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A HYBRID EXTRAGRADIENT-LIKE APPROXIMATION METHOD WITH REGULARIZATION FOR SOLVING SPLIT FEASIBILITY AND FIXED POINT PROBLEMS

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Dedicated to Professor Mau-Hsiang Shih on the occasion of his 65th birthday

Abstract. The purpose of this paper is to investigate the problem of finding a common element of the fixed point set $\text{Fix}(S)$ of a nonexpansive mapping S and the solution set Γ of the split feasibility problem (SFP) in real Hilbert spaces. We introduce a hybrid extragradient-like approximation method with regularization, which is based on the well-known extragradient method, a hybrid (or outer approximation) method, and a regularization method. The method produces three sequences which are shown to converge strongly to the same common element of $\text{Fix}(S) \cap \Gamma$. In addition, we also prove a new weak convergence theorem by a modified extragradient method with regularization for finding an element of $\text{Fix}(S) \cap \Gamma$. These results represent the supplementation, improvement, extension and development of the corresponding results in the very recent literature.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let K be a nonempty closed convex subset of $\mathcal{H}$. The (nearest point or metric) projection from $\mathcal{H}$ onto K, denoted by P_K. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \longrightarrow x$ to indicate that the sequence $\{x_n\}$ converges strongly to x.

Let C and Q be nonempty closed convex subsets of infinite-dimensional real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The split feasibility problem (SFP) is to find a point x^* with the property:

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.1}$$

where $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $B(\mathcal{H}_1, \mathcal{H}_2)$ denotes the family of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

In 1994, the SFP was first introduced by Censor and Elfving [8], in finite-dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. A number of image reconstruction

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problems can be formulated as the SFP; see, e.g., [2] and the references therein. Recently, it is found that the SFP can also be applied to study intensity-modulated radiation therapy; see, e.g., [6, 7, 9] and the references therein. In the recent past, a wide variety of iterative methods have been used in signal processing and image reconstruction and for solving the SFP; see, e.g., [2, 3, 6, 7, 9, 23, 25, 27–30] and the references therein. A special case of the SFP is the following convex constrained linear inverse problem [12] of finding an element x such that

$$x \in C \quad \text{and} \quad Ax = b. \tag{1.2}$$

It has been extensively investigated in the literature using the projected Landweber iterative method [17]. Comparatively, the SFP has received much less attention so far, due to the complexity resulting from the set Q. Therefore, whether various versions of the projected Landweber iterative method [17] can be extended to solve the SFP remains an interesting open topics. For example, it is yet not clear whether the dual approach to (1.2) of [22] can be extended to the SFP. The original algorithm given in [8] involves the computation of the inverse A^{-1} (assuming the existence of the inverse of A), and thus has not become popular. A seemingly more popular algorithm that solves the SFP is the CQ algorithm of Byrne [2, 3] which is found to be a gradient-projection method (GPM) in convex minimization. It is also a special case of the proximal forward-backward splitting method [11]. The CQ algorithm only involves the computation of the projections P_C and P_Q onto the sets C and Q, respectively, and is therefore implementable in the case where P_C and P_Q have closed-form expressions, for example, C and Q are closed balls or half-spaces. However, it remains a challenge how to implement the CQ algorithm in the case where the projections P_C and/or P_Q fail to have closed-form expressions, though theoretically we can prove the (weak) convergence of the algorithm.

Very recently, Xu [28] gave a continuation of the study on the CQ algorithm and its convergence. She applied Mann's algorithm to the SFP and proposed an averaged CQ algorithm which was proved to be weakly convergent to a solution of the SFP. She also established the strong convergence result, which shows that the minimum-norm solution can be obtained.

Furthermore, Korpelevich [16] introduced the so-called extragradient method for finding a solution of a saddle point problem. He proved that the sequences generated by the proposed iterative algorithm converge to a solution of the saddle point problem.

Throughout this paper, assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Let $f : \mathcal{H}_1 \longrightarrow R$ be a continuous differentiable function. The minimization problem

$$\min_{x \in C} f(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 \tag{1.3}$$

is ill-posed. Therefore, Xu [28] considered the following Tikhonov regularization problem:

$$\min_{x \in C} f_\alpha(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2 + \frac{1}{2}\alpha\|x\|^2, \tag{1.4}$$

where $\alpha > 0$ is the regularization parameter. The regularized minimization (1.4) has a unique solution which is denoted by x_α. The following results are easy to prove.

Proposition 1.1 (see [4, Proposition 3.1]). *Given $x^* \in \mathcal{H}_1$, the following statements are equivalent:*

(i) *x^* solves the SFP;*

(ii) *x^* solves the fixed point equation*

$$P_C(I - \lambda\nabla f)x^* = x^*,$$

where $\lambda > 0$, $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A;

(iii) *x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that*

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.5}$$

It is clear from Proposition 1.1 that

$$\varGamma = \mathrm{Fix}(P_C(I - \lambda\nabla f)) = \mathrm{VI}(C, \nabla f)$$

for all $\lambda > 0$, where $\mathrm{Fix}(P_C(I - \lambda\nabla f))$ and $\mathrm{VI}(C, \nabla f)$ denote the set of fixed points of $P_C(I - \lambda\nabla f)$ and the solution set of VIP (1.5), respectively.

Proposition 1.2 (see [4]). *There hold the following statements:*

(i) *the gradient*

$$\nabla f_\alpha = \nabla f + \alpha I = A^*(I - P_Q)A + \alpha I$$

is $(\alpha + \|A\|^2)$-Lipschitz continuous and α-strongly monotone;

(ii) *the mapping $P_C(I - \lambda\nabla f_\alpha)$ is a contraction with coefficient*

$$\sqrt{1 - \lambda(2\alpha - \lambda(\|A\|^2 + \alpha)^2)} \ (\leq \sqrt{1 - \alpha\lambda} \leq 1 - \frac{1}{2}\alpha\lambda),$$

where $0 < \lambda \leq \frac{\alpha}{(\|A\|^2+\alpha)^2}$;

(iii) *if the SFP is consistent, then the strong $\lim_{\alpha \longrightarrow 0} x_\alpha$ exists and is the minimum-norm solution of the SFP.*

Very recently, Ceng, Ansari and Yao [4] proposed an extragradient algorithm with regularization, and proved that the sequences generated by the proposed algorithm converge weakly to an element of $\mathrm{Fix}(S) \cap \varGamma$, where $S : C \longrightarrow C$ is a nonexpansive mapping.

Theorem 1.3 (see [4, Theorem 3.1]). *Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \varGamma \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following extragradient algorithm:*

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \quad \forall n \geq 0, \end{cases} \tag{1.6}$$

where $\sum_{n=0}^{\infty} \alpha_n < \infty$, $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\|A\|^2})$ and $\{\beta_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in \mathrm{Fix}(S) \cap \varGamma$.

On the other hand, assume that C is a nonempty, closed and convex subset of $\mathcal{H}$ and $A : C \longrightarrow \mathcal{H}$ is a nonlinear mapping. The variational inequality problem (VIP) on C is stated as:

$$\text{find } x^* \in C \text{ such that } \quad \langle Ax^*, x - x^* \rangle \geq 0, \ \forall x \in C.$$

The set of solutions of the VIP is denoted by Ω_A; see [15] for more details. In 2006, Nadezhkina and Takahashi [18] first proved the weak convergence of the sequences generated by their proposed extragradient method, to an element of $\mathrm{Fix}(S) \cap \Omega_A$ under approximate assumptions. In the meantime, by combining a hybrid-type method with an extragradient-type method, Nadezhkina and Takahashi [19] introduced the following iterative method for finding an element of $\mathrm{Fix}(S) \cap \Omega_A$ and established the following strong convergence theorem.

Theorem 1.4 (see [19, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $A : C \longrightarrow \mathcal{H}$ be a monotone and k-Lipschitz continuous mapping and let $S : C \longrightarrow C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Omega_A \neq \emptyset$. *Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences generated by*

$$\left\{ \begin{array}{l} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{array} \right. \tag{1.7}$$

for all $n \geq 0$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{k})$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$. Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ converge strongly to the same point $q = P_{\mathrm{Fix}(S) \cap \Omega_A} x$.

Very recently, Ceng, Hadjisavvas and Wong [5] introduced a hybrid extragradient-like approximation method which is based on the above extragradient method and the hybrid (or outer approximation) method, for finding an element of $\mathrm{Fix}(S) \cap \Omega_A$, and derived the following strong convergence theorem.

Theorem 1.5 (see [5, Theorem 5]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $A : C \longrightarrow \mathcal{H}$ be a monotone, k-Lipschitz continuous mapping and let $S : C \longrightarrow C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Omega_A \neq \emptyset$. *We define inductively the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ by*

$$\left\{ \begin{array}{l} x_0 \in C, \\ y_n = (1 - \gamma_n) x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ z_n = (1 - \alpha_n - \beta_n) x_n + \alpha_n y_n + \beta_n S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + (3 - 3\gamma_n + \alpha_n) b^2 \|A x_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{array} \right. \tag{1.8}$$

for all $n \geq 0$, where $\{\lambda_n\}$ is a sequence in $[a, b]$ with $a > 0$ and $b < \frac{1}{2k}$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leq 1$ *for all* $n \geq 0$*;*
(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0$*;*
(iii) $\liminf_{n\longrightarrow\infty} \beta_n > 0$*;*
(iv) $\lim_{n\longrightarrow\infty} \gamma_n = 1$ *and* $\gamma_n > \frac{3}{4}$ *for all* $n \geq 0$*.*

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ *are well-defined and converge strongly to the same point* $q = P_{\mathrm{Fix}(S)\cap\Omega_A} x_0$*.*

In this paper, inspired by the iterative scheme (1.6) and utilizing the hybrid extragradient-like approximation method in [5, Theorem 5], we propose the following hybrid extragradient-like iterative algorithm with regularization:

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)), \\ z_n = (1 - \beta_n - \gamma_n)x_n + \beta_n y_n + \gamma_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + 2\alpha_n\lambda_n\kappa(\kappa + \|y_n\|)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} x_0 \end{cases} \tag{1.9}$$

for all $n \geq 0$, where $\sup_{p\in \mathrm{Fix}(S)\cap\Gamma} \|p\| \leq \kappa$ for some $\kappa \geq 0$, and the following conditions hold for four sequences $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\beta_n\}, \{\gamma_n\} \subset [0, 1]$:

(i) $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\|A\|^2})$;
(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0$;
(iii) $\beta_n + \gamma_n \leq 1$ for all $n \geq 0$;
(iv) $\lim_{n\longrightarrow\infty} \beta_n = 0$ and $\liminf_{n\longrightarrow\infty} \gamma_n > 0$.

It is shown that the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ are well-defined and converge strongly to the same point $q = P_{\mathrm{Fix}(S)\cap\Gamma} x_0$. It is worth emphasizing that our result is new and novel for Hilbert spaces. The main result represents the supplementation, improvement, extension and development of the corresponding results in the very recent literature, for example, [28, Theorem 5.7] and [4, Theorem 3.1] to a great extent. Moreover, the hybrid extragradient-like approximation method given in [5, Theorem 5] is extended to develop our hybrid extragradient-like approximation method with regularization. In addition, we also prove a new weak convergence theorem by a modified extragradient method with regularization for the split feasibility problem and fixed point problem, which essentially includes [4, Theorem 3.1] as a special case.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Throughout the paper, unless otherwise specified, we denote by $x_n \longrightarrow x$ (respectively, $x_n \rightharpoonup x$), the strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x. In addition, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; namely,

$$\omega_w(x_n) := \{x : x_{n_k} \rightharpoonup x \text{ for some subsequence } \{x_{n_k}\} \text{ of } \{x_n\}\}.$$

Let K be a nonempty, closed and convex subset of $\mathcal{H}$. Now we present some known results and definitions which will be used in the sequel.

The metric (or nearest point) projection from $\mathcal{H}$ onto K is the mapping $P_K : \mathcal{H} \longrightarrow K$ which assigns to each point $x \in \mathcal{H}$ the unique point $P_K x \in K$ satisfying the property

$$\|x - P_K x\| = \inf_{y \in K} \|x - y\| =: d(x, K).$$

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1 (see [13]). *For given $x \in \mathcal{H}$ and $z \in K$:*

(i) $z = P_K x \;\Leftrightarrow\; \langle x - z, y - z\rangle \leq 0,\ \forall y \in K$;
(ii) $z = P_K x \;\Leftrightarrow\; \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \forall y \in K$;
(iii) $\langle P_K x - P_K y, x - y\rangle \geq \|P_K x - P_K y\|^2,\ \forall y \in \mathcal{H}$, *which hence implies that P_K is nonexpansive and monotone.*

Definition 2.2. A mapping $T : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be:

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H};$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in \mathcal{H};$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : \mathcal{H} \longrightarrow \mathcal{H}$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.3. Let T be a nonlinear operator whose domain is $D(T) \subseteq \mathcal{H}$ and whose range is $R(T) \subseteq \mathcal{H}$.

(a) T is said to be monotone if

$$\langle x - y, Tx - Ty\rangle \geq 0, \quad \forall x, y \in D(T).$$

(b) Given a number $\beta > 0$, T is said to be β-strongly monotone if

$$\langle x - y, Tx - Ty\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in D(T).$$

(c) Given a number $\nu > 0$, T is said to be ν-inverse strongly monotone (ν-ism) if

$$\langle x - y, Tx - Ty\rangle \geq \nu\|Tx - Ty\|^2, \quad \forall x, y \in D(T).$$

It can be easily seen that if S is nonexpansive, then $I - S$ is monotone. It is also easy to see that a projection P_K is 1-ism.

Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields, for instance, in traffic assignment problems; see, e.g., [1, 14].

A mapping $T : \mathcal{H} \longrightarrow \mathcal{H}$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S$$

where $\alpha \in (0,1)$ and $S : \mathcal{H} \longrightarrow \mathcal{H}$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$-averaged maps.

Proposition 2.4 (see [3])**.** *Let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$-ism.*
(ii) *If T is ν-ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$-ism.*
(iii) *T is averaged if and only if the complement $I-T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0,1)$, T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-ism.*

Proposition 2.5 (see [3,10])**.** *Let $S, T, V : \mathcal{H} \longrightarrow \mathcal{H}$ be given operators.*

(i) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is averaged and V is nonexpansive, then T is averaged.*
(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*
(iii) *If $T = (1-\alpha)S + \alpha V$ for some $\alpha \in (0,1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*
(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \circ T_2 \circ \cdots \circ T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0,1)$, then the composite $T_1 \circ T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*
(v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

The notation $\mathrm{Fix}(T)$ denotes the set of all fixed points of the mapping T, that is, $\mathrm{Fix}(T) = \{x \in \mathcal{H} : Tx = x\}$.

The following so-called demiclosedness principle for nonexpansive mappings will often be used.

Lemma 2.6 (see [13, Demiclosedness Principle])**.** *Let K be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $S : K \longrightarrow K$ be a nonexpansive mapping with $\mathrm{Fix}(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in K converging weakly to x and if $\{(I-S)x_n\}$ converges strongly to y, then $(I-S)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(S)$.*

The following elementary result in the real Hilbert spaces is quite well-known.

Lemma 2.7 (see [13])**.** *Let $\mathcal{H}$ be a real Hilbert space. Then for every $\lambda,\ \mu,\ \nu \in [0,1]$ with $\lambda + \mu + \nu = 1$, we have*

$$\begin{aligned} \|\lambda x + \mu y + \nu z\|^2 =& \lambda\|x\|^2 + \mu\|y\|^2 + \nu\|z\|^2 - \lambda\mu\|x-y\|^2 \\ & - \lambda\nu\|x-z\|^2 - \mu\nu\|y-z\|^2, \quad \forall x,\ y,\ z \in \mathcal{H}. \end{aligned}$$

To prove a new weak convergence theorem by a modified extragradient method with regularization for the split feasibility problem and fixed point problem, we need the following lemma due to Osilike et al. [21].

Lemma 2.8 (see [21, p. 80])**.** *Let* $\{a_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1+\alpha_n)a_n + \delta_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \alpha_n < \infty$ *and* $\sum_{n=1}^{\infty} \delta_n < \infty$, *then* $\lim_{n\longrightarrow\infty} a_n$ *exists. If, in addition,* $\{a_n\}_{n=1}^{\infty}$ *has a subsequence which converges to zero, then* $\lim_{n\longrightarrow\infty} a_n = 0$.

Corollary 2.9 (see [26, p. 303])**.** *Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{\delta_n\}_{n=0}^{\infty}$ *be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + \delta_n, \quad \forall n \geq 0.$$

If $\sum_{n=0}^{\infty} \delta_n$ *converges, then* $\lim_{n\longrightarrow\infty} a_n$ *exists.*

The following fact is straightforward but useful.

Lemma 2.10. *There holds the following inequality in an inner product space* X*:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$$

Let K be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and let $F : K \longrightarrow \mathcal{H}$ be a monotone mapping. The variational inequality problem (VIP) is to find $x \in K$ such that

$$\langle Fx, y - x\rangle \geq 0, \quad \forall y \in K.$$

The solution set of the VIP is denoted by $\mathrm{VI}(K, F)$. It is well-known that

$$x \in \mathrm{VI}(K,F) \;\Leftrightarrow\; x = P_K(x - \lambda Fx),\ \forall \lambda > 0.$$

Recall that a Banach space X is said to satisfy the Opial condition [20] if for any sequence $\{x_n\}$ in X the condition that $\{x_n\}$ converges weakly to $x \in X$ implies that the inequality

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$$

holds for every $y \in X$ with $y \neq x$. It is well-known that every Hilbert space satisfies the Opial condition.

A set-valued mapping $T : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is called monotone if for all $x, y \in \mathcal{H}$, $f \in Tx$ and $g \in Ty$ imply

$$\langle x - y, f - g\rangle \geq 0.$$

A monotone mapping $T : \mathcal{H} \longrightarrow 2^{\mathcal{H}}$ is called maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in \mathcal{H}\times\mathcal{H}$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let $F : K \longrightarrow \mathcal{H}$ be a monotone and k-Lipschitz continuous mapping and let $N_K v$ be the normal cone to K at $v \in K$, that is,

$$N_K v = \{w \in \mathcal{H} : \langle v - y, w\rangle \geq 0,\ \forall y \in K\}.$$

Define

$$Tv = \begin{cases} Fv + N_K v, & \text{if } v \in K, \\ \emptyset, & \text{if } v \notin K. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(K, F)$; see [24] for more details.

3. Main results

We are now in a position to prove that the sequences generated by the proposed hybrid extragradient-like approximation method with regularization, converge strongly to an element of $\mathrm{Fix}(S) \cap \Gamma$.

Theorem 3.1. *Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma$ is a nonempty bounded subset of C. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences in C generated by the following hybrid extragradient-like iterative algorithm with regularization:*

$$\left\{\begin{array}{l} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)), \\ z_n = (1 - \beta_n - \gamma_n)x_n + \beta_n y_n + \gamma_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + 2\alpha_n \lambda_n \kappa(\kappa + \|y_n\|)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0 \end{array}\right. \tag{3.1}$$

for all $n \geq 0$, where $\sup_{p \in \mathrm{Fix}(S) \cap \Gamma} \|p\| \leq \kappa$ for some $\kappa \geq 0$, and the following conditions hold for four sequences $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\beta_n\}, \{\gamma_n\} \subset [0, 1]$:

(i) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{\|A\|^2})$;
(ii) $\lim_{n \longrightarrow \infty} \alpha_n = 0$;
(iii) $\beta_n + \gamma_n \leq 1$ *for all* $n \geq 0$;
(iv) $\lim_{n \longrightarrow \infty} \beta_n = 0$ *and* $\liminf_{n \longrightarrow \infty} \gamma_n > 0$.

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ are well-defined and converge strongly to the same point $q = P_{\mathrm{Fix}(S) \cap \Gamma} x_0$.

Proof. First, let us show that $P_C(I - \lambda \nabla f_\alpha)$ is ζ-averaged for each $\lambda \in (0, \frac{2}{\alpha + \|A\|^2})$, where

$$\zeta = \frac{2 + \lambda(\alpha + \|A\|^2)}{4}.$$

Indeed, it is easy to see that $\nabla f = A^*(I - P_Q)A$ is $\frac{1}{\|A\|^2}$-ism, that is,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{\|A\|^2} \|\nabla f(x) - \nabla f(y)\|^2.$$

Observe that

$$\begin{aligned} &(\alpha + \|A\|^2)\langle \nabla f_\alpha(x) - \nabla f_\alpha(y), x - y \rangle \\ &= (\alpha + \|A\|^2)[\alpha\|x - y\|^2 + \langle \nabla f(x) - \nabla f(y), x - y \rangle] \\ &= \alpha^2\|x - y\|^2 + \alpha\langle \nabla f(x) - \nabla f(y), x - y \rangle + \alpha\|A\|^2\|x - y\|^2 \\ &\quad + \|A\|^2\langle \nabla f(x) - \nabla f(y), x - y \rangle \\ &\geq \alpha^2\|x - y\|^2 + 2\alpha\langle \nabla f(x) - \nabla f(y), x - y \rangle + \|\nabla f(x) - \nabla f(y)\|^2 \\ &= \|\alpha(x - y) + \nabla f(x) - \nabla f(y)\|^2 \\ &= \|\nabla f_\alpha(x) - \nabla f_\alpha(y)\|^2. \end{aligned}$$

Hence, it follows that $\nabla f_\alpha = \alpha I + A^*(I - P_Q)A$ is $\frac{1}{\alpha+\|A\|^2}$-ism. Thus, $\lambda\nabla f_\alpha$ is $\frac{1}{\lambda(\alpha+\|A\|^2)}$-ism according to Proposition 2.4 (ii). By Proposition 2.4 (iii) the complement $I - \lambda\nabla f_\alpha$ is $\frac{\lambda(\alpha+\|A\|^2)}{2}$-averaged. Therefore, noting that P_C is $\frac{1}{2}$-averaged and utilizing Proposition 2.5 (iv), we know that for each $\lambda \in (0, \frac{2}{\alpha+\|A\|^2})$, $P_C(I - \lambda\nabla f_\alpha)$ is ζ-averaged with

$$\zeta = \frac{1}{2} + \frac{\lambda(\alpha + \|A\|^2)}{2} - \frac{1}{2} \cdot \frac{\lambda(\alpha + \|A\|^2)}{2} = \frac{2 + \lambda(\alpha + \|A\|^2)}{4} \in (0, 1).$$

This shows that $P_C(I - \lambda\nabla f_\alpha)$ is nonexpansive. Furthermore, for $\{\lambda_n\} \subset [a, b]$ with $a, b \in (0, \frac{1}{\|A\|^2})$, we have

$$a \le \inf_{n\ge0} \lambda_n \le \sup_{n\ge0} \lambda_n \le b < \frac{1}{\|A\|^2} = \lim_{n\longrightarrow\infty} \frac{1}{\alpha_n + \|A\|^2}.$$

Without loss of generality we may assume that

$$a \le \inf_{n\ge0} \lambda_n \le \sup_{n\ge0} \lambda_n \le b < \frac{1}{\alpha_n + \|A\|^2}, \quad \forall n \ge 0.$$

Consequently, it follows that for each integer $n \ge 0$, $P_C(I - \lambda_n\nabla f_{\alpha_n})$ is ζ_n-averaged with

$$\zeta_n = \frac{1}{2} + \frac{\lambda_n(\alpha_n + \|A\|^2)}{2} - \frac{1}{2} \cdot \frac{\lambda_n(\alpha_n + \|A\|^2)}{2} = \frac{2 + \lambda_n(\alpha_n + \|A\|^2)}{4} \in (0, 1).$$

This immediately implies that $P_C(I - \lambda_n\nabla f_{\alpha_n})$ is nonexpansive for all $n \ge 0$.

Next we divide the remainder of the proof into several steps.

Step 1. Assuming that x_n is a well-defined element of C for some $n \ge 0$, we show that $\mathrm{Fix}(S) \cap \Gamma \subset C_n$.

Since x_n is defined, y_n, z_n are obviously well-defined elements of C. Now, take a fixed $p \in \mathrm{Fix}(S) \cap \Gamma$ arbitrarily. Then, we get $Sp = p$ and $P_C(I - \lambda\nabla f)p = p$ for $\lambda \in (0, \frac{2}{\|A\|^2})$. From (3.1) it follows that

$$\begin{aligned}
\|y_n - p\| &= \|P_C(I - \lambda_n\nabla f_{\alpha_n})x_n - P_C(I - \lambda_n\nabla f)p\| \\
&\le \|P_C(I - \lambda_n\nabla f_{\alpha_n})x_n - P_C(I - \lambda_n\nabla f_{\alpha_n})p\| \\
&\quad + \|P_C(I - \lambda_n\nabla f_{\alpha_n})p - P_C(I - \lambda_n\nabla f)p\| \\
&\le \|x_n - p\| + \|(I - \lambda_n\nabla f_{\alpha_n})p - (I - \lambda_n\nabla f)p\| \\
&\le \|x_n - p\| + \lambda_n\alpha_n\|p\|.
\end{aligned} \tag{3.2}$$

Utilizing Lemma 2.10 we also have
(3.3)

$$\begin{aligned}\|y_n - p\|^2 &= \|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n - P_C(I-\lambda_n\nabla f)p\|^2\\ &= \|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n - P_C(I-\lambda_n\nabla f_{\alpha_n})p\\ &\quad + P_C(I-\lambda_n\nabla f_{\alpha_n})p - P_C(I-\lambda_n\nabla f)p\|^2\\ &\le \|P_C(I-\lambda_n\nabla f_{\alpha_n})x_n - P_C(I-\lambda_n\nabla f_{\alpha_n})p\|^2\\ &\quad + 2\langle P_C(I-\lambda_n\nabla f_{\alpha_n})p - P_C(I-\lambda_n\nabla f)p, y_n - p\rangle\\ &\le \|x_n - p\|^2 + 2\|P_C(I-\lambda_n\nabla f_{\alpha_n})p - P_C(I-\lambda_n\nabla f)p\|\|y_n-p\|\\ &\le \|x_n - p\|^2 + 2\|(I-\lambda_n\nabla f_{\alpha_n})p - (I-\lambda_n\nabla f)p\|\|y_n-p\|\\ &= \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n-p\|.\end{aligned}$$

Put $t_n = P_C(x_n - \lambda_n\nabla f_{\alpha_n}(y_n))$. Then, by Proposition 2.1 (ii), we have

$$\begin{aligned}&\|t_n - p\|^2 \le \|x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - p\|^2 - \|x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - t_n\|^2\\ &= \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle\nabla f_{\alpha_n}(y_n), p - t_n\rangle\\ &= \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n(\langle\nabla f_{\alpha_n}(y_n) - \nabla f_{\alpha_n}(p), p - y_n\rangle\\ &\quad + \langle\nabla f_{\alpha_n}(p), p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - t_n\rangle)\\ &\le \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n(\langle\nabla f_{\alpha_n}(p), p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - t_n\rangle)\\ &= \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n[\langle(\alpha_n I + \nabla f)p, p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - t_n\rangle]\\ &\le \|x_n - p\|^2 - \|x_n - t_n\|^2 + 2\lambda_n[\alpha_n\langle p, p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - t_n\rangle]\\ &= \|x_n - p\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2\\ &\quad + 2\lambda_n[\alpha_n\langle p, p - y_n\rangle + \langle\nabla f_{\alpha_n}(y_n), y_n - t_n\rangle]\\ &= \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - y_n, t_n - y_n\rangle\\ &\quad + 2\lambda_n\alpha_n\langle p, p - y_n\rangle.\end{aligned}$$

Further, by Proposition 2.1 (i), we have

$$\begin{aligned}&\langle x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - y_n, t_n - y_n\rangle\\ &= \langle x_n - \lambda_n\nabla f_{\alpha_n}(x_n) - y_n, t_n - y_n\rangle + \langle\lambda_n\nabla f_{\alpha_n}(x_n) - \lambda_n\nabla f_{\alpha_n}(y_n), t_n - y_n\rangle\\ &\le \langle\lambda_n\nabla f_{\alpha_n}(x_n) - \lambda_n\nabla f_{\alpha_n}(y_n), t_n - y_n\rangle\\ &\le \lambda_n\|\nabla f_{\alpha_n}(x_n) - \nabla f_{\alpha_n}(y_n)\|\|t_n - y_n\|\\ &\le \lambda_n(\alpha_n + \|A\|^2)\|x_n - y_n\|\|t_n - y_n\|.\end{aligned}$$

So, we obtain
(3.4)

$$\begin{aligned}&\|t_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2\\ &\quad + 2\langle x_n - \lambda_n\nabla f_{\alpha_n}(y_n) - y_n, t_n - y_n\rangle + 2\lambda_n\alpha_n\langle p, p - y_n\rangle\\ &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n(\alpha_n + \|A\|^2)\|x_n - y_n\|\|t_n - y_n\|\\ &\quad + 2\lambda_n\alpha_n\|p\|\|y_n - p\|\\ &\le \|x_n - p\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2\\ &\quad + \lambda_n^2(\alpha_n + \|A\|^2)^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\|\\ &= \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\| + (\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1)\|x_n - y_n\|^2\\ &\le \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\|.\end{aligned}$$

Consequently, utilizing Lemma 2.7, from (3.2), (3.3) and the last inequality, we conclude that

(3.5)

$$\begin{aligned}
&\|z_n - p\|^2\\
&= \|(1-\beta_n-\gamma_n)x_n + \beta_n y_n + \gamma_n St_n - p\|^2\\
&= (1-\beta_n-\gamma_n)\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|St_n - p\|^2\\
&\quad -(1-\beta_n-\gamma_n)\beta_n\|x_n - y_n\|^2 - (1-\beta_n-\gamma_n)\gamma_n\|x_n - St_n\|^2 - \beta_n\gamma_n\|y_n - St_n\|^2\\
&\le (1-\beta_n-\gamma_n)\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|t_n - p\|^2\\
&\quad -(1-\beta_n-\gamma_n)\beta_n\|x_n - y_n\|^2 - (1-\beta_n-\gamma_n)\gamma_n\|x_n - St_n\|^2 - \beta_n\gamma_n\|y_n - St_n\|^2\\
&\le (1-\beta_n-\gamma_n)\|x_n - p\|^2 + \beta_n(\|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\|)\\
&\quad +\gamma_n[\|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\| + (\lambda_n^2(\alpha_n + \|A\|^2)^2 - 1)\|x_n - y_n\|^2]\\
&\quad -(1-\beta_n-\gamma_n)\beta_n\|x_n - y_n\|^2 - (1-\beta_n-\gamma_n)\gamma_n\|x_n - St_n\|^2 - \beta_n\gamma_n\|y_n - St_n\|^2\\
&\le (1-\beta_n-\gamma_n)\|x_n - p\|^2 + (\beta_n+\gamma_n)(\|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\|)\\
&\quad -(1-\beta_n-\gamma_n)\beta_n\|x_n - y_n\|^2 - (1-\beta_n-\gamma_n)\gamma_n\|x_n - St_n\|^2 - \beta_n\gamma_n\|y_n - St_n\|^2\\
&\le \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\|\\
&\quad -(1-\beta_n-\gamma_n)\beta_n\|x_n - y_n\|^2 - (1-\beta_n-\gamma_n)\gamma_n\|x_n - St_n\|^2 - \beta_n\gamma_n\|y_n - St_n\|^2\\
&\le \|x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\|\\
&\le \|x_n - p\|^2 + 2\lambda_n\alpha_n\kappa(\|y_n\| + \kappa).
\end{aligned}$$

This implies that $p \in C_n$. Thus $\mathrm{Fix}(S) \cap \varGamma \subset C_n$.

Step 2. We show that the sequence $\{x_n\}$ is well-defined and $\mathrm{Fix}(S)\cap\varGamma \subset C_n\cap Q_n$ for all $n \ge 0$.

We show this assertion by mathematical induction. For $n = 0$ we have $Q_0 = C$. Hence by Step 1 we get $\mathrm{Fix}(S) \cap \varGamma \subset C_0 \cap Q_0$. Assume that x_k is defined and $\mathrm{Fix}(S) \cap \varGamma \subset C_k \cap Q_k$ for some $k \ge 0$. Then y_k, z_k are well-defined elements of C. Note that C_k is a closed convex subset of C since

$$C_k = \{z \in C : \|z_k - x_k\|^2 + 2\langle z_k - x_k, x_k - z\rangle \le 2\lambda_k\alpha_k\kappa(\|y_k\| + \kappa)\}.$$

Also, it is obvious that Q_k is closed and convex. Thus, $C_k\cap Q_k$ is a closed convex subset, which is nonempty since by assumption it contains $\mathrm{Fix}(S)\cap\varGamma$. Consequently, $x_{k+1} = P_{C_k\cap Q_k}x_0$ is well-defined.

The definitions of x_{k+1} and of Q_{k+1} imply that $C_k\cap Q_k \subseteq Q_{k+1}$. Hence, $\mathrm{Fix}(S)\cap \varGamma \subseteq Q_{k+1}$. Using Step 1 we infer that $\mathrm{Fix}(S) \cap \varGamma \subseteq C_{k+1} \cap Q_{k+1}$.

Step 3. We show that the following statements hold:

(1) $\{x_n\}$ is bounded, $\lim_{n\longrightarrow\infty} \|x_n - x_0\|$ exists, and $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0$;

(2) $\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0$.

Indeed, take any $p \in \mathrm{Fix}(S) \cap \varGamma$. Using $x_{n+1} = P_{C_n\cap Q_n}x_0$ and $p \in \mathrm{Fix}(S) \cap \varGamma \subset C_n \cap Q_n$, we obtain

$$\|x_{n+1} - x_0\| \le \|p - x_0\|, \quad \forall n \ge 0. \tag{3.6}$$

Therefore, $\{x_n\}$ is bounded and so are both $\{y_n\}$ and $\{z_n\}$. From the definition of Q_n it is clear that $x_n = P_{Q_n}x_0$. Since $x_{n+1} \in C_n \cap Q_n \subset Q_n$, we have

$$\|x_{n+1} - x_n\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \quad \forall n \ge 0.$$

In particular, $\|x_{n+1} - x_0\| \geq \|x_n - x_0\|$. Hence $\lim_{n\longrightarrow\infty} \|x_n - x_0\|$ exists. Then it is easy to see that

$$\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + 2\alpha_n\lambda_n\kappa(\kappa + \|y_n\|).$$

Since $\{y_n\}$ is bounded, $\{\lambda_n\} \subset [a, b]$ and $\lim_{n\longrightarrow\infty} \alpha_n = 0$, we deduce from (3.7) that $\lim_{n\longrightarrow\infty} \|z_n - x_{n+1}\| = 0$. Again from (3.7) it follows that $\lim_{n\longrightarrow\infty} \|z_n - x_n\| = 0$.

Step 4. We show that the following statements hold:
(1) $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$;
(2) $\lim_{n\longrightarrow\infty} \|Sx_n - x_n\| = 0$.
Indeed, from inequality (3.5) we infer that

$$\begin{aligned}\|z_n - p\|^2 - \|x_n - p\|^2 &\leq (-\beta_n - \gamma_n)\|x_n - p\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|St_n - p\|^2 \\ &\leq 2\alpha_n\lambda_n\kappa(\kappa + \|y_n\|).\end{aligned}$$

Since $\alpha_n \longrightarrow 0$, $\beta_n \longrightarrow 0$, $\|z_n - x_n\| \longrightarrow 0$, $\{\lambda_n\} \subset [a, b]$, and $\{x_n\}$, $\{y_n\}$ are bounded, we deduce from the last inequality that

$$\lim_{n\longrightarrow\infty} \gamma_n(\|St_n - p\|^2 - \|x_n - p\|^2) = 0.$$

Using $\liminf_{n\longrightarrow\infty} \gamma_n > 0$ we get $\lim_{n\longrightarrow\infty}(\|St_n - p\|^2 - \|x_n - p\|^2) = 0$. Then inequality (3.4) implies that

$$\begin{aligned}\lim_{n\longrightarrow\infty}(\|St_n - p\|^2 - \|x_n - p\|^2) &\leq \lim_{n\longrightarrow\infty}(\|t_n - p\|^2 - \|x_n - p\|^2) \\ &\leq \lim_{n\longrightarrow\infty} 2\lambda_n\alpha_n\|p\|\|y_n - p\| = 0.\end{aligned}$$

Thus, $\lim_{n\longrightarrow\infty}(\|t_n - p\|^2 - \|x_n - p\|^2) = 0$. Now from inequality (3.4) we have

$$\begin{aligned}(1 - b^2(\alpha_n + \|A\|^2)^2)\|x_n - y_n\|^2 &\leq (1 - \lambda_n^2(\alpha_n + \|A\|^2)^2)\|x_n - y_n\|^2 \\ &\leq \|x_n - p\|^2 - \|t_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|y_n - p\|.\end{aligned}$$

This implies that

$$\lim_{n\longrightarrow\infty}(1 - b^2(\alpha_n + \|A\|^2)^2)\|x_n - y_n\|^2 = 0.$$

Taking into account $[a, b] \subset (0, \frac{1}{\|A\|^2})$, we have $1 - b^2\|A\|^4 > 0$. Consequently, $\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0$. Further, again from (3.4) we have

$$\begin{aligned}\|y_n - t_n\|^2 &\leq \|x_n - p\|^2 - \|t_n - p\|^2 - \|x_n - y_n\|^2 \\ &\quad +2\lambda_n(\alpha_n + \|A\|^2)\|x_n - y_n\|\|t_n - y_n\| + 2\lambda_n\alpha_n\|p\|\|y_n - p\| \\ &\leq \|x_n - p\|^2 - \|t_n - p\|^2 + 2\lambda_n(\alpha_n + \|A\|^2)\|x_n - y_n\|\|t_n - y_n\| \\ &\quad +2\lambda_n\alpha_n\|p\|\|y_n - p\|.\end{aligned}$$

Since $\alpha_n \longrightarrow 0$, $\|x_n - y_n\| \longrightarrow 0$ and $\{t_n\}$ is bounded, we derive $\lim_{n\longrightarrow\infty} \|y_n - t_n\| = 0$. Hence, $\lim_{n\longrightarrow\infty} \|x_n - t_n\| = 0$. Using the nonexpansivity of S, we get $\lim_{n\longrightarrow\infty} \|Sx_n - St_n\| = 0$.

We rewrite the definition of z_n as

$$z_n - x_n = -\beta_n x_n + \beta_n y_n + \gamma_n(St_n - x_n).$$

From $\lim_{n\longrightarrow\infty}\|z_n - x_n\| = 0$, $\lim_{n\longrightarrow\infty}\beta_n = 0$, $\liminf_{n\longrightarrow\infty}\gamma_n > 0$, and the boundedness of $\{x_n\}, \{y_n\}$, we infer that $\lim_{n\longrightarrow\infty}\|St_n - x_n\| = 0$. Thus finally $\lim_{n\longrightarrow\infty}\|Sx_n - x_n\| = 0$.

Step 5. We claim that $\omega_w(x_n) \subset \mathrm{Fix}(S) \cap \Gamma$, where $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{u \in \mathcal{H}_1 : x_{n_j} \rightharpoonup u \text{ for some subsequence } \{x_{n_j}\} \text{ of } \{x_n\}\}.$$

Indeed, since $\{x_n\}$ is bounded, it has a subsequence which converges weakly to some point in C and hence $\omega_w(x_n) \neq \emptyset$. Let $u \in \omega_w(x_n)$ be arbitrary. Then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to u. Since we also have $\lim_{j\longrightarrow\infty}(x_{n_j} - Sx_{n_j}) = 0$, from the demiclosedness principle it follows that $(I - S)u = 0$. Thus $u \in \mathrm{Fix}(S)$. We now show that $u \in \Gamma$.

Since $\|x_n - t_n\| \longrightarrow 0$ and $\|y_n - t_n\| \longrightarrow 0$, it is known that $t_{n_j} \rightharpoonup u$ and $y_{n_j} \rightharpoonup u$. Let

$$Tv = \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases}$$

where $N_C v = \{w \in \mathcal{H}_1 : \langle v - y, w\rangle \geq 0,\ \forall y \in C\}$. Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C, \nabla f)$; see [24] for more details. Let $(v, w) \in G(T)$. Then, we have

$$w \in Tv = \nabla f(v) + N_C v$$

and hence,

$$w - \nabla f(v) \in N_C v.$$

So, we have

$$\langle v - y, w - \nabla f(v)\rangle \geq 0, \quad \forall y \in C.$$

On the other hand, from

$$t_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) \text{ and } v \in C,$$

we have

$$\langle x_n - \lambda_n \nabla f_{\alpha_n}(y_n) - t_n, t_n - v\rangle \geq 0,$$

and hence,

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + \nabla f_{\alpha_n}(y_n)\rangle \geq 0.$$

Therefore, from

$$w - \nabla f(v) \in N_C v \text{ and } t_{n_j} \in C,$$

we have

$$\begin{aligned}
\langle v - t_{n_j}, w\rangle &\geq \langle v - t_{n_j}, \nabla f(v)\rangle \\
&\geq \langle v - t_{n_j}, \nabla f(v)\rangle - \langle v - t_{n_j}, \tfrac{t_{n_j} - x_{n_j}}{\lambda_{n_j}} + \nabla f_{\alpha_{n_j}}(y_{n_j})\rangle \\
&= \langle v - t_{n_j}, \nabla f(v)\rangle - \langle v - t_{n_j}, \tfrac{t_{n_j} - x_{n_j}}{\lambda_{n_j}} + \nabla f(y_{n_j})\rangle - \alpha_{n_j}\langle v - t_{n_j}, y_{n_j}\rangle \\
&= \langle v - t_{n_j}, \nabla f(v) - \nabla f(t_{n_j})\rangle + \langle v - t_{n_j}, \nabla f(t_{n_j}) - \nabla f(y_{n_j})\rangle \\
&\quad - \langle v - t_{n_j}, \tfrac{t_{n_j} - x_{n_j}}{\lambda_{n_j}}\rangle - \alpha_{n_j}\langle v - t_{n_j}, y_{n_j}\rangle \\
&\leq \langle v - t_{n_j}, \nabla f(t_{n_j}) - \nabla f(y_{n_j})\rangle - \langle v - t_{n_j}, \tfrac{t_{n_j} - x_{n_j}}{\lambda_{n_j}}\rangle - \alpha_{n_j}\langle v - t_{n_j}, y_{n_j}\rangle.
\end{aligned}$$

So, we obtain

$$\langle v-u, w\rangle \geq 0, \quad \text{as } j \longrightarrow \infty.$$

Since T is maximal monotone, we have $u \in T^{-1}0$, and hence, $u \in \mathrm{VI}(C, \nabla f)$. Thus it is clear that $u \in \Gamma$. This shows that $\omega_w(x_n) \subset \mathrm{Fix}(S) \cap \Gamma$.

Step 6. We show that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to the same point $q = P_{\mathrm{Fix}(S)\cap\Gamma} x_0$.

Assume that $\{x_n\}$ does not converge strongly to q. Then there exists $\varepsilon > 0$ and a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\|x_{n_i} - q\| > \varepsilon$ for all i. Without loss of generality we may assume that $\{x_{n_i}\}$ converges weakly to some point u. By Step 5, $u \in \mathrm{Fix}(S) \cap \Gamma$. Utilizing $q = P_{\mathrm{Fix}(S)\cap\Gamma} x_0$, the weak lower semicontinuity of $\|\cdot\|$, and relation (3.6) for $p = q$, we obtain

$$\|q - x_0\| \leq \|u - x_0\| \leq \liminf_{i\longrightarrow\infty} \|x_{n_i} - x_0\| = \lim_{n\longrightarrow\infty} \|x_n - x_0\| \leq \|q - x_0\|. \tag{3.8}$$

This implies that $\|q - x_0\| = \|u - x_0\|$. Hence $u = q$ since q is the unique element in $\mathrm{Fix}(S) \cap \Gamma$ that minimizes the distance from x_0. Also, relation (3.8) leads to $\lim_{i\longrightarrow\infty} \|x_{n_i} - x_0\| = \|q - x_0\|$. Since $\{x_{n_i} - x_0\}$ converges weakly to $q - x_0$, this shows that $\{x_{n_i} - x_0\}$ converges strongly to $q - x_0$, and hence $\{x_{n_i}\}$ converges strongly to q, a contradiction.

Therefore, $\{x_n\}$ converges strongly to q. It is easy to see that both $\{y_n\}$ and $\{z_n\}$ converge strongly to the same point q. This completes the proof. □

Corollary 3.2. *Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma$ is a nonempty bounded subset of C. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences in C generated by the following hybrid extragradient iterative algorithm with regularization:*

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)), \\ z_n = (1-\gamma_n)x_n + \gamma_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2 + 2\alpha_n\lambda_n\kappa(\kappa + \|y_n\|)\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n} x_0 \end{cases} \tag{3.9}$$

for all $n \geq 0$, where $\sup_{p\in\mathrm{Fix}(S)\cap\Gamma} \|p\| \leq \kappa$ for some $\kappa \geq 0$, and the following conditions hold for three sequences $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\gamma_n\} \subset [0, 1]$:

(i) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{\|A\|^2})$;
(ii) $\lim_{n\longrightarrow\infty} \alpha_n = 0$;
(iii) $\liminf_{n\longrightarrow\infty} \gamma_n > 0$.

Then the sequences $\{x_n\}, \{y_n\}, \{z_n\}$ are well-defined and converge strongly to the same point $q = P_{\mathrm{Fix}(S)\cap\Gamma} x_0$.

Proof. In Theorem 3.1, put $\beta_n = 0$ for all $n \geq 0$. Then iterative algorithm (3.1) reduces to (3.9), and conditions (iii) and (iv) in Theorem 3.1 reduce to condition (iii) in Corollary 3.2. Thus, by Theorem 3.1 we obtain the conclusion. □

Remark 3.3. we remark that [28, Theorem 5.7] and [4, Theorem 3.1] are weak convergence results for solving the SFP. Therefore, Theorem 3.1 as a strong convergence result is quite interesting. This result represents the supplementation, improvement, extension and development of the corresponding results in the very recent literature, for example, [28, Theorem 5.7] and [4, Theorem 3.1] to a great extent.

We now propose a modified extragradient method with regularization and prove that the sequences generated by the proposed method converge weakly to an element of $\mathrm{Fix}(S) \cap \Gamma$.

Theorem 3.4. *Let $S : C \longrightarrow C$ be a nonexpansive mapping such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following modified extragradient iterative algorithm with regularization:*

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily}, \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = (1 - \beta_n - \gamma_n)x_n + \beta_n y_n + \gamma_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) \end{cases} \tag{3.10}$$

for all $n \geq 0$, where the following conditions hold for four sequences $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\beta_n\}, \{\gamma_n\} \subset [0, 1]$:

(i) *$\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{\|A\|^2})$;*
(ii) *$\sum_{n=0}^{\infty} \alpha_n < \infty$;*
(iii) *$\beta_n + \gamma_n \leq 1$ for all $n \geq 0$;*
(iv) *$0 < \liminf_{n \longrightarrow \infty} \gamma_n \leq \limsup_{n \longrightarrow \infty} (\beta_n + \gamma_n) < 1$.*

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$.

Proof. First, as shown in the proof of Theorem 3.1, $P_C(I - \lambda \nabla f_\alpha)$ is ζ-averaged for each $\lambda \in (0, \frac{2}{\alpha + \|A\|^2})$, where $\zeta = \frac{2 + \lambda(\alpha + \|A\|^2)}{4} \in (0, 1)$. It is known that $P_C(I - \lambda \nabla f_\alpha)$ is nonexpansive. Further, for $\{\lambda_n\} \subset [a, b]$ with $a, b \in (0, \frac{1}{\|A\|^2})$, $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is ζ_n-averaged with $\zeta_n = \frac{2 + \lambda_n(\alpha_n + \|A\|^2)}{4} \in (0, 1)$. It is known immediately that $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is nonexpansive for all $n \geq 0$.

Next, we show that the sequence $\{x_n\}$ is bounded. Indeed, take a fixed $p \in \mathrm{Fix}(S) \cap \Gamma$ arbitrarily. Then, we get $Sp = p$ and $P_C(I - \lambda \nabla f)p = p$ for $\lambda \in (0, \frac{2}{\|A\|^2})$.

Utilizing Lemma 2.7, from (3.10) and inequalities (3.2), (3.3) and (3.4), we conclude that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \|(1 - \beta_n - \gamma_n)x_n + \beta_n y_n + \gamma_n S t_n - p\|^2 \\
&= (1 - \beta_n - \gamma_n)\|x_n - p\|^2 + \beta_n \|y_n - p\|^2 + \gamma_n \|S t_n - p\|^2 \\
&\quad -(1 - \beta_n - \gamma_n)\beta_n \|x_n - y_n\|^2 - (1 - \beta_n - \gamma_n)\gamma_n \|x_n - S t_n\|^2 - \beta_n \gamma_n \|y_n - S t_n\|^2 \\
&\leq (1 - \beta_n - \gamma_n)\|x_n - p\|^2 + \beta_n \|y_n - p\|^2 + \gamma_n \|t_n - p\|^2
\end{aligned}$$

$$\begin{aligned}
&-(1-\beta_n-\gamma_n)\beta_n\|x_n-y_n\|^2-(1-\beta_n-\gamma_n)\gamma_n\|x_n-St_n\|^2-\beta_n\gamma_n\|y_n-St_n\|^2\\
&\le(1-\beta_n-\gamma_n)\|x_n-p\|^2+\beta_n(\|x_n-p\|^2+2\lambda_n\alpha_n\|p\|\|y_n-p\|)\\
&+\gamma_n[\|x_n-p\|^2+2\lambda_n\alpha_n\|p\|\|y_n-p\|+(\lambda_n^2(\alpha_n+\|A\|^2)^2-1)\|x_n-y_n\|^2]\\
&-(1-\beta_n-\gamma_n)\beta_n\|x_n-y_n\|^2-(1-\beta_n-\gamma_n)\gamma_n\|x_n-St_n\|^2-\beta_n\gamma_n\|y_n-St_n\|^2\\
&\le(1-\beta_n-\gamma_n)\|x_n-p\|^2+(\beta_n+\gamma_n)(\|x_n-p\|^2+2\lambda_n\alpha_n\|p\|\|y_n-p\|)\\
&-\gamma_n(1-\lambda_n^2(\alpha_n+\|A\|^2)^2)\|x_n-y_n\|^2-(1-\beta_n-\gamma_n)\gamma_n\|x_n-St_n\|^2\\
&\le\|x_n-p\|^2+2\lambda_n\alpha_n\|p\|\|y_n-p\|\\
&-\gamma_n(1-\lambda_n^2(\alpha_n+\|A\|^2)^2)\|x_n-y_n\|^2-(1-\beta_n-\gamma_n)\gamma_n\|x_n-St_n\|^2\\
&\le\|x_n-p\|^2+2\lambda_n\alpha_n\|p\|\|y_n-p\|\\
&\le\|x_n-p\|^2+2\lambda_n\alpha_n\|p\|(\|x_n-p\|+\lambda_n\alpha_n\|p\|)\\
&\le\|x_n-p\|^2+\alpha_n(\lambda_n^2\|p\|^2+\|x_n-p\|^2)+2\lambda_n^2\alpha_n^2\|p\|^2\\
&=(1+\alpha_n)\|x_n-p\|^2+\alpha_n\lambda_n^2\|p\|^2(1+2\alpha_n)\\
&=(1+\alpha_n)\|x_n-p\|^2+\delta_n,
\end{aligned}\tag{3.11}$$

where $\delta_n=\alpha_n\lambda_n^2\|p\|^2(1+2\alpha_n)$. Since $\sum_{n=0}^{\infty}\alpha_n<\infty$ and $\{\lambda_n\}\subset[a,b]$ for some $a,b\in(0,\frac{1}{\|A\|^2})$, we derive $\sum_{n=0}^{\infty}\delta_n<\infty$. Therefore, by Lemma 2.8 we obtain that

$$\lim_{n\longrightarrow\infty}\|x_n-p\|\text{ exists for each }p\in\mathrm{Fix}(S)\cap\varGamma,\tag{3.12}$$

and the sequences $\{x_n\}$ and $\{y_n\}$ are bounded. From (3.11) we also obtain

$$\begin{aligned}
&\gamma_n(1-b^2(\alpha_n+\|A\|^2)^2)\|x_n-y_n\|^2+(1-\beta_n-\gamma_n)\gamma_n\|x_n-St_n\|^2\\
&\le\gamma_n(1-\lambda_n^2(\alpha_n+\|A\|^2)^2)\|x_n-y_n\|^2+(1-\beta_n-\gamma_n)\gamma_n\|x_n-St_n\|^2\\
&\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\lambda_n\alpha_n\|p\|\|y_n-p\|.
\end{aligned}$$

Since $0<\liminf_{n\longrightarrow\infty}\gamma_n\le\limsup_{n\longrightarrow\infty}(\beta_n+\gamma_n)<1$, we deduce from (3.12) and $\alpha_n\longrightarrow0$ that

$$\lim_{n\longrightarrow\infty}\|x_n-y_n\|=\lim_{n\longrightarrow\infty}\|x_n-St_n\|=0.\tag{3.13}$$

Furthermore, we get

$$\begin{aligned}
\|y_n-t_n\|&=\|P_C(x_n-\lambda_n\nabla f_{\alpha_n}(x_n))-P_C(x_n-\lambda_n\nabla f_{\alpha_n}(y_n))\|\\
&\le\|(x_n-\lambda_n\nabla f_{\alpha_n}(x_n))-(x_n-\lambda_n\nabla f_{\alpha_n}(y_n))\|\\
&=\lambda_n\|\nabla f_{\alpha_n}(x_n)-\nabla f_{\alpha_n}(y_n)\|\\
&\le\lambda_n(\alpha_n+\|A\|^2)\|x_n-y_n\|.
\end{aligned}$$

This together with (3.13) implies that

$$\lim_{n\longrightarrow\infty}\|y_n-t_n\|=0.\tag{3.14}$$

Note that

$$\|t_n-St_n\|\le\|t_n-y_n\|+\|y_n-x_n\|+\|x_n-St_n\|.$$

This together with (3.13) and (3.14) implies that

$$\lim_{n\longrightarrow\infty}\|t_n-St_n\|=0.\tag{3.15}$$

Also, from

$$\|x_n-t_n\|\le\|x_n-y_n\|+\|y_n-t_n\|,$$

we get

$$\lim_{n\longrightarrow\infty}\|x_n - t_n\| = 0. \tag{3.16}$$

Since $\nabla f = A^*(I - P_Q)A$ is Lipschitz continuous, we have

$$\lim_{n\longrightarrow\infty}\|\nabla f(y_n) - \nabla f(t_n)\| = 0. \tag{3.17}$$

As $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some $\hat{x}$. Thus $\omega_w(x_n) \neq \emptyset$. Repeating the same argument as in Step 5 of the proof of Theorem 3.1, we obtain that $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$.

Furthermore, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup \bar{x}$. Then, $\bar{x} \in \mathrm{Fix}(S) \cap \Gamma$. Let us show that $\hat{x} = \bar{x}$. Assume that $\hat{x} \neq \bar{x}$. From the Opial condition [20], we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n - \hat{x}\| &= \liminf_{i\longrightarrow\infty}\|x_{n_i} - \hat{x}\| < \liminf_{i\longrightarrow\infty}\|x_{n_i} - \bar{x}\| \\ &= \lim_{n\longrightarrow\infty}\|x_n - \bar{x}\| = \liminf_{j\longrightarrow\infty}\|x_{n_j} - \bar{x}\| \\ &< \liminf_{j\longrightarrow\infty}\|x_{n_j} - \hat{x}\| = \lim_{n\longrightarrow\infty}\|x_n - \hat{x}\|.\end{aligned}$$

This is a contraction. Thus, we have $\hat{x} = \bar{x}$. This implies

$$x_n \rightharpoonup \bar{x} \in \mathrm{Fix}(S) \cap \Gamma.$$

Further, from $\|x_n - y_n\| \longrightarrow 0$, it follows that $y_n \rightharpoonup \bar{x}$. This shows that both sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$. This completes the proof. □

Corollary 3.5. *Let $S : C \longrightarrow C$ be a nonexpansive mapping such that* $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$*. Let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following extragradient iterative algorithm with regularization:*

$$\begin{cases} x_0 = x \in C \text{ chosen arbitrarily,} \\ y_n = P_C(x_n - \lambda_n \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = (1 - \gamma_n)x_n + \gamma_n S P_C(x_n - \lambda_n \nabla f_{\alpha_n}(y_n)) \end{cases} \tag{3.18}$$

for all $n \geq 0$, where the following conditions hold for three sequences $\{\alpha_n\} \subset (0, \infty)$, $\{\lambda_n\} \subset (0, \frac{1}{\|A\|^2})$ and $\{\gamma_n\} \subset [0, 1]$:

(i) $\{\lambda_n\} \subset [a, b]$ *for some* $a, b \in (0, \frac{1}{\|A\|^2})$*;*
(ii) $\sum_{n=0}^{\infty} \alpha_n < \infty$*;*
(iii) $0 < \liminf_{n\longrightarrow\infty} \gamma_n \leq \limsup_{n\longrightarrow\infty} \gamma_n < 1$.

Then, both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $\hat{x} \in \mathrm{Fix}(S) \cap \Gamma$.

Proof. In Theorem 3.4, put $\beta_n = 0$ for all $n \geq 0$. Then iterative algorithm (3.10) reduces to (3.18), and conditions (iii) and (iv) in Theorem 3.4 reduce to condition (iii) in Corollary 3.5. Thus, by Theorem 3.4 we obtain the conclusion. □

Remark 3.6. Compared with [4, Theorem 3.1], Corollary 3.5 essentially coincides with it. Thus, Theorem 3.4 essentially includes [4, Theorem 3.1] as a special case.

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