X. In addition, by (2), $x_\alpha \notin P_\alpha(x) = H\text{-}coP_\alpha(x)$ for each $x \in X$. Thus, P_α is of class $\mathcal{L}_{\pi_\alpha}$, and hence is $\mathcal{L}_{\pi_\alpha}$-majorized. Therefore, the set-valued mapping $A_\alpha \cap P_\alpha$ is also $\mathcal{L}_{\pi_\alpha}$-majorized.

Note that for each nonempty compact subset K of X, since $A_\alpha^{-1}(z_\alpha) \cap K$ is open, the set

$$\begin{aligned} \{x \in X \mid (A_\alpha \cap P_\alpha)(x) \neq \emptyset\} \cap K &= \left(\bigcup_{z_\alpha \in X_\alpha} (A_\alpha \cap P_\alpha)^{-1}(z_\alpha) \right) \cap K \\ &= \bigcup_{z_\alpha \in X_\alpha} \left(A_\alpha^{-1}(z_\alpha) \cap K) \cap P_\alpha^{-1}(z_\alpha) \right) \end{aligned}$$

is open in K; i.e., the set $\{x \in X \mid (A_\alpha \cap P_\alpha)(x) \neq \emptyset\}$ is compactly open in X. Further, the set-valued mapping $\mathrm{cl}A_\alpha$ is upper semicontinuous and Q_α-condensing. Consequently, by virtue of Theorem 2.1 with $B_\alpha = A_\alpha$, there exists $\hat{x} \in X$ such that $\hat{x}_\alpha \in \mathrm{cl}A_\alpha(\hat{x})$ and $A_\alpha(\hat{x}) \cap P_\alpha(\hat{x}) = \emptyset$ for each $\alpha \in I$. It follows that

$$\sup_{y_\alpha \in T_\alpha(\hat{x})} \phi_\alpha(\hat{x}, y_\alpha, z_\alpha) \geq 0, \quad \forall\ z_\alpha \in A_\alpha(\hat{x}).$$

Since each ϕ_α is upper semicontinuous and $T_\alpha(\hat{x})$ is compact, it follows that for each $z_\alpha \in A_\alpha(\hat{x})$, there exists $y_\alpha(z_\alpha) \in T_\alpha(\hat{x})$ such that $\phi_\alpha(\hat{x}, y_\alpha(z_\alpha), z_\alpha) \geq 0$. This leads us to define a set-valued mapping $G_\alpha : A_\alpha(\hat{x}) \longrightarrow 2^{T_\alpha(\hat{x})}$ by

$$G_\alpha(z_\alpha) := \{y_\alpha \in T_\alpha(\hat{x}) \mid \phi_\alpha(\hat{x}, y_\alpha, z_\alpha) \geq 0\}, \quad \forall\ z_\alpha \in A_\alpha(\hat{x}).$$

By the upper semicontinuity of ϕ_α, the graph of G_α is closed. Moreover, since $T_\alpha(\hat{x})$ is compact, G_α is upper semicontinuous, with nonempty acyclic values by (3).

Assume that the conclusion of Theorem 3.1 is false. Then there exists $\beta \in I$ such that for each $y_\beta \in T_\beta(\hat{x})$, there exists a point $z_\beta \in A_\beta(\hat{x})$ satisfying $\phi_\beta(\hat{x}, y_\beta, z_\beta) < 0$. Let the set-valued mapping $S_\beta : T_\beta(\hat{x}) \longrightarrow 2^{A_\beta(\hat{x})}$ be defined by

$$S_\beta(y_\beta) := \{z_\beta \in A_\beta(\hat{x}) \mid \phi_\beta(\hat{x}, y_\beta, z_\beta) < 0\}, \quad \forall\ y_\beta \in T_\beta(\hat{x}).$$

Then, by (1) and the H-convexity of $A_\beta(\hat{x})$, S_β has nonempty H-convex values. For each $z_\beta \in A_\beta(\hat{x})$, the set

$$S_\beta^{-1}(z_\beta) = \{y_\beta \in T_\beta(\hat{x}) \mid z_\beta \in S_\beta(y_\beta)\} = \{y_\beta \in T_\beta(\hat{x}) \mid \phi_\beta(\hat{x}, y_\beta, z_\beta) < 0\}$$

is open in $T_\beta(\hat{x})$. By [3, Theorem 3.1], there exists a coincidence $(\bar{x}_\beta, \bar{y}_\beta)$ for S_β and G_β; that is, $\bar{x}_\beta \in S_\beta(\bar{y}_\beta)$ and $\bar{y}_\beta \in G_\beta(\bar{x}_\beta)$. It follows that $\phi_\beta(\hat{x}, \bar{y}_\beta, \bar{x}_\beta) < 0$ and $\phi_\beta(\hat{x}, \bar{y}_\beta, \bar{x}_\beta) \geq 0$, which is a trivial contradiction each other. Therefore, we complete the proof. □

Remark that Theorem 3.1 generalizes and improves [11, Theorem 8] as follows:

(1) The space X need not be perfectly normal.
(2) Theorem 3.1 is concerning a system of generalized quasivariational inequalities.
(3) Theorem 3.1 need not have an extra acyclic condition [11, Theorem 8 (iii)].

As a consequence, the following Corollary extends [11, Corollary 9].

Corollary 3.2. *Let $(X_\alpha)_{\alpha\in I}$ be a family of locally convex topological vector spaces, X_α^* the conjugate space with respect to X_α, and $X := \prod_{\alpha\in I} X_\alpha$, $X^* := \prod_{\alpha\in I} X_\alpha^*$. For each $\alpha \in I$, let $A_\alpha : X \longrightarrow 2^{X_\alpha}$ be an almost upper semicontinuous Q_α-condensing set-valued mapping with nonempty convex values, satisfying $A_\alpha^{-1}(x_\alpha)$ is compactly open for each $x_\alpha \in X_\alpha$, and $T_\alpha : X \longrightarrow 2^{X_\alpha^*}$ be upper semicontinuous with nonempty compact convex values. Then there exists $(\hat{x}, \hat{y}) \in X \times X^*$ such that for each $\alpha \in I$,*

$$\hat{x}_\alpha \in clA_\alpha(\hat{x}),\ \hat{y}_\alpha \in T_\alpha(\hat{x}),\ \text{and}\ Re\langle \hat{y}_\alpha, z_\alpha - \hat{x}_\alpha\rangle \geq 0 \text{ for all } z_\alpha \in A_\alpha(\hat{x}).$$

Proof. Let $\mathcal{N}_\alpha$ be the family of all neighborhoods of zero in X_α. For each $N_\alpha \in \mathcal{N}_\alpha$, we define $U_\alpha := \{(x_\alpha, y_\alpha) \in X_\alpha \times X_\alpha \mid x_\alpha - y_\alpha \in N_\alpha\}$. Then $(X_\alpha, \{coD_\alpha\})$ $(D_\alpha \in \langle X_\alpha\rangle)$ is an $l.c.$-space with precompact polytopes, whose uniformity is $\mathcal{U}_\alpha := \{ U_\alpha \mid N_\alpha \in \mathcal{N}_\alpha\}$. For each $(x, y_\alpha, z_\alpha) \in X \times X_\alpha^* \times X_\alpha$, let $\phi_\alpha(x, y_\alpha, z_\alpha) = Re\langle y_\alpha, z_\alpha - \pi_\alpha(x)\rangle$. Then ϕ_α is continuous and satisfies all the conditions of Theorem 3.1. Consequently, the conclusion follows from Theorem 3.1. □

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