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WEAK CONVERGENCE THEOREM BY A MODIFIED EXTRAGRADIENT METHOD FOR VARIATIONAL INCLUSIONS,VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. In this paper, we investigate the problem of finding common solutions of variational inclusions, variational inequalities and fixed point problems in real Hilbert spaces. Motivated by Nadezhkina and Takahashi's extragradient method [N. Nadezhkina, W. Takahashi, Weak convergence theorem by an extragradient method for nonexpansive mappings and monotone mappings, J. Optim. Theory Appl. 128 (2006) 191-201], we propose and analyze a modified extragradient algorithm for finding common solutions. It is proven that three sequences generated by this algorithm converge weakly to the same common solution under very mild conditions by virtue of the Opial condition of Hilbert spaces, the demiclosedness principle for nonexpansive mappings and the coincidence of solutions of variational inequalities with zeros of maximal monotone operators.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Let P_C be the metric projection of H onto C. A single-valued mapping $A : C \to H$ is called monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C.$$

The variational inequality is to find $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of problem (1.1) is denoted by $\mathrm{VI}(C, A)$. A single-valued mapping A is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It is obvious that any α-inverse strongly monotone mapping A is monotone and Lipschitz continuous. A self-mapping $S : C \to C$ is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

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We denote by $F(S)$ the fixed point set of S; namely, $F(S) = \{x \in C : Sx = x\}$.

A set-valued mapping M with domain $D(M)$ and range $R(M)$ in H is called monotone if its graph $G(M) = \{(x, f) \in H \times H : x \in D(M), f \in Mx\}$ is a monotone set in $H \times H$; namely, M is monotone if and only if

$$(x, f), (y, g) \in G(M) \ \Rightarrow\ \langle x - y, f - g\rangle \geq 0.$$

A monotone set-valued mapping M is called maximal if its graph $G(M)$ is not properly contained in the graph of any other monotone mapping in H.

Let $A : C \to H$ be a single-valued mapping and M be a multivalued mapping with $D(M) = C$. Consider the following variational inclusion: find $\bar{x} \in C$, such that

$$0 \in A\bar{x} + M\bar{x}. \tag{1.2}$$

We denote by Ω the solution set of problem (1.2).

In 1998, Huang [3] studied problem (1.2) in the case where M is maximal monotone and A is strongly monotone and Lipschitz continuous with $D(M) = C = H$. Subsequently, Zeng, Guu and Yao [13] further studied problem (1.2) in the case which is more general than Huang's one [3]. Moreover, the authors [13] obtained the same strong convergence conclusion as in Huang's result [3]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions.

In 2003, for finding an element of $F(S) \cap \mathrm{VI}(C, A)$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is α-inverse strongly monotone, Takahashi and Toyoda [11] introduced the following iterative algorithm:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n Ax_n), \tag{1.3}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C$ chosen arbitrarily, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap \mathrm{VI}(C, A)$ is nonempty, the sequence $\{x_n\}$ generated by (1.3) converges weakly to some $z \in F(S) \cap \mathrm{VI}(C, A)$. On the other hand, for solving the variational inequality in the finite-dimensional Euclidean space R^n under the assumption that a set $C \subset R^n$ is nonempty, closed and convex, a mapping $A : C \to R^n$ is monotone and k-Lipschitz continuous and $\mathrm{VI}(C, A)$ is nonempty, Korpelevich [5] introduced the following so-called extragradient method:

$$\begin{cases} y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases} \tag{1.4}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C$ chosen arbitrarily, and $\lambda \in (0, \frac{1}{k})$. He showed that the sequences $\{x_n\}$ and $\{y_n\}$ generated by (1.4) converge to the same point $z \in \mathrm{VI}(C, A)$.

Recently, motivated by Korpelevich's extragradient method, Nadezhkina and Takahashi [6] introduced an iterative algorithm for finding a common element of the fixed point set of a nonexpansive mapping and the solution set of the variational inequality for a monotone, Lipschitz continuous mapping in a real Hilbert space. They gave a weak convergence theorem for two sequences generated by this algorithm.

Theorem NT (see [6]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone k-Lipschitz continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \end{cases} \tag{1.5}$$

for every $n = 0, 1, 2, ...,$ where $x_0 = x \in C$ chosen arbitrarily, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}, \{y_n\}$ converges weakly to the same point $z \in F(S) \cap \mathrm{VI}(C, A)$, where $z = \|\cdot\|$-$\lim_{n\to\infty} P_{F(S)\cap \mathrm{VI}(C,A)} x_n$.

In this paper, let $A : C \to H$ be an α-inverse strongly monotone mapping, M be a maximal monotone mapping with $D(M) = C$ and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \cap \mathrm{VI}(C, A) \neq \emptyset$. Inspired by Nadezhkina and Takahashi's extragradient algorithm (1.5) we introduce the following modified extragradient algorithm

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ z_n = J_{M,\lambda_n}(t_n - \lambda_n A t_n), \\ x_{n+1} = (1 - \alpha_n - \hat{\alpha}_n) x_n + \alpha_n z_n + \hat{\alpha}_n S z_n, \end{cases} \tag{1.6}$$

for every $n = 0, 1, 2, ...,$ where $J_{M,\lambda_n} = (I + \lambda_n M)^{-1}$, $x_0 = x \in C$ chosen arbitrarily, $\{\lambda_n\} \subset (0, \alpha)$, and $\{\alpha_n\}, \{\hat{\alpha}_n\} \subset (0, 1)$ such that $\alpha_n + \hat{\alpha}_n \leq 1$. It is proven that under very mild conditions three sequences $\{x_n\}, \{y_n\}, \{z_n\}$ generated by (1.6) converge weakly to the same point $z \in F(S) \cap \Omega \cap \mathrm{VI}(C, A)$, where $z = \|\cdot\|$-$\lim_{n\to\infty} P_{F(S)\cap\Omega\cap \mathrm{VI}(C,A)} x_n$. Our result improves and extends Nadezhkina and Takahashi's corresponding one [6], namely, the above Theorem NT.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and C be a nonempty closed convex subset of H. We write $\to$ to indicate that the sequence $\{x_n\}$ converges strongly to x and $\rightharpoonup$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; namely,

$$\omega_w(x_n) := \{x : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall x \in C.$$

P_C is called the metric projection of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C; namely, there holds the following relation

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H.$$

Consequently, P_C is nonexpansive and monotone. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \tag{2.1}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.2}$$

for all $x \in H, y \in C$; see [2] for more details. Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality, this implies that

$$x \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad x = P_C(x - \lambda Ax),\ \forall \lambda > 0. \tag{2.3}$$

It is also known that H satisfies the Opial condition [8]; namely, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.4}$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $M : D(M) \subset H \to 2^H$ is called monotone if for all $x, y \in D(M)$, $f \in Mx$ and $g \in My$ imply

$$\langle f - g, x - y\rangle \geq 0.$$

A set-valued mapping M is called maximal monotone if M is monotone and $(I + \lambda M)D(M) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(M)$ the graph of M. It is known that a monotone mapping M is maximal if and only if, for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$.

Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$; namely,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0,\ \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [9].

Assume that $M : D(M) \subset H \to 2^H$ is a maximal monotone mapping. Then, for $\lambda > 0$, associated with M, the resolvent operator $J_{M,\lambda}$ can be defined as

$$J_{M,\lambda} x = (I + \lambda M)^{-1} x, \quad \forall x \in H.$$

In terms of Huang [3] (see also [13]), there holds the following property for the resolvent operator $J_{M,\lambda} : H \to H$.

Lemma 2.1. *$J_{M,\lambda}$ is single-valued and firmly nonexpansive; namely,*

$$\langle J_{M,\lambda} x - J_{M,\lambda} y, x - y\rangle \geq \|J_{M,\lambda} x - J_{M,\lambda} y\|^2, \quad \forall x \in H.$$

Consequently, $J_{M,\lambda}$ is is nonexpansive and monotone.

Lemma 2.2 (see [10]). *. There holds the relation:*

$$\|\lambda x + \mu y + \nu z\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 - \lambda\mu\|x - y\| - \mu\nu\|y - z\|^2 - \lambda\nu\|x - z\|^2$$

for all $\lambda, \mu, \nu \in [0, 1]$ with $\lambda + \mu + \nu = 1$ and all $x, y, z \in H$.

Lemma 2.3 (see [2]). *Demiclosedness Principle. Assume that S is a nonexpansive self-mapping on a nonempty closed convex subset C of a Hilbert space H. If S has a fixed point, then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity mapping of H.*

Lemma 2.4 (see [3, 13]). *Let M be a maximal monotone mapping with $D(M) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem (1.1) if and only if $u \in C$ satisfies*

$$u = J_{M,\lambda}(u - \lambda Au).$$

Proposition 2.5 (see [13]). *Let M be a maximal monotone mapping with $D(M) = C$ and let $V : C \to H$ be a strongly monotone, continuous and single-valued mapping. Then for each $z \in H$, the equation $z \in Vx + \lambda Mx$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.6. *Let M be a maximal monotone mapping with $D(M) = C$ and $A : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(M + A))C = H$ for each $\lambda > 0$. In this case, $M + A$ is maximal monotone.*

Proof. For each fixed $\lambda > 0$, put $V = I + \lambda A$. Then $V : C \to H$ is a strongly monotone, continuous and single-valued mapping. In terms of Proposition 2.5, we obtain $(V + \lambda M)C = H$. That is, $(I + \lambda(M + A))C = H$. It is clear that $M + A$ is monotone. Therefore, $M + A$ is maximal monotone. □

3. Weak convergence theorem

The following lemma was proved by Takahashi and Toyoda [11].

Lemma 3.1. *Let H be a real Hilbert space and let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leq \|x_n - u\|,$$

for every $n = 0, 1, 2, \ldots.$ Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

Now, we are in a position to state and prove a weak convergence theorem.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse strongly monotone mapping, M be a maximal monotone mapping with $D(M) = C$ and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap \Omega \cap \mathrm{VI}(C, A) \neq \emptyset$. For $x_0 = x \in C$ chosen arbitrarily, let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by*

$$\left\{\begin{array}{l} y_n = P_C(x_n - \lambda_n A x_n), \\ t_n = P_C(x_n - \lambda_n A y_n), \\ z_n = J_{M,\lambda_n}(t_n - \lambda_n A t_n), \\ x_{n+1} = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n z_n + \hat{\alpha}_n S z_n, \end{array}\right. \tag{3.1}$$

for every $n = 0, 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \alpha)$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$, and $\{\hat{\alpha}_n\} \subset [\hat{c}, \hat{d}]$ for some $\hat{c}, \hat{d} \in (0, 1)$ such that $d + \hat{d} < 1$. Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega \cap \mathrm{VI}(C, A)$, where

$$z = \|\cdot\| - \lim_{n\to\infty} P_{F(S)\cap\Omega\cap\mathrm{VI}(C,A)} x_n.$$

Proof. Take a fixed $u \in F(S) \cap \Omega \cap \mathrm{VI}(C, A)$ arbitrarily. From (2.2) we have

$$\begin{aligned}
\|t_n - u\|^2 &\le \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 \\
&\quad + 2\lambda_n (\langle A y_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle A y_n, y_n - t_n\rangle) \\
&\le \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\
&\quad + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle.
\end{aligned}$$

Further, from (2.1) we have

$$\begin{aligned}
\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle &= \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle \\
&\quad + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\
&\le \lambda_n k \|x_n - y_n\| \|t_n - y_n\|,
\end{aligned}$$

where $k = \frac{1}{\alpha}$. So, we obtain

$$\begin{aligned}
\|t_n - u\|^2 &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\
&\quad + 2\lambda_n k \|x_n - y_n\| \|t_n - y_n\| \\
&\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2 \|x_n - y_n\|^2 \\
&\quad + \|y_n - t_n\| \\
&= \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\
&\le \|x_n - u\|^2.
\end{aligned}$$

Hence, utilizing Lemma 2.2 we get

$$\begin{aligned}
&\|x_{n+1}-u\|^2 = \|(1-\alpha_n-\hat{\alpha}_n)(x_n-u)+\alpha_n(z_n-u)+\hat{\alpha}_n(Sz_n-u)\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+\alpha_n\|z_n-u\|^2+\hat{\alpha}_n\|Sz_n-u\|^2 \\
&\ -(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2-\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)\|z_n-u\|^2 \\
&\ -(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2-\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&= (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)\|J_{M,\lambda_n}(t_n-\lambda_nAt_n)-J_{M,\lambda_n}(u-\lambda_nAu)\|^2 \\
&\ -(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2-\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)\|(t_n-\lambda_nAt_n)-(u-\lambda_nAu)\|^2 \\
&\ -(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2-\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)[\|t_n-u\|^2+\lambda_n(\lambda_n-2\alpha)\|At_n-Au\|^2] \\
&\ -(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2-\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)\|t_n-u\|^2 \\
&\ -(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2-\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&\le (1-\alpha_n-\hat{\alpha}_n)\|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)[\|x_n-u\|^2+(\lambda_n^2k^2-1)\|x_n-y_n\|^2] \\
&\ -(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2-\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&= \|x_n-u\|^2+(\alpha_n+\hat{\alpha}_n)(\lambda_n^2k^2-1)\|x_n-y_n\|^2 \\
&\ -(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2-\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&\le \|x_n-u\|^2,
\end{aligned}$$

due to the conditions that $\{\lambda_n\}\subset[a,b]$ for some $a,b\in(0,\alpha)$, $\{\alpha_n\}\subset[c,d]$ for some $c,d\in(0,1)$, and $\{\hat{\alpha}_n\}\subset[\hat{c},\hat{d}]$ for some $\hat{c},\hat{d}\in(0,1)$, such that $d+\hat{d}<1$. Thus there holds the limit $\lim_{n\to\infty}\|x_n-u\|$ and the sequences $\{x_n\},\{y_n\},\{z_n\}$ are bounded. From the last relations, we also obtain

$$\begin{aligned}
&(c+\hat{c})(1-b^2k^2)\|x_n-y_n\|^2+(1-d-\hat{d})c\|x_n-z_n\|^2+c\hat{c}\|z_n-Sz_n\|^2 \\
&\le (\alpha_n+\hat{\alpha}_n)(1-\lambda_n^2k^2)\|x_n-y_n\|^2 \\
&\quad +(1-\alpha_n-\hat{\alpha}_n)\alpha_n\|x_n-z_n\|^2+\alpha_n\hat{\alpha}_n\|z_n-Sz_n\|^2 \\
&\le \|x_n-u\|^2-\|x_{n+1}-u\|^2.
\end{aligned}$$

So we have

$$\lim_{n\to\infty}\|x_n-y_n\|=\lim_{n\to\infty}\|x_n-z_n\|=\lim_{n\to\infty}\|z_n-Sz_n\|=0. \tag{3.2}$$

In the meantime, we obtain

$$\begin{aligned}
\|y_n-t_n\| &= \|P_C(x_n-\lambda_nAx_n)-P_C(x_n-\lambda_nAy_n)\| \\
&\le \|(x_n-\lambda_nAx_n)-(x_n-\lambda_nAy_n)\| \\
&= \lambda_n\|Ax_n-Ay_n\|.
\end{aligned}$$

This together with the Lipschitz continuity of A and $\|x_n-y_n\|\to 0$, implies that

$$\lim_{n\to\infty}\|y_n-t_n\|=0.$$

Noting that

$$\|x_n-t_n\|\le\|x_n-y_n\|+\|y_n-t_n\|.$$

we have

$$\lim_{n\to\infty}\|x_n-t_n\|=0.$$

Thus, from (3.2) it follows that

$$\lim_{n\to\infty}\|t_n-z_n\|=0.$$

Again from the Lipschitz continuity of A, we get

$$\lim_{n\to\infty} \|Ay_n - At_n\| = 0.$$

As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some z. We assert that $z \in F(S) \cap \Omega \cap \mathrm{VI}(C, A)$. Indeed, first, we show that $z \in \mathrm{VI}(C, A)$. Since $x_n - t_n \to 0$ and $y_n - t_n \to 0$, we obtain that $t_{n_i} \rightharpoonup z$ and $y_{n_i} \rightharpoonup z$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, A)$; see [9]. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So we have

$$\langle v - u, w - Av\rangle \geq 0, \quad \forall u \in C. \tag{3.3}$$

Furthermore, noting that $t_n = P_C(x_n - \lambda_n Ay_n)$ and $v \in C$, we have

$$\langle x_n - \lambda_n Ay_n - t_n, t_n - v\rangle \geq 0,$$

and hence,

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ay_n\rangle \geq 0.$$

Therefore, from (3.3) and $t_n \in C$ it follows that

$$\begin{aligned} \langle v - t_{n_i}, w\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \langle v - t_{n_i}, \tfrac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle. \end{aligned}$$

Letting $i \to \infty$, we get

$$\langle v - z, w\rangle \geq 0.$$

Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in \mathrm{VI}(C, A)$.

Secondly, we show that $z \in F(S)$. Indeed, since $x_{n_i} \rightharpoonup z$, from (3.2) we deduce that $z_{n_i} \rightharpoonup z$ and $z_{n_i} - Sz_{n_i} \to 0$. By Lemma 2.3 we obtain $z \in F(S)$.

Next, we show that $z \in \Omega$. Indeed, since A is α-inverse strongly monotone and M is maximal monotone, by Lemma 2.6 we know that $M + A$ is maximal monotone. Take a fixed $(y, g) \in G(M + A)$ arbitrarily. Then we have $g \in (M + A)y$. So, we have $g - Ay \in My$. Since $z_{n_i} = J_{M,\lambda_{n_i}}(t_{n_i} - \lambda_{n_i} At_{n_i})$ yields $\frac{1}{\lambda_{n_i}}(t_{n_i} - z_{n_i} - \lambda_{n_i} At_{n_i}) \in Mz_{n_i}$, we have

$$\langle y - z_{n_i}, g - Ay - \frac{1}{\lambda_{n_i}}(t_{n_i} - z_{n_i} - \lambda_{n_i} At_{n_i})\rangle \geq 0,$$

which hence yields

$$\begin{aligned} \langle y - z_{n_i}, g\rangle &\geq \langle y - z_{n_i}, Ay + \tfrac{1}{\lambda_{n_i}}(t_{n_i} - z_{n_i} - \lambda_{n_i} At_{n_i})\rangle \\ &= \langle y - z_{n_i}, Ay - At_{n_i}\rangle + \langle y - z_{n_i}, \tfrac{1}{\lambda_{n_i}}(t_{n_i} - z_{n_i})\rangle \\ &\geq \alpha\|Ay - Az_{n_i}\|^2 + \langle y - z_{n_i}, Az_{n_i} - At_{n_i}\rangle + \langle y - z_{n_i}, \tfrac{1}{\lambda_{n_i}}(t_{n_i} - z_{n_i})\rangle \\ &\geq \langle y - z_{n_i}, Az_{n_i} - At_{n_i}\rangle + \langle y - z_{n_i}, \tfrac{1}{\lambda_{n_i}}(t_{n_i} - z_{n_i})\rangle. \end{aligned}$$

Letting $i \to \infty$, we deduce from $\|t_n - z_n\| \to 0$ and the Lipschitz continuity of A that

$$\langle y - z, g \rangle \geq 0.$$

This shows that $0 \in (A + M)z$. Hence, $z \in \Omega$. Therefore, $z \in F(S) \cap \Omega \cap \mathrm{VI}(C, A)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup z'$. Then we obtain $z' \in F(S) \cap \Omega \cap \mathrm{VI}(C, A)$. Let us show that $z = z'$. Assume that $z \neq z'$. Utilizing the Opial condition [8], we have

$$\begin{aligned} \lim_{n\to\infty} \|x_n - z\| &= \liminf_{n\to\infty} \|x_{n_i} - z\| < \liminf_{n\to\infty} \|x_{n_i} - z'\| \\ &= \lim_{n\to\infty} \|x_n - z'\| = \liminf_{j\to\infty} \|x_{n_j} - z'\| \\ &< \liminf_{j\to\infty} \|x_{n_j} - z\| = \lim_{n\to\infty} \|x_n - z\|. \end{aligned}$$

This is a contradiction. So, $z = z'$. This implies that $x_n \rightharpoonup z$. Since $x_n - y_n \to 0$ and $x_n - z_n \to 0$, we also have that $y_n \rightharpoonup z$ and $z_n \rightharpoonup z$.

Finally, let us show that $u_n \to z$, where $u_n = P_{F(S)\cap\Omega\cap\mathrm{VI}(C,A)} x_n$ for every $n = 0, 1, 2, \ldots.$ Indeed, since

$$u_n = P_{F(S)\cap\Omega\cap\mathrm{VI}(C,A)} x_n \quad \text{and} \quad z \in F(S) \cap \Omega \cap \mathrm{VI}(C, A),$$

we have

$$\langle z - u_n, u_n - x_n \rangle \geq 0,$$

for every $n = 0, 1, 2, \ldots.$ Utilizing Lemma 3.1, from (3.3) we obtain that $\{u_n\}$ converges strongly to some $\tilde{z} \in F(S) \cap \Omega \cap \mathrm{VI}(C, A)$. Then, we have

$$\langle z - \tilde{z}, \tilde{z} - z \rangle \geq 0$$

and hence $z = \tilde{z}$. This completes the proof. □

Remark 3.3. Compared with Theorem 3.2 in Nadezhkina and Takahashi [6], our Theorem 3.2 improves and extends Nadezhkina and Takahashi [6, Theorem 3.1] in the following aspects:

(i) Nadezhkina and Takahashi's extragradient algorithm in [6, Theorem 3.1] is extended to develop the modified extragradient algorithm in our Theorem 3.2.

(ii) the technique of proving weak convergence in our Theorem 3.2 is very different from that in Nadezhkina and Takahashi [6, Theorem 3.1] because our technique depends on the properties for maximal monotone mappings and their resolvent operators, and the demiclosedness principle for nonexpansive mappings in Hilbert spaces.

(iii) our problem of finding an element of $\mathrm{Fix}(S) \cap \Omega \cap \mathrm{VI}(C, A)$ is more general than Nadezhkina and Takahashi's problem of finding an element of $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in [6, Theorem 3.1].

Furthermore, utilizing our Theorem 3.2 as above, we can immediately obtain the following weak convergence results:

Corollary 3.4. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone k-Lipschitz continuous mapping, $M : H \to 2^H$ be a maximal monotone mapping* *and $S : H \to H$ be a nonexpansive mapping such that $F(S) \cap \Omega \cap A^{-1}0 \neq \emptyset$. For $x_0 = x \in H$ chosen arbitrarily, let $\{x_n\}, \{t_n\}$ be the sequences generated by*

$$\begin{cases} t_n = x_n - \lambda_n A(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n J_{M,\lambda_n}(t_n - \lambda_n A t_n) + \hat{\alpha}_n S J_{M,\lambda_n}(t_n - \lambda_n A t_n), \end{cases}$$

for every $n = 0, 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \alpha)$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$, and $\{\hat{\alpha}_n\} \subset [\hat{c}, \hat{d}]$ for some $\hat{c}, \hat{d} \in (0, 1)$ such that $d + \hat{d} < 1$. Then the sequences $\{x_n\}, \{t_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega \cap A^{-1}0$, where

$$z = \|\cdot\| - \lim_{n\to\infty} P_{F(S)\cap\Omega\cap A^{-1}0} x_n.$$

Proof. We have $C = D(M) = H$, $A^{-1}0 = \mathrm{VI}(H, A)$ and $P_H = I$. By Theorem 3.2, we obtain the desired result. □

Remark 3.5. Whenever $M = I$ the identity mapping of H, we have $\Omega = H$ and $F(S) \cap \Omega \cap A^{-1}0 = F(S) \cap A^{-1}0 \subset \mathrm{VI}(F(S), A)$. See also Yamada [12] for the case when $A : H \to H$ is a strongly monotone and Lipschitz continuous mapping and $S : H \to H$ is a nonexpansive mapping.

Corollary 3.6. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone k-Lipschitz continuous mapping and $B, M : H \to 2^H$ be two maximal monotone mappings such that $A^{-1}0 \cap B^{-1}0 \cap \Omega \neq \emptyset$. For $x_0 = x \in H$ chosen arbitrarily, let $\{x_n\}, \{t_n\}$ be the sequences generated by*

$$\begin{cases} t_n = x_n - \lambda_n A(x_n - \lambda_n A x_n), \\ x_{n+1} = (1 - \alpha_n - \hat{\alpha}_n)x_n + \alpha_n J_{M,\lambda_n}(t_n - \lambda_n A t_n) + \hat{\alpha}_n J_{B,r} J_{M,\lambda_n}(t_n - \lambda_n A t_n), \end{cases}$$

for every $n = 0, 1, 2, ...,$ where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \alpha)$, $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$, and $\{\hat{\alpha}_n\} \subset [\hat{c}, \hat{d}]$ for some $\hat{c}, \hat{d} \in (0, 1)$ such that $d + \hat{d} < 1$. Then the sequences $\{x_n\}, \{t_n\}$ converge weakly to the same point $z \in A^{-1}0 \cap B^{-1}0 \cap \Omega$, where

$$z = \|\cdot\| - \lim_{n\to\infty} P_{A^{-1}0\cap B^{-1}0\cap\Omega} x_n.$$

Proof. We have $C = D(M) = H$, $A^{-1}0 = \mathrm{VI}(H, A)$ and $F(J_{B,r}) = B^{-1}0$. Putting $P_H = I$, by Theorem 3.2 we obtain the desired result. □

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