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A NEW INTERPRETATION OF THE SHAPLEY VALUE

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Abstract. It is well-known that the Shapley value of a convex TU game is the center of gravity of the extreme points of the core. However, the above geometric interpretation is not applied to the Shapley value of a game with empty core. In this article, we provide a new geometric interpretation for the Shapley value of both TU games with non-empty core and TU games with empty core.

1. Introduction

The Shapley value was proposed by Lloyd Shapley in his 1953 Ph.D. dissertation [5]. He originally adopted the axiomatic method and proved that there is a unique single-valued solution to TU games satisfying the three axioms: symmetry, additivity and carrier efficiency. It is well-known that carrier efficiency axiom can be replaced by two axioms, namely efficiency axiom and dummy axiom. He had obtained the explicit formula of this unique solution(see Shapley [6], 1953). The Shapley value has applications in many fields such as economics, political sciences, accounting, and even military sciences.

The core(Gillies [2], 1953) of an n-person TU game is a m-dimensional polytope(might be empty), where m is less than or equal to $n-1$. Shapley [8](1971) showed that the Shapley value for a convex TU game is the center of gravity of the extreme points of the core(also see Ichiishi [4],1981). However, the Bondareva-Shapley theorem(Bondareva [1], 1962 and Shapley [7], 1967) said that the core of a game is non-empty if and only if the game is balanced. In other words, the core of a game could be empty and the above geometric interpretation is not applied to the Shapley value of a game with empty core.

In this article, enlightened by the "snowballing" or "bandwagon" effect mentioned in the paper of Shapley([8],1971), we propose a new class of TU games called coalitional regular in average games, abbreviated as CRIA games.

Observing the structure of the core, we introduce the concepts of k^{th} semi-cores and k^{th} pseudo-cores of an n-person game, for $k = 1, 2, \ldots, n-1$. When all pseudo-cores of an n-person TU game are non-empty, enlightened by the concept of compromise, a middle way between two extremes, we propose the middle-way solution of the TU game as the geometric centroid of the $n-1$ mass centers of the pseudo-cores of the game. Surprisingly, we find that the middle-way solution is exactly the Shapley value. This gives the Shapley value a new geometric interpretation and a new characterization, or say, a new intuitive interpretation, as the middle-way solution. In section 4 of this article, we have a real-world example to explain the intuitive meaning of the middle-way solution.

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Also, we show that a game is CRIA if and only if none of its k^{th} pseudo-cores is empty. Furthermore, a CRIA game might have empty core, therefore, our geometric interpretation is applied to the Shapley value for games with empty core. Finally, our middle-way solution is different from the core-center(Gonzalez-Diaz& Sanchez-Rodriguez [3], 2007).

2. Preliminaries

We adopt the notation as in most of mathematics text book, a vector in $\mathbb{R}^n$ is usually denoted by a bold face letter, $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and the order relation $\mathbf{x} \geq \mathbf{y}$ means that $x_i \geq y_i$ for all $i \in N$. Following [2],[6] and [8], we have definitions and notations as follows.

Let $N = \{1, 2, \ldots, n\}$ be the set of players. The collection of coalitions(subsets) in N is denoted by 2^N. The coalition N is called the grand coalition. The number of players in coalition S is denoted by $|S|$.

A cooperative n-person game with transferable utility, shortly TU game, in characteristic function is a pair (N, v), where $v : 2^N \to \mathbb{R}$ is a function assigning, to each coalition $S \in 2^N$, its worth $v(S)$ satisfying $v(\emptyset) = 0$. The set of all n-person games defined on N is denoted by G^N.

A game (N, v) is said to be superadditive if and only if $v(T \cup S) \geq v(T) + v(S)$, for all disjoint $S, T \in 2^N$. Moreover, (N, v) is called a proper game if it is superadditive. A proper game is clearly monotonic in the following sense: if $S \subseteq T$ then $v(S) \leq v(T)$. A game (N, v) is convex if, $v(T \cup S) + v(T \cap S) \geq v(T) + v(S)$, for every $S, T \in 2^N$.

A payoff vector(allocation) $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called an *imputation* if $\sum_{i \in N} x_i = v(N)$ (group rationality or efficiency), and $x_i \geq v(\{i\})$, for each $i \in N$ (individual rationality). The set of all imputations of v is denoted by $I(v)$.

Given a payoff vector $\mathbf{x}$, we define $\mathbf{x}(S) = \sum_{j \in S} x_j$, for each $S \subseteq N$, and define $\mathbf{x}(\emptyset) = 0$.

The core(Gillies, 1953) of an n-person game (N, v) is the set of payoff vectors(allocations)

$$C(N, v) = \{\mathbf{x} \in R^n : \mathbf{x}(S) \geq v(S), \text{ for every } S \subseteq N, \mathbf{x}(N) = v(N)\}. \tag{2.1}$$

The Shapley value on G^N is well-known as the mapping $\phi = (\phi_1, \ldots, \phi_n) : G^N \to \mathbb{R}^n$ such that

$$\phi_i(v) = \sum_{\substack{S \subset N \\ i \notin S}} \frac{|S|!(n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S)), i = 1, \ldots, n.$$

3. The coalitional regular in average games

Given an n-person game (N, v), for each integer k= 0, 1, 2, $\ldots$, n, we define $\Omega^k = \{T \mid |T| = k, T \subseteq N\}$. (*Note:* The indices $k+1$, k, $k-1$ and $k-2$ always denote the size of a coalition through out this article.)

It is easy to see that the size of Ω^k is $|\Omega^k| = \binom{n}{k}$. For each k, we define the average of all the worths of the coalitions $T \in \Omega^k$ by the following:

$$ave(k, v) = \frac{1}{\binom{n}{k}} \sum_{\substack{|T|=k \\ T \subseteq N}} v(T).$$

It is easy to see $ave(0, v) = 0$ and $ave(n, v) = v(N)$. Suppose that each probability function P_k defined on sample space Ω^k is uniformly distributed and the players in a coalition T share the worth $v(T)$ equally, i.e. each player in T shares $\frac{1}{|T|} \cdot v(T)$, then the expected payoff for a player in a coalition of size k is $\frac{1}{k} \cdot ave(k, v)$. Enlightened by the "snowballing" or "bandwagon" effect mentioned in the paper of Shapley([5],1971), we propose the following definition.

Definition 3.1. A TU game (N, v) is called a *coalitional regular in average game*, abbreviated as CRIA game if

$$\frac{1}{n} \cdot v(N) \geq \frac{1}{k} \cdot ave(k, v) = \frac{1}{k} \cdot \frac{1}{\binom{n}{k}} \sum_{\substack{|T|=k \\ T \subseteq N}} v(T), \text{ for each } k = 1, 2, \ldots, n-1. \tag{3.1}$$

In the investigation of the solutions of a cooperative game, researchers usually assume that the grand coalition N is formed, then study how to distribute $v(N)$ among all the players in N. A player has incentive to participate in a bigger coalition if the expected payoff is bigger. The inequality (3.1) makes players have incentive to participate in N, i.e. there is a "snowballing" or "bandwagon" effect in a CRIA game. Therefore it is more acceptable in a CRIA game than in an improper game to assume that the grand coalition is formed. As illustrations, we provide three examples.

Example 3.2. Let $N = \{1, 2, 3\}, v(\{1\}) = v(\{2\}) = v(\{3\}) = 0, v(\{1, 2\}) = 1$, $v(\{1, 3\}) = 3, v(\{2, 3\}) = 2, v(\{1, 2, 3\}) = 3$. Then $ave(1, v) = 0, ave(2, v) = 2$, and hence $ave(1, v) \leq \frac{1}{3}v(N), ave(2, v) \leq \frac{2}{3}v(N)$. Hence, (N, v) is CRIA. On the other hand, since for $A = \{1, 3\}, B = \{2, 3\}, v(A \cup B) + v(A \cap B) = 3 + 0 \ngeq 3 + 2 = v(A) + V(B)$, (N, v) is not a convex game. Therefore a CRIA game is not necessarily a convex game. Later, we will show that a convex game is a CRIA game.

Example 3.3. Let $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 2, v(\{1, 3\}) = 2, v(\{2, 3\}) = 2$, and $v(\{1, 2, 3\}) = 2.5$. (N, v) is clearly a proper game. On the other hand, since $\frac{1}{2} \times \frac{1}{\binom{3}{2}} \times (v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\})) = \frac{1}{2} \cdot \frac{1}{3}(2 + 2 + 2) \nleq \frac{1}{3} \times v(N) = \frac{1}{3} \times 2.5$, (N, v) is not a CRIA game. Therefore, a proper game is not necessarily a CRIA game.

Example 3.4. Let $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = 3, v(\{1, 3\}) = 1, v(\{2, 3\}) = 1$ and $v(\{1, 2, 3\}) = 2.5$. (N, v) is clearly not a proper game. On the other hand, since $ave(1, v) = 0$, $ave(2, v) = \frac{5}{3}$, and hence $ave(1, v) \leq \frac{1}{3}v(N)$, $\frac{1}{2}ave(2, v) \leq \frac{1}{3}v(N)$, hence (N, v) is a CRIA game. Therefore, a CRIA game is not necessarily a proper game.

4. Pseudo cores and middle-way solution

Observing the core $C(v)$ as (2.1), for $k = 1, 2, \ldots, n-1$, we defined the k^{th} semi-cores $C^k(v)$ as follows.

Definition 4.1. Given an integer k with $1 \leq k \leq n-1$, the k^{th} *semi-core* of an n-person game (N, v), denoted by $C^k(v)$, is defined as follows.

$$C^k(v) = \{\mathbf{x} \mid \mathbf{x}(S) \geq v(S), \text{ for all } S \subseteq N \text{ with } |S| = k \text{ and } \mathbf{x}(N) = v(N)\} \tag{4.1}$$

Apparently, the first semi-core $C^1(v)$ is the set of all imputations of v. Also, it is clear that the core of v is the intersection of all the k^{th} semi-cores of v, i.e.

$$C(v) = \bigcap_{k=1}^{k=n-1} C^k(v).$$

We have the following example where all k^{th} semi-cores $C^k(v) \neq \emptyset$ but the core $C(v) = \emptyset$.

Example 4.2. Let $N = \{1, 2, 3\}$ and v be the TU game defined on 2^N such that $v(\{1,2\}) = 2$, $v(\{3\}) = 2$, $v(\{1,2,3\}) = 3$ and $v(S) = 0$ for any other coalition S.

Then, the core $C(v) = \emptyset$, the first semi-core $C^1(v) = \{(x_1, x_2, x_3) \mid x_1 \geq 0, x_2 \geq 0, x_3 \geq 2 \text{ and } x_1 + x_2 + x_3 = 3\} \neq \emptyset$ and the 2nd semi-core $C^2(v) = \{(x_1, x_2, x_3) \mid x_1 + x_2 \geq 2, x_1 + x_3 \geq 0, x_2 + x_3 \geq 0 \text{ and } x_1 + x_2 + x_3 = 3\} \neq \emptyset$. Since the core of v is empty, we cannot say that the Shapley value of v is the geometric center of the core. However, since $C^1(v) \neq \emptyset$ and $C^2(v) \neq \emptyset$, the players may consider a middle-way solution which is out of $C^1(v)$ and $C^2(v)$.

Given an n-person game (N, v), we consider the $(k+1)^{th}$ semi-core of the game (N, v) and a fixed player $\ell \in N$. Write $\mathbf{x} = (x_1, x_2, \ldots, x_{\ell-1}, x_\ell, x_{\ell+1}, \ldots, x_n) \in C^{(k+1)}(v)$. Observing
$C^{(k+1)}(v) = \{\mathbf{x} \mid \mathbf{x}(S) \geq v(S) \text{ for all } S \subset N \text{ with } |S| = k+1 \text{ and } \mathbf{x}(N) = v(N)\}$, by a simple combinatorial calculation, we can see that the player ℓ appears $\binom{n-1}{k}$ times in the coalitions S where $|S| = k+1$ related to the inequalities $\mathbf{x}(S) \geq v(S)$. Also, by a simple combinatorial calculation, we can see that any other player $i \neq \ell$ appears $\binom{n-2}{k-1}$ times in the coalitions S with $\ell \in S$ and $|S| = k+1$. Therefore, given $S = T \cup \{\ell\}$ with $\ell \notin T$ and $|S| = k+1$, there are $\binom{n-1}{k}$ inequalities as follows.

$$x_\ell + \sum_{\substack{i \in S \\ i \neq \ell}} x_i \geq v(T \cup \{\ell\}) = v(S), \text{ for each } T \subseteq N - \{\ell\} \text{ with } |T| = k \tag{4.2}$$

Now, for all the inequalities corresponding to S where $S = T \cup \{\ell\}$ with $|T| = k$ and $\ell \notin T$ in (4.2), we have the following.

Case 1. When $|T| = k = 0$, $T = \emptyset$, then

$$x_\ell + 0 \geq \sum_{T=\emptyset} v(T \cup \{\ell\}) = v(\{\ell\})$$

$$= \frac{n-1}{n-k-1} \sum_{T=\emptyset} v(T \cup \{\ell\}) - \frac{k}{n-k-1} v(N) \tag{4.3}$$

$$= \frac{1}{1}v(\{\ell\}) - \frac{0}{n-k-1}v(N) = w_\ell^0. \tag{4.4}$$

Case 2. When $|T| = k = 1, 2, \ldots, n-2$, sum all the inequalities corresponding to S where $S = T \cup \{\ell\}$ with $|T| = k$ and $\ell \notin T$ in (4.2) together, we have

$$\binom{n-1}{k} \cdot x_\ell + \binom{n-2}{k-1} \cdot \sum_{i \in N - \{\ell\}} x_i \geq \sum_{\substack{\ell \in S \\ |S| = k+1}} v(S).$$

Since $\binom{n-1}{k} = \binom{n-2}{k} + \binom{n-2}{k-1}$, we have

$$\begin{aligned} &\binom{n-1}{k} \cdot x_\ell + \binom{n-2}{k-1} \cdot \sum_{i \in N - \{\ell\}} x_i \\ &= \binom{n-2}{k} \cdot x_\ell + \binom{n-2}{k-1} \cdot x_\ell + \binom{n-2}{k-1} \cdot \sum_{i \in N - \{\ell\}} x_i \\ &= \binom{n-2}{k} \cdot x_\ell + \binom{n-2}{k-1} \cdot \sum_{i \in N} x_i \\ &= \binom{n-2}{k} \cdot x_\ell + \binom{n-2}{k-1} \cdot v(N) \geq \sum_{\substack{\ell \in S \\ |S| = k+1}} v(S). \end{aligned}$$

Therefore by some simple algebraic computations, we have

$$\begin{aligned} x_\ell &\geq \frac{1}{\binom{n-2}{k}} \sum_{\substack{|T| = k \\ T \subset N - \{\ell\}}} v(T \cup \{\ell\}) + \left(\frac{-k}{n-1-k}\right) v(N) \\ &= \frac{n-1}{n-k-1} \left[\frac{1}{\binom{n-1}{k}} \sum_{\substack{T \subset N - \{\ell\} \\ |T| = k}} v(T \cup \{\ell\}) \right] + \frac{-k}{n-k-1} v(N). \end{aligned} \tag{4.5}$$

Combine (4.4) and (4.5) we have the following proposition.

Proposition 4.3. *Given an n-person game (N, v) and its $(k+1)^{th}$ semi-core $C^{(k+1)}(v)$. let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in C^{(k+1)}(v)$, then*

$$x_\ell \geq \frac{n-1}{n-k-1} \left[\frac{1}{\binom{n-1}{k}} \sum_{\substack{T \subset N - \{\ell\} \\ |T| = k}} v(T \cup \{\ell\}) \right] - \frac{k}{n-k-1} v(N) \overset{def}{=\!=} w_\ell^{k+1}, \tag{4.6}$$

for each $k = 0, 1, 2, \ldots, n-2$ and each $\ell \in N$.

We denote the greatest lower bound of x_ℓ in inequality (4.6) by w_ℓ^{k+1} where ℓ stands for player ℓ and k stands for the number of the other players who participate in a coalition of size $k+1$ with player ℓ. we called w_ℓ^{k+1} the "**worst**" payoff for player ℓ in $C^{k+1}(v)$.

We do not define w_ℓ^k for $k = n-1$, however, when $k = n-1$, we may regard $C^{k+1}(v)$ as the hyperplane $x_1 + x_2 + \cdots + x_n = v(N)$ and $x_\ell \in \mathbb{R}$.

Remark 4.4. Fixed a player ℓ, let $\Omega_\ell^{k+1} = \{S : |S| = k+1, \ell \in S \subseteq N\}$, it is easy to see the number of element $|\Omega_\ell^{k+1}| = \binom{n-1}{k}$. We define the average of all the worths of coalitions S's of size $k+1$ with $\ell \in S$ as follows.

$$ave(k+1, v, \ell) = \frac{1}{\binom{n-1}{k}} \sum_{\substack{|T|=k \\ T \subseteq N-\{\ell\}}} v(T \cup \{\ell\}).$$

If the probability defined on Ω_ℓ^{k+1} is uniformly distributed and the players in each coalition S share the worth $v(S)$ equally, then player ℓ's expected payoff is $(\frac{1}{k+1}) \cdot ave(k+1, v, \ell)$. We have the following interesting observation.

$$w_\ell^{k+1} = \frac{1}{k+1} ave(k+1, v, \ell) - n \cdot \frac{k}{n-k-1}\Big[\frac{1}{n} ave(n, v, \ell) - \frac{1}{k+1} ave(k+1, v, \ell)\Big],$$

which implies $w_\ell^{k+1} = \frac{1}{k+1} ave(k+1, v, \ell)$ if and only if $\frac{1}{n} ave(n, v, \ell) = \frac{1}{k+1} ave(k+1, v, \ell)$. Intuitively, when a player ℓ's expected payoff in coalition of size $(k+1)$ is the same as that in the grand coalition, the player at worst can get his expected payoff in coalition of size $(k+1)$, w.r.t. any payoff vector $\mathbf{x} \in C^{(k+1)}(v)$.

For convenience of combinatorial computations, we let $k = 0, 1, \ldots, n-2$ and make the following definition.

Definition 4.5. The $(k+1)^{th}$ *pseudo-core* of an n-person game (N, v) is defined as follows.

(4.7)

$$PC^{(k+1)}(v) = \{(x_1, \ldots, x_n) \mid x_\ell \geq w_\ell^{k+1}, \text{ for each } \ell \in N \text{ and } \mathbf{x}(N) = v(N)\},$$

where

$$w_\ell^{k+1} = (\frac{n-1}{n-k-1})\left[\frac{1}{\binom{n-1}{k}} \sum_{\substack{T \subset N-\{\ell\} \\ |T|=k}} v(T \cup \{\ell\})\right] - \frac{k}{n-k-1} v(N).$$

Remark 4.6. Please note that $PC^1(v) = C^1(v)$ and $PC^{(k+1)}(v) \supseteq C^{(k+1)}(v)$, however in general $C^{(k+1)}(v)$ is a proper subset of $PC^{(k+1)}(v)$.

We denote $H^0 = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}(N) = v(N)\}$. Given $k \in \{0, 1, 2, \ldots, n-2\}$, for each $\ell \in N$, we denote $H_\ell^{(k+1)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_\ell = w_\ell^{k+1}\}$.

We can easily see that H^0 and $H_\ell^{(k+1)}$, for each $k \in \{0, 1, 2, \ldots, n-2\}$ and each $\ell \in N$ are hyperplanes in $\mathbb{R}^n$.

Next, given a fixed $r \in N$, we define $\mathbf{y}^{(k+1),r} = (y_1^{(k+1),r}, y_2^{(k+1),r}, \ldots, y_n^{(k+1),r}) \in PC^{(k+1)}(v)$ to be the payoff vector where each player $\ell \in N - \{r\}$ get his"worst" payoff w_ℓ^{k+1}, i.e.

$$y_\ell^{(k+1),r} = w_\ell^{k+1}, \text{ for each } \ell \in N - \{r\}, \tag{4.8}$$

then the player r gets

$$y_r^{(k+1),r} = v(N) - \sum_{\ell \in N-\{r\}} y_\ell^{(k+1),r} = v(N) - \sum_{\ell \in N-\{r\}} w_\ell^{k+1}$$

$$= v(N) + (n-1)\cdot\frac{k}{n-1-k}v(N) - \frac{1}{\binom{n-2}{k}}\sum_{\ell\in N-\{r\}}\Bigg(\sum_{\substack{|T|=k\\T\subset N-\{\ell\}}} v(T\cup\{\ell\})\Bigg). \tag{4.9}$$

Intuitively, since all the players other than r get their "worst" payoff's w.r.t. $\mathbf{y}^{(k+1),r}$, then with restriction to (4.6) and the efficiency restriction $\mathbf{y}^{(k+1),r}(N) = v(N)$, $y_r^{(k+1),r}$ is the least upper bound of the inequality

$$x_r \le v(N) - \sum_{\ell\in N-\{r\}} w_\ell^{k+1},$$

it comes from

$$x_r = v(N) - \sum_{\ell\in N-\{r\}} x_\ell \text{ and } x_\ell \ge w_\ell^{k+1}\ \forall\ \ell\ne r.$$

We call $y_r^{(k+1),r}$ the "**best**" payoff of player r in $PC^{(k+1)}(v)$. Also, the payoff vector $\mathbf{y}^{(k+1),r}$ is said to be the "best" payoff vector *in favor of* player r, or player r's "best" payoff vector in $PC^{(k+1)}(v)$.

Since each player r takes his "best" payoff only once w.r.t. $\mathbf{y}^{(k+1),r}$ and takes his "worst" payoff w_r^{k+1} in the rest of $n-1$ times w.r.t. $\mathbf{y}^{(k+1),i}, i\in N-\{r\}$, in $PC^{(k+1)}(v)$ it is easy to see the following.

(4.10)

$$y_r^{(k+1),1} = y_r^{(k+1),2} = \cdots = y_r^{(k+1),(r-1)} = y_r^{(k+1),(r+1)} = \cdots = y_r^{(k+1),n} = w_r^{k+1}.$$

From the setting of $\mathbf{y}^{(k+1),r}$, by (4.8), (4.9) and (4.10), we know that the intersection of the n hyperplanes

$$\Big(\bigcap_{\ell\in N-\{r\}} H_\ell^{(k+1)}\Big)\cap H^0$$

is exactly the singleton $\{\mathbf{y}^{(k+1),r}\}$.

Definition 4.7. Given a TU game (N, v) and all its pseudo cores, let all the players in N take turns to be the one who get his "best" payoff in $PC^{(k+1)}(v)$, then we have constructed n points $\mathbf{y}^{(k+1),1}, \mathbf{y}^{(k+1),2}, \ldots, \mathbf{y}^{(k+1),n}$, in favor of player $1, 2, \ldots, n$, respectively, in $PC^{(k+1)}(v) \subseteq H^0$. It is certainly a middle-way method for the players in N to take turns to get their "best" payoff, then accept the average of the n "best" payoff vectors, denoted by $\overline{\mathbf{y}^{(k+1),\cdot}}$ where

$$\overline{\mathbf{y}^{(k+1),\cdot}} = \frac{\sum\limits_{r\in N}\mathbf{y}^{(k+1),r}}{n}.$$

We call $\overline{\mathbf{y}^{(k+1),\cdot}}$ the $(k+1)^{th}$-*middle-way payoff vector*(allocation).

Again, we have the following definition.

Definition 4.8. The *middle-way solution* of an n-person game is the average of all the $(k+1)^{th}$-middle-way payoff vector, $\overline{\mathbf{y}^{(k+1),\cdot}}$, $k = 0, 1, \dots, n-2$. We denote the middle-way solution by $\mu(v) = (\mu_1(v), \cdot, \mu_n(v)) = \overline{\mathbf{y}}$ where

$$\overline{\mathbf{y}} = \frac{\sum_{k=0}^{n-2} \overline{\mathbf{y}^{(k+1),\cdot}}}{n-1}.$$

Later, we will prove that the middle-way solution $\mu(v)$ is exactly the Shapley value.

Now, the following example justifies the value of this paper.

Example 4.9. There are there children, 1:Alice, 2:Bob and 3:Cathy. Alice has an ordinary jumping rope, she can play jumping rope by herself and get $v(\{1\}) = 10$ unit of happiness. Bob has an ordinary jumping rope too and $v(\{2\}) = 10$. Cathy has a valuable kid's rocking chair, she can play with the chair by herself and get $v(\{3\}) = 15$. If Alice and Bob play together, they can use the two jumping ropes to make a simple swing and have more fun, say $v(\{1, 2\}) = 27$. Only one jumping rope cannot do much with the kid's rocking chair, hence we let $v(\{1, 3\}) = v(\{2, 3\}) = 26$. Suppose the three children play together, they may use the two jumping ropes and the chair to make a comfortable swing with chair and have much more fun, say $v(\{1, 2, 3\}) = 41$. But, the swing will be broken if two or more children ride on it, therefore, the children must take turns to ride on the swing.

Now, we use our mathematical model to explain the game. The player set is $N = \{1, 2, 3\}$ and the game (N, v) is CRIA.

Case 1: The upper index 1 of w_i^1 denotes the case that each child plays alone. We have $w_1^1 = v(\{1\}) = 10$, $w_2^1 = v(\{2\}) = 10$ and $w_3^1 = v(\{3\}) = 15$. Intuitively, when a child plays alone, at worst the child can get $w_i^1 = v(\{i\})$. Hence, $\mathbf{y}^{1,1} = (16, 10, 15)$ means that when Alice takes her turn to ride on the swing and enjoys 16 unit of happiness, Bob and Cathy are happily waiting for their turns, Bob is as happy as playing alone, Cathy is also as happy as playing alone. By similar calculations, $\mathbf{y}^{1,2} = (10, 16, 15)$ and $\mathbf{y}^{1,3} = (10, 10, 21)$, hence $\overline{\mathbf{y}^{1,\cdot}} = (\frac{36}{3}, \frac{36}{3}, \frac{51}{3}) = (12, 12, 17)$.

Case 2: The upper index 2 of w_i^2 denotes the case that two children play together. Now, $w_1^2 = 12 > 10 = v(\{1\})$ and $w_2^2 = 12 > 10 = v(\{2\})$ mean that a child who has an ordinary jumping rope is better off whenever there is another child playing with him or her. The very interesting case which $w_3^2 = 11 < 15 = v(\{3\})$ means that Cathy is a little reluctant to share her valuable kid's rocking chair with others. Therefore, $\mathbf{y}^{2,1} = (18, 12, 11) \notin I(v)$ means that when Alice takes her turn to ride on the swing and enjoys 18 unit of happiness, Bob and Cathy are happily waiting for their turns, Bob is happier than playing alone, but Cathy is a little reluctant to share her valuable rocking chair with others. By similar calculation, $\mathbf{y}^{2,2} = (12, 18, 11)$ and $\mathbf{y}^{2,3} = (12, 12, 17)$, hence $\overline{\mathbf{y}^{2,\cdot}} = (\frac{42}{3}, \frac{42}{3}, \frac{39}{3}) = (14, 14, 13)$.

Finally, suppose the game is repeated 3 times, and each child has his or her turn to ride on the comfortable swing, the middle-way solution is $(13, 13, 15)$ which is the Shapley value of (N, v). It is easy to see that the core of (N, v) is empty.

In game (N, v), Cathy has a very subtle situation, her payoff w.r.t the middle-way solution $(13, 13, 15)$ is 15. If she plays alone, she has at least 15 unit of happiness.

If she plays with Alice and Bob, she has to share her valuable kid's rocking chair with others taking turns to ride on the comfortable swing and has the risks of $\mathbf{y}^{2,1} = (18, 12, 11)$, $\mathbf{y}^{2,2} = (12, 18, 11)$ and $\overline{\mathbf{y}^{2,\cdot}} = (14, 14, 13) \notin I(v)$. Nevertheless, on the other hand, if she plays with Alice and Bob, she has opportunity to get $\mathbf{y}^{2,3} = (12, 12, 17)$ or even $\mathbf{y}^{1,3} = (10, 10, 21)$ and $\overline{\mathbf{y}^{1,\cdot}} = (12, 12, 17)$. Suppose the game is repeated 3 times, the children take turns to ride on the comfortable swing, it is still rational for Cathy to accept the risks of $\mathbf{y}^{2,1}$, $\mathbf{y}^{2,2}$ and $\overline{\mathbf{y}^{2,\cdot}}$ and then has chance to get $\mathbf{y}^{2,3}$ or even $\mathbf{y}^{1,3}$ and $\overline{\mathbf{y}^{1,\cdot}}$. It is still worthwhile studying a payoff vector which is not in $I(v)$.

Before we give the main results, we need the following lemma.

Lemma 4.10. *Given a game* (N, v) *and an* $i \in N = \{1, 2, 3, \ldots, n\}$ *then*

$$\begin{aligned}&\sum_{\ell \in N-\{i\}} \Bigg(\sum_{\substack{|T|=k \\ T \subset N-\{\ell\}}} v(T \cup \{\ell\}) \Bigg) \\ =&k \cdot \sum_{\substack{|T|=k \\ T \subset N-\{i\}}} v(T \cup \{i\}) + (k+1) \cdot \sum_{\substack{|T|=k+1 \\ T \subset N-\{i\}}} v(T). \end{aligned} \tag{4.11}$$

Proof. We have the following combinatorial calculations.

$$\begin{aligned}&\sum_{\ell \in N-\{i\}} \Bigg(\sum_{\substack{|T|=k \\ T \subset N-\{\ell\}}} v(T \cup \{\ell\}) \Bigg) \\ =&\Bigg\{ \sum_{\ell \in N-\{i\}} \Bigg(\sum_{\substack{|T|=k \\ T \subset N-\{\ell\}}} v(T \cup \{\ell\}) \Bigg) + \sum_{\substack{|T|=k \\ T \subset N-\{i\}}} v(T \cup \{i\}) \Bigg\} - \sum_{\substack{|T|=k \\ T \subset N-\{i\}}} v(T \cup \{i\}) \\ =&\sum_{\ell \in N} \Bigg(\sum_{\substack{|T|=k \\ T \subset N-\{\ell\}}} v(T \cup \{\ell\}) \Bigg) - \sum_{\substack{|T|=k \\ T \subset N-\{i\}}} v(T \cup \{i\}) \end{aligned} \tag{4.12}$$

Now, observe the double summation

$$\sum_{\ell \in N} \Bigg(\sum_{\substack{|T|=k \\ T \subset N-\{\ell\}}} v(T \cup \{\ell\}) \Bigg). \tag{4.13}$$

Given any coalition with $k+1$ players, say $\{j_1, j_2, \ldots, j_{k+1}\}$, we can write the same coalition in turns of $k+1$ different forms of $T \cup \{\ell\}$ by choosing $\ell = j_1, j_2, \ldots, j_{k+1}$ respectively. In other words, each $v(\{j_1, j_2, \ldots, j_{k+1}\})$ of the same $k+1$ players is counted $k+1$ times in the double summation (4.13).

Therefore,

$$\sum_{\ell \in N} \Bigg(\sum_{\substack{|T|=k \\ T \subset N-\{\ell\}}} v(T \cup \{\ell\}) \Bigg) = (k+1) \cdot \sum_{\substack{|T|=k+1 \\ T \subset N}} v(T)$$

$$= (k+1) \cdot \sum_{\substack{|T|=k \\ T \subset N-\{i\}}} v(T \cup \{i\}) + (k+1) \cdot \sum_{\substack{|T|=k+1 \\ T \subset N-\{i\}}} v(T). \tag{4.14}$$

Combine (4.12), (4.13) and (4.14), we get (4.11) and complete the proof. □

The following theorem modifies the pseudo-cores and is one of our main results.

Theorem 4.11. *Let (N, v) be an n-person game and $\mathbf{y}^{(k+1),1}, \ldots, \mathbf{y}^{(k+1),n}, 0 \leq k \leq n-2$ be "best" payoff vectors in favor of players $1, 2, \ldots, n$ respectively. Then*

(i) *$\mathbf{y}^{(k+1),1}, \ldots, \mathbf{y}^{(k+1),n}$ are either distinct or identical.*

(ii) *$PC^{(k+1)}(v)$ is an empty set or a singleton or an $(n-1)$ simplex of n vertices $\mathbf{y}^{(k+1),1}, \ldots, \mathbf{y}^{(k+1),n}$ in the $(n-1)$-dimensional hyperplane H^0.*

In fact

(ii.a) *$PC^{(k+1)}(v) \neq \emptyset$ if and only if*

$$\frac{1}{n} v(N) \geq \frac{1}{k+1} \frac{1}{\binom{n}{k+1}} \sum_{\substack{|T|=k+1 \\ T \subset N}} v(T) = \frac{1}{k+1} \cdot ave(k+1, v).$$

Or, equivalently,

$$PC^{(k+1)}(v) = \emptyset \textit{ if and only if } \frac{1}{n} v(N) < \frac{1}{k+1} \cdot \frac{1}{\binom{n}{k+1}} \sum_{\substack{|T|=k+1 \\ T \subset N}} v(T).$$

(ii.b) *The following equality holds*

$$\frac{1}{n} v(N) = \frac{1}{k+1} \frac{1}{\binom{n}{k+1}} \sum_{\substack{|T|=k+1 \\ T \subset N}} v(T) \textit{ if and only if } PC^{(k+1)}(v) = \{\mathbf{y}^{(k+1),j}\},$$

where $\mathbf{y}^{(k+1),j} = \mathbf{y}^{(k+1),\ell}$, for all $\ell \in N$, moreover, the i^{th} coordinate of $\mathbf{y}^{(k+1),j}$ is of the form: $y_i^{(k+1),j} = w_i^{k+1}$, for each $i \in N$. (In this case, $w_1^{k+1} + w_2^{k+1} + \cdots + w_n^{k+1} = v(N)$.)

(ii.c) *The following inequality holds*

$$\frac{1}{n} v(N) > \frac{1}{k+1} \cdot \frac{1}{\binom{n}{k+1}} \sum_{\substack{|T|=k+1 \\ T \subset N}} v(T)$$

if and only if $PC^{(k+1)}(v)$ is an $(n-1)$ simplex of n vertices $\mathbf{y}^{(k+1),1}, \ldots, \mathbf{y}^{(k+1),n}$ in the $(n-1)$-dimensional hyperplane H^0.

Proof. First observe for each fixed i

(4.15)

$$\mathbf{y}^{(k+1),i} \in PC^{(k+1)}(v) \Longleftrightarrow$$

$$y_i^{(k+1),i} \geq w_i^{k+1}, \text{ where } w_i^{k+1} = \frac{1}{\binom{n-2}{k}} \sum_{\substack{|T|=k \\ T\subset N-\{i\}}} v(T\cup\{i\}) - \frac{k}{n-1-k} v(N) \iff \tag{4.16}$$

$$v(N) + (n-1)\cdot\frac{k}{n-1-k}v(N) - \frac{1}{\binom{n-2}{k}} \sum_{\ell\in N-\{i\}} \Bigg(\sum_{\substack{|T|=k \\ T\subset N-\{\ell\}}} v(T\cup\{\ell\}) \Bigg) \tag{4.17}$$

$$\geq -\frac{k}{n-1-k}v(N) + \frac{1}{\binom{n-2}{k}} \sum_{\substack{|T|=k \\ T\subset N-\{i\}}} v(T\cup\{i\}) \qquad \text{(by (4.9))} \iff$$

$$(n-1)\cdot\frac{k+1}{n-1-k}v(N) \tag{4.18}$$

$$\geq \frac{1}{\binom{n-2}{k}} \left[\sum_{\substack{|T|=k \\ T\subset N-\{i\}}} v(T\cup\{i\}) + \sum_{\ell\subset N-\{i\}} \Bigg(\sum_{\substack{|T|=k \\ T\subset N-\{\ell\}}} v(T\cup\{\ell\}) \Bigg) \right]$$

$$= \frac{1}{\binom{n-2}{k}} \left[\sum_{\substack{|T|=k \\ T\subset N-\{i\}}} v(T\cup\{i\}) + \qquad \text{(by Lemma 4.10)} \right.$$

$$\left. \Bigg(k\cdot \sum_{\substack{|T|=k \\ T\subset N-\{i\}}} v(T\cup\{i\}) + (k+1)\cdot \sum_{\substack{|T|=k+1\leq n \\ T\subset N-\{i\}}} v(T) \Bigg) \right]$$

$$= \frac{(k+1)}{\binom{n-2}{k}} \left[\sum_{\substack{|T|=k \\ T\subset N-\{i\}}} v(T\cup\{i\}) + \sum_{\substack{|T|=k+1 \\ T\subset N-\{i\}}} v(T) \right]$$

$$= \frac{(k+1)}{\binom{n-2}{k}} \sum_{|T|=k+1} v(T)$$

$$\iff \frac{1}{n}v(N) \geq \frac{1}{n}\cdot\frac{(n-1-k)}{n-1}\cdot\frac{k!(n-2-k)!}{(n-2)!}\cdot\sum_{|T|=k+1} v(T)$$

$$= \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}} \sum_{|T|=k+1} v(T) \tag{4.19}$$

Since i is arbitrary we see that

$$\mathbf{y}^{(k+1),i} \in PC^{(k+1)}(v) \text{ for some } i \iff \frac{1}{n}v(N) \geq \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}} \sum_{|T|=k+1, T\subset N} v(T)$$

$$\iff \mathbf{y}^{(k+1),j} \in PC^{(k+1)}(v), \quad \forall j \in N. \tag{4.20}$$

Especially, by (4.20) we prove the "$\Leftarrow$" statement of (ii.a). On the other hand, we also get the following:

$$\begin{aligned}
&\frac{1}{n}v(N) < \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1} v(T)\\
&\Longleftrightarrow \mathbf{y}^{(k+1),i} \notin PC^{(k+1)}(v),\ \text{for some } i\\
&\Longleftrightarrow \mathbf{y}^{(k+1),j} \notin PC^{(k+1)}(v), \forall j \in N\\
&\Longleftrightarrow y_i^{(k+1),i} < w_i^{k+1},\ \text{for some } i \Longleftrightarrow\ y_j^{(k+1),j} < w_j^{k+1}, \quad \forall j\in N
\end{aligned} \tag{4.21}$$

Next, if we claim that $PC^{(k+1)}(v) \neq \emptyset$, say $(x_1, \ldots, x_n) \in PC^{(k+1)}(v)$, implies $\frac{1}{n}v(N) \geq \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1} v(T)$, then the proof of (ii.a) is complete.

Suppose the conclusion fails, that is, $\frac{1}{n}v(N) < \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1} v(T)$, which by (4.21), implies $y_j^{(k+1),j} < w_j^{k+1} \quad \forall j \in N$. We shall get a contradiction. Now by the definitions of $PC^{(k+1)}(v)$ and $\mathbf{y}^{(k+1),i}$ we have

$$\begin{aligned}
&(x_1, x_2, \ldots, x_n) \in PC^{(k+1)}(v)\\
\Longrightarrow\ & w_j^{k+1} \leq x_j = v(N) - (x_1 + x_2 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_n)\\
&\leq v(N) - (w_1^{k+1} + w_2^{k+1} + \cdots + w_{j-1}^{k+1} + w_{j+1}^{k+1} + \cdots + w_n^{k+1})\\
&= v(N) - (y_j^{(k+1),1} + y_j^{(k+1),2} \cdots + y_j^{(k+1),j-1} + y_j^{(k+1),j+1} + \cdots + y_j^{(k+1),n})\\
&= y_j^{(k+1),j}, \qquad \forall j \in N.
\end{aligned} \tag{4.22}$$

Therefore, $w_i^{k+1} \leq y_i^{(k+1),i} < w_i^{k+1}$, a contradiction occurs. We have proved our claim.

Next observe

$$\begin{aligned}
&y_i^{(k+1),i} = w_i^{k+1} (\text{ notice that } y_i^{k+1,j} = w_i^{k+1},\ \text{for all } j \neq i)\\
\Longleftrightarrow\ & v(N) + (n-1)\cdot\frac{k}{n-1-k}v(N) - \frac{1}{\binom{n-2}{k}}\sum_{\ell\in N-\{i\}}\Bigg(\sum_{\substack{|T|=k\\ T\subset N-\{\ell\}}} v(T\cup\{\ell\})\Bigg)\\
&= -\frac{k}{n-1-k}v(N) + \frac{1}{\binom{n-2}{k}}\sum_{\substack{|T|=k\\ T\subset N-\{i\}}} v(T\cup\{i\}).
\end{aligned} \tag{4.23}$$

Equation (4.23) is exactly the same as (4.16) if the notation "$\geq$" is replaced by the notation "$=$" in inequality (4.16). Similar computation from (4.16) to (4.19) with all those "$\geq$" being replaced by "$=$", we have for each fixed i,

$$y_i^{(k+1),i} = w_i^{k+1}$$

$$\iff \frac{1}{n}v(N) = \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{\substack{|T|=k+1\\T\subset N}} v(T). \tag{4.24}$$

Notice that by definition $y_i^{(k+1),j} = w_i^{k+1}$, for each $j \neq i$, therefore in the case of (4.24), since i is arbitrary chosen, we have for each fixed j,

$$\begin{aligned}\mathbf{y}^{(k+1),j} &= (y_1^{(k+1),j},\ldots,y_{j-1}^{(k+1),j},y_j^{(k+1),j},y_{j+1}^{(k+1),j},\ldots,y_n^{(k+1),j})\\ &= (w_1^{k+1},\ldots,w_{j-1}^{k+1},w_j^{k+1},w_{j+1}^{k+1},\ldots,w_n^{k+1}), (\text{ choose } i=j \text{ in } (4.24))\end{aligned}$$

and hence

$$\begin{aligned}&\mathbf{y}^{(k+1),j} = (w_1^{k+1},\ldots,w_{j-1}^{k+1},w_j^{k+1},w_{j+1}^{k+1},\ldots,w_n^{k+1}), \text{ for some } j\\ \iff &\frac{1}{n}v(N) = \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{\substack{|T|=k+1\\T\subset N}} v(T)\\ \iff &\mathbf{y}^{(k+1),1} = \mathbf{y}^{(k+1),2} = \ldots = \mathbf{y}^{(k+1),n} = (w_1^{k+1},w_2^{k+1},\ldots,w_n^{k+1}).\end{aligned} \tag{4.25}$$

On the other hand, the statement in (4.24) is equivalent to

$$\begin{aligned}&y_i^{(k+1),i} \neq w_i^{k+1} \text{ (notice that by definition } y_i^{(k+1),j} = w_i^{k+1} \text{ for all } j\neq i)\\ \iff &\frac{1}{n}v(N) \neq \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1} v(T).\end{aligned} \tag{4.26}$$

But if for each fixed i, the inequalities $y_i^{(k+1),i} \neq y_i^{(k+1),j}$ for all $j \neq i$, $i = 1,\ldots,n$, hold then it is equivalent to say $\mathbf{y}^{(k+1),i} \neq \mathbf{y}^{(k+1),j}$ for all $j \neq i$, where $i = 1,2\ldots,n$, which means that all points $\mathbf{y}^{(k+1),i}, i = 1,\ldots,n$ are distinct. According to the above argument as well as (4.26), we have proved the following:

$$\begin{aligned}&\text{All points } \mathbf{y}^{(k+1),i}, i=1,\ldots,n, \text{ are distinct}\\ \iff &\frac{1}{n}v(N) \neq \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1} v(T)\end{aligned} \tag{4.27}$$

By (4.25) and (4.27), we complete the proof of (i). Next suppose $\frac{1}{n}v(N) = \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1} v(T)$, then by the fact "(4.15)$\iff$ (4.19)" and (4.25), we have $PC^{(k+1)}(v) \supseteq \{\mathbf{y}^{(k+1),*}\}$, where $\mathbf{y}^{(k+1),*} = \mathbf{y}^{(k+1),1} = \cdots = \mathbf{y}^{(k+1),j} = \cdots = \mathbf{y}^{(k+1),n} = (w_1^{k+1},w_2^{k+1},\ldots,w_n^{k+1})$. We shall claim $PC^{(k+1)}(v) = \{\mathbf{y}^{(k+1),*}\}$. Suppose $(x_1,\ldots,x_n) \in PC^{(k+1)}(v) - \{\mathbf{y}^{(k+1),*}\}$, then by (4.22) and (4.24), we get $(x_1,\ldots,x_n) = (w_1^{k+1},\ldots,w_n^{k+1}) = \{\mathbf{y}^{(k+1),*}\}$, this contradicts to $(x_1,\ldots,x_n) \in PC^{(k+1)}(v) - \{\mathbf{y}^{(k+1),*}\}$

Finally, by (4.20) and (4.27) we have

$$\frac{1}{n}v(N) > \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1} v(T) \tag{4.28}$$

$\iff PC^{(k+1)}(v)$ contains $\mathbf{y}^{(k+1),1,}\ldots,\mathbf{y}^{(k+1),n},$ which are distinct points.

Therefore if we claim that

$$\frac{1}{n}v(N) > \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1}v(T)$$

implies that $PC^{(k+1)}(v)$ is an $(n-1)$ simplex of vertices $\mathbf{y}^{(k+1),1},\ldots,\mathbf{y}^{(k+1),n}$ in H^0. Then we can complete the proof of (ii.c) and hence complete the proof of theorem 4.11.

So let us suppose

$$\frac{1}{n}v(N) > \frac{1}{k+1}\cdot\frac{1}{\binom{n}{k+1}}\sum_{|T|=k+1}v(T),$$

then by (4.28) we have $PC^{(k+1)}(v)\neq\emptyset$.

Let $(x_1,x_2,\ldots,x_n)\in PC^{(k+1)}(v)$, then by (4.22), there exist $t_1,\ldots,t_n$ with $0\le t_i\le 1$ such that $x_i=t_iy_i^{(k+1),i}+(1-t_i)w_i^{k+1}, i=1,\ldots,n$.

Now,

$$\begin{aligned}
v(N) &= \sum_{i=1}^n x_i = \sum_{i=1}^n t_i\cdot y_i^{(k+1),i}+\sum_{i=1}^n w_i^{k+1}-\sum_{i=1}^n t_i\cdot w_i^{k+1}\\
&= \sum_{i=1}^n t_i(v(N)-\sum_{\ell\in N-\{i\}}y_\ell^{(k+1),i})\\
&\qquad +\sum_{\ell=1}^n w_\ell^{k+1}-\sum_{i=1}^n t_iw_i^{k+1} \qquad\qquad (\text{because } \mathbf{y}^{(k+1),i)}\in H_0)\\
&\quad = \sum_{i=1}^n t_i(v(N)-\sum_{\ell\in N-\{i\}}w_\ell^{k+1})+\sum_{\ell=1}^n w_\ell^{k+1}+\sum_{i=1}^n t_iw_i^{k+1}\\
&\quad = \sum_{i=1}^n t_iv(N)-\sum_{i=1}^n(t_i\sum_{\ell=1}^n w_\ell^{k+1})+\sum_{\ell=1}^n w_\ell^{k+1}\\
&\quad = (\sum_{i=1}^n t_i)v(N)-(\sum_{i=1}^n t_i)\sum_{\ell=1}^n w_\ell^{k+1}+\sum_{\ell=1}^n w_\ell^{k+1}\\
\Longrightarrow (1-\sum_{i=1}^n t_i)v(N) &= (1-\sum_{i=1}^n t_i)\sum_{\ell=1}^n w_\ell^{k+1}.
\end{aligned}$$

If $1-\sum_{i=1}^n t_i\neq 0$ then $v(N)=\sum_{\ell=1}^n w_\ell^{k+1}$. But $v(N)=y_j^{(k+1),j}+\sum_{\ell\in N-\{j\}}w_\ell^{k+1}$ (by definition of "best" payoff vector), $j=1,2,\ldots,n$. Therefore, $w_j^{k+1}=y_j^{(k+1),j}$, $j=1,\ldots,n$. By (4.24) $PC^{(k+1)}(v)$ contains only one point $(w_1^{k+1},\ldots,w_n^{k+1})=\mathbf{y}^{(k+1),j}, j=1,\ldots,n$, this contradicts $\mathbf{y}_j^{(k+1),j}, j=1,\ldots,n$, being distinct points. Hence, $1-\sum_{i=1}^n t_i=0$, that is, $\sum_{i=1}^n t_i=1$. Now for $i=1,\ldots,n$,

$$x_i = t_i y_i^{(k+1),i} + (1-t_i)w_i^{k+1} = t_i y_i^{(k+1),i} + (\sum_{j\neq i} t_j) w_i^{k+1}$$
$$= t_i y_i^{(k+1),i} + \sum_{j\neq i}(t_j w_i^{k+1}) = t_i y_i^{(k+1),i} + \sum_{j\neq i} t_j y_i^{(k+1),j} = \sum_{j=1}^{n} t_j y_i^{(k+1),j},$$

therefore, $(x_1,\ldots,x_n) = \sum_{j=1}^n t_j \mathbf{y}^{(k+1),j}$. This proves that $PC^{(k+1)}(v)$ is the convex hull of n points $\mathbf{y}^{(k+1),1},\ldots,\mathbf{y}^{(k+1),n}$. But $\mathbf{y}^{(k+1),1},\ldots,\mathbf{y}^{(k+1),n}$ are distinct points in H^0 and H^0 is $(n-1)$-dimensional, we have that $PC^{(k+1)}(v)$ is an $(n-1)$ simplex of vertices $\mathbf{y}^{(k+1),1},\ldots,\mathbf{y}^{(k+1),n}$ in H^0. This proves the claim. □

Remark 4.12. Intuitively, $PC^k(v)$ is empty whenever $y_i^{k,i} < w_i^k$, i.e. the "best" payoff of player i is less than the "worst" payoff of player i.

By Theorem 4.11 we have the following proposition immediately.

Proposition 4.13. *Suppose that (N,v) is an n-person convex game, then it is a CRIA game.*

5. A new interpretation of the Shapley value

While each player is willing to participate in a coalition with any other k players and share the worth of the coalition according to a payoff vector in $PC^{(k+1)}(v)$. Intuitively, it is a middle-way method for the players if the players take turns to get the "best" payoff and find the average of all the payoff vectors of all the turns. In geometric, the average of all the payoff vectors of all the turns(either distinct or identical, see Theorem 4.11) is the center of mass of $PC^{(k+1)}(v)$(either a $(n-1)$-simplex or a singleton) provided that $PC^{(k+1)}(v)$ is nonempty. Considering the vertices $\mathbf{y}^{(k+1),1},\ldots,\mathbf{y}^{(k+1),n}$(either distinct or identical) as vectors, the centroid is

$$\overline{\mathbf{y}^{(k+1),\cdot}} = \frac{\mathbf{y}^{(k+1),1}+\cdots+\mathbf{y}^{(k+1),n}}{n}$$

Now, for all the players in N there are $n-1$ possible ways to share $v(N)$, say, $\overline{\mathbf{y}^{1,\cdot}}, \overline{\mathbf{y}^{2,\cdot}},\ldots,\overline{\mathbf{y}^{(n-1),\cdot}}$. Again, it is a middle-way method to choose the average of the above $n-1$ payoff vectors which is the geometric center of the $n-1$ points in the hyperplane H^0. Surprisingly, it is the Shapely value.

Theorem 5.1. *Given an n-person CRIA cooperative TU game (N,v), the Shapely value of v is the geometric center of the $n-1$ points $\overline{\mathbf{y}^{1,\cdot}}, \overline{\mathbf{y}^{2,\cdot}},\ldots,\overline{\mathbf{y}^{(n-1),\cdot}}$ in $\mathbb{R}^n$, where each $\overline{\mathbf{y}^{k,\cdot}}$ is the center of mass of $PC^k(v)$, the k^{th} pseudo-core of v which is either an $(n-1)$-simplex or a singleton, for $k=1,2,\ldots,n-1$. In other words, the Shapely value is the geometric centroid of the $n-1$ mass centers of pseudo-cores $PC^k(v)$, $k=1,2,\ldots,n-1$.*

Proof. Let

$$(\mu_1,\mu_2,\ldots,\mu_n) = \frac{\overline{\mathbf{y}^{1,\cdot}}+\overline{\mathbf{y}^{2,\cdot}}+\cdots+\overline{\mathbf{y}^{(n-1),\cdot}}}{n-1},$$

and write $\overline{\mathbf{y}^{k,\cdot}} = (\overline{y_1^{k,\cdot}},\ldots,\overline{y_i^{k,\cdot}},\ldots,\overline{y_n^{k,\cdot}})$ for $k=1,2,\ldots n-1$.

Then

$$\mu_i = \frac{\overline{y_i^{1,\cdot}} + \overline{y_i^{2,\cdot}} + \overline{y_i^{3,\cdot}} + \cdots + \overline{y_i^{(n-1),\cdot}}}{n-1},$$

for $i = 1, \ldots, n$. We are going to show that

$$\mu_i = \sum_{\substack{S \subset N \\ i \notin S}} \frac{|S|!(n-|S|-1)!}{n!}(v(S \cup \{i\}) - v(S)).$$

Now since $\overline{\mathbf{y}^{(k+1),\cdot}}$ is the center of mass of $PC^k(v)$ with vertices $\mathbf{y}^{(k+1),1}, \mathbf{y}^{(k+1),2}, \ldots,$ $\mathbf{y}^{(k+1),n}$(either distinct or identical), it follows from the definition of "best payoff" vectors, we have

$$\overline{y_i^{(k+1),\cdot}} = \frac{y_i^{(k+1),1} + y_i^{(k+1),2} + \cdots + y_i^{(k+1),n}}{n}$$

$$= \frac{(y_i^{(k+1),1} + \cdots + y_i^{(k+1),i-1} + y_i^{(k+1),i-1} + \cdots + y_i^{(k+1),n}) + y_i^{(k+1),i}}{n}$$

$$= \frac{1}{n}\Bigg\{(n-1)\cdot\Bigg[-\frac{k}{n-1-k}v(N) + \frac{1}{\binom{n-2}{k}}\sum_{\substack{|T|=k \\ T \subset N-\{i\}}} v(T \cup \{i\})\Bigg] \text{ (see (4.5))}$$

(5.1)

$$+ v(N) + (n-1)\cdot\frac{k}{n-1-k}v(N) - \frac{1}{\binom{n-2}{k}}\sum_{\ell \in N-\{i\}}\Bigg(\sum_{\substack{|T|=k \\ T \subset N-\{\ell\}}} v(T \cup \{\ell\})\Bigg)\Bigg\}.$$

It follows from (5.1) and lemma 4.10 that

$$\overline{y_i^{(k+1),\cdot}} = \frac{1}{n}\Bigg\{\frac{n-1}{\binom{n-2}{k}}\sum_{\substack{|T|=k \\ T \subset N-\{i\}}} v(T \cup \{i\}) + v(N)$$

$$- \frac{1}{\binom{n-2}{k}}\Bigg[k \cdot \sum_{\substack{|T|=k \\ T \subset N-\{i\}}} v(T \cup \{i\}) + (k+1)\cdot\sum_{\substack{|T|=k+1 \\ T \subset N-\{i\}}} v(T)\Bigg]\Bigg\}$$

$$= \frac{1}{n}\left\{v(N) + \sum_{\substack{|T|=k \\ T \subset N-\{i\}}}\left[\frac{(n-1)}{\binom{n-2}{k}} - \frac{k}{\binom{n-2}{k}}\right]v(T \cup \{i\})\right.$$

$$\left. - \frac{(k+1)}{\binom{n-2}{k}}\sum_{\substack{|T|=k+1 \\ T \subset N-\{i\}}} v(T)\right\}$$

$$= \frac{1}{n}v(N) + \frac{(n-1)}{n}$$

$$\cdot \left[\frac{1}{\binom{n-1}{k}} \sum_{\substack{|T|=k \\ T\subset N-\{i\}}} v(T\cup\{i\}) - \frac{1}{\binom{n-1}{k+1}} \sum_{\substack{|T|=k+1 \\ T\subset N-\{i\}}} v(T) \right]. \tag{5.2}$$

Let

$$a_{k+1} = \frac{1}{\binom{n-1}{k}} \sum_{\substack{|T|=k \\ T\subset N-\{i\}}} v(T\cup\{i\}), k = 0,\dots,n-2, \tag{5.3}$$

and

$$b_{k+1} = \frac{1}{\binom{n-1}{k+1}} \sum_{\substack{|T|=k+1 \\ T\subset N-\{i\}}} v(T), k = 0,\dots,n-2, \tag{5.4}$$

then

$$\overline{y_i^{(k+1),\cdot}} = \frac{1}{n}v(N) + \frac{n-1}{n}(a_{k+1} - b_{k+1}). \tag{5.5}$$

Next, observe

$$\begin{aligned} a_{k+1} - b_k &= \frac{1}{\binom{n-1}{k}} \sum_{\substack{|T|=k \\ T\subset N-\{i\}}} [v(T\cup\{i\}) - v(T)] \\ &= \frac{1}{\frac{(n-1)!}{k!(n-1-k)!}} \sum_{\substack{|S|=k, S\subset N \\ i\notin S}} [v(S\cup\{i\}) - v(S)] \\ &= \sum_{\substack{|S|=k, S\subset N \\ i\notin S}} \frac{|S|!(n-|S|-1)!}{(n-1)!}[v(S\cup\{i\}) - v(S)], \end{aligned} \tag{5.6}$$

for $k = 1,\dots,n-2$,

$$\begin{aligned} a_1 &= \frac{1}{\binom{n-1}{0}} \cdot v(\{i\}) = \frac{1}{\frac{(n-1)!}{0!(n-1)!}}[v(\{i\}) - v(\emptyset)] \\ &= \sum_{\substack{|S|=0, S\subset N \\ i\notin S}} \frac{|S|!(n-|S|-1)!}{(n-1)!}[v(S\cup\{i\}) - v(S)] \text{ (in fact here } S=\emptyset) \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} -b_{n-1} + v(N) &= -\frac{1}{\binom{n-1}{n-1}} \sum_{\substack{|T|=n-1 \\ T\subset N-\{i\}}} v(T) + v(N) \\ &= \frac{1}{\frac{(n-1)!}{(n-1)!0!}}[v(N) - v(N-\{i\})] \end{aligned}$$

(5.8)
$$= \sum_{\substack{|S|=n-1, S\subset N\\ i\notin S}} \frac{|S|!(n-|S|-1)!}{(n-1)!}[v(S\cup\{i\})-v(S)](\text{ in fact here } S=N-\{i\}).$$

It follows from (5.2), (5.3), (5.4), (5.6), (5.7) and (5.8)

$$\begin{aligned}
\mu_i =& \frac{\overline{y_i^{1,\cdot}}+\cdots+\overline{y_i^{k,\cdot}}+\overline{y_i^{(k+1),\cdot}}+\cdots+\overline{y_i^{(n-1),\cdot}}}{n-1} \\
=& \frac{1}{n-1}\Big(\big[\frac{v(N)}{n}+\frac{n-1}{n}(a_1-b_1)\big]+\cdots+\big[\frac{v(N)}{n}+\frac{n-1}{n}(a_k-b_k)\big] \\
&+\big[\frac{v(N)}{n}+\frac{n-1}{n}(a_{k+1}-b_{k+1})\big]+\cdots+\big[\frac{v(N)}{n}+\frac{n-1}{n}(a_{n-1}-b_{n-1})\big]\Big) \\
&= \frac{1}{n-1}\frac{n-1}{n}\big[a_1+(-b_1+a_2)+\cdots+(-b_k+a_{k+1}) \\
&\qquad +\cdots+(-b_{n-2}+a_{n-1})+(-b_{n-1}+v(N))\big] \\
&= \sum_{\substack{S\subset N\\ i\notin S}} \frac{|S|!(n-|S|-1)!}{n!}[v(S\cup\{i\})-v(S)]. \qquad (5.9)
\end{aligned}$$

□

Theorem 5.1 shows that the geometric centroid of the $n-1$ mass centers of the pseudo-cores of the game (N, v) is the Shapley value. Further, even when $PC^{(k+1)}(v)=\emptyset$, the points $\mathbf{y}^{(k+1),r}$, $r=1,\ldots,n$ are still in the hyperplane $x_1+x_2+\cdots+x_n=v(N)$, then by the proof of Theorem 5.1, we have Theorem 5.2.

Theorem 5.2. *Given an n-person cooperative game (N, v), the Shapely value is the geometric center of the $n-1$ points $\overline{\mathbf{y}^{1,\cdot}}, \overline{\mathbf{y}^{2,\cdot}}, \ldots, \overline{\mathbf{y}^{(n-1),\cdot}}$ in $\mathbb{R}^n$, where $\overline{\mathbf{y}^{k,\cdot}}$ is the geometric center of the n vertices $\mathbf{y}^{k,1}, \mathbf{y}^{k,2}, \ldots, \mathbf{y}^{k,n}$, for $k=1,2,\ldots,n-1$.*

Conclusion Let a cooperative TU game (N, v) be repeated n times, then we may not only propose a new interpretation for the Shapley value but also propose the investigation of a payoff vector which is not even an imputation. In our Example 4.9, Cathy justifies the worth of studying a payoff vector which is not an imputation.

We introduce a new class of CRIA games which are not necessarily proper, also proper games are not necessarily CRIA. We show that a game is CRIA if and only if none of its pseudo cores is empty. In a CRIA game the players have incentive to participate in the grand coalition. It is more acceptable in a CRIA game than in an improper game to assume that the grand coalition is formed.

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