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CONVEX VECTOR FUNCTIONS AND SOME APPLICATIONS

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Abstract. We investigate some properties of convex vector functions and applications to vector optimization. By introducing the definition of the concept of C_- definite operator for operators from $\mathbb{R}^n$ to $\mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ (where, $C \subset \mathbb{R}^m$ is a convex cone) which generalizes the concept of positively semi-definite matrix, we show out a second order characterization of convexity for vector functions. In addition, a first order characterization via the monotonicity of directional derivatives is also presented. For continuity, we show that the closedness is sufficient for convex vector functions to be continuous relative to any locally simplicial subset of their domains. Finally, a definition of recession maps of convex vector functions is proposed and by investigating properties of this object we obtain existence conditions for optimal solutions of vector problems with constraints.

1. Introduction

Convex functions play an important role in nonlinear analysis, especially in optimization. In the vector case, a lot of attention was paid to convex vector functions in order to enlighten the structure of this class of vector functions and apply to vector optimization ([2]-[11], [14]-[17]). In ([2], [5], [8], [10]), characterizations of convexity are expressed in terms of scalarization and in terms of first order generalized derivatives. But there are almost no results on the second order characterizations. One of the useful properties of convex vector functions is the locally Lipschitz continuity on the relative interiors of their domains ([10]). However we are also interested in what conditions under which the continuity property is still valid at the boundary points. In optimization, to get sufficient conditions for optimal solutions, we need either a second order condition or a convexity assumption. Beside this, an approach of the study on the existence condition of optimal solutions is based on recession maps ([12], [13]). The difficulty in the extension and the study of recession maps in the vector case is the set-valued structure of such maps.

The aim of this article is to investigate above problems. By introducing the definition of the concept of C_- definite operator for operators from $\mathbb{R}^n$ to $\mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ (where, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the space of continuous linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $C \subset \mathbb{R}^m$ is a convex cone) which generalizes the concept of positively semi-definite matrix, we show out a second order characterization of convexity of twice continuous differentiable vector functions. In addition, a first order characterization via the

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monotonicity of directional derivatives is also presented. For continuity, we show that the closedness is sufficient for convex vector functions to be continuous relative to any locally simplicial subset of the domains. Finally, a definition of recession maps of convex vector functions is proposed and by investigating properties of this object we get existence conditions for optimal solutions of vector problems with and without constraints. Several examples are also presented to illustrate the results.

The paper is organized as follows. In the next section we present some preliminaries on cone orders in the Eucidean space $\mathbb{R}^m$. Section 3 deals to the continuity of convex vector functions. Section 4 is devoted to characterizations of convexity. In section 5 we investigate properties of recession maps of convex vector functions. The final section presents existence conditions of optimal solutions of vector problems in terms of recession maps.

2. Preliminaries

We recall that a nonempty set $C \subset \mathbb{R}^m$ is said to be a cone if $tx \in C,\ \forall x \in C, t \geq 0$. A cone C is called pointed if $C \cap (-C) = \{0\}$. We say a set $B \subset \mathbb{R}^m$ generates a cone C if $C = \{tb \mid b \in B, t \geq 0\}$ and denote $C = \text{con}B$. The polar cone of a cone C is defined as the set $C' := \{\xi \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}) \ :\ \xi(x) \geq 0,\ \forall x \in C\}$. We list here some properties of cones from ([4], [8]) which will be used in the sequel.

Lemma 2.1. Let $C \subset \mathbb{R}^m$ be a cone.

1) If C is closed, convex and pointed, then $\text{int}C' \neq \emptyset$.
2) Assume that the cone C is closed and convex. Let $c \in \mathbb{R}^m$. Then
   (i) $c \in C$ if and only if $\xi(c) \geq 0,\ \forall \xi \in C' \setminus \{0\}$.
   (ii) Supposing that $intC \neq \emptyset$. Then $c \in intC$ if and only if $\xi(c) > 0,\ \forall \xi \in C' \setminus \{0\}$.
3) Assume that C is a closed, convex and pointed cone. Then for every neighborhood W of the origin in $\mathbb{R}^m$, there exists another neighborhood V of the origin such that
$$(V + C) \cap (V - C) \subset W.$$

A convex cone C specifies in $\mathbb{R}^m$ a partial order "$\preceq_C$" defined by
$$x, y \in \mathbb{R}^m, x \preceq_C y \Leftrightarrow y - x \in C.$$
When $\text{int}C \neq \emptyset$ we write $x \ll y$ if $y - x \in \text{int}C$. We recall here the concepts of efficiency.

Definition 2.2 ([8], Definition 2.1)**.** Let $A \subset \mathbb{R}^m$ be a nonempty set and let $a \in A$. We say that

i) a is an ideal efficient (or, ideal minimal) element of A with respect to C if $a \preceq_C x,\ \forall x \in A$. The set of ideal efficient elements of A is denoted by $\text{IMin}(A|C)$.

ii) a is an efficient (or, Pareto minimal) element of A with respect to C if $\forall x \in A,\ x \preceq_C a \Rightarrow a \preceq_C x$. The set of efficient elements of A is denoted by $\text{Min}(A|C)$.

We note that if $\text{IMin}(A|C)$ is nonempty then $\text{Min}(A|C)$=$\text{IMin}(A|C)$. In addition, if C is pointed, then $\text{IMin}(A|C)$ is a singleton. Concepts of Max and IMax are defined analogously. Clearly, $-\text{Min}A = \text{Max}(-A)$.

Let $A \subset \mathbb{R}^m$ be a nonempty set and let $a \in \mathbb{R}^m$. We say that a is an upper bound of A with respect to C if

$$x \preceq_C a, \ \forall x \in A.$$

The set of upper bounds of A is denoted by $\mathrm{Ub}(A|C)$. We say that A is bounded from above if $\mathrm{Ub}(A|C) \neq \emptyset$. The concept of lower bounds is defined analogously and the set of them is denoted by $\mathrm{Lb}(A|C)$.

Definition 2.3 ([17], Definition 2.3)**.** Let $A \subset \mathbb{R}^m$ be a nonempty set and let $a \in \mathbb{R}^m$. We say that

i) a is an ideal supremal point of A with respect to C if $a \in \mathrm{IMin}(\mathrm{Ub}A|C)$, i.e.,

$$\begin{cases} x \preceq_C a, \ \forall x \in A \\ a \preceq_C y, \ \forall y \in \mathrm{Ub}(A|C). \end{cases}$$

The set of ideal supremal points of A is denoted by $\mathrm{ISup}(A|C)$.

ii) a is a supremal point of A with respect to C if $a \in \mathrm{Min}(\mathrm{Ub}A|C)$, i.e.,

$$\begin{cases} x \preceq_C a, \ \forall x \in A \\ \forall y \in \mathrm{Ub}(A|C), y \preceq_C a \Rightarrow a \preceq_C y. \end{cases}$$

The set of supremal points of A is denoted by $\mathrm{Sup}(A|C)$.

iii) a is an ideal infimal point of A with respect to C if $a \in \mathrm{IMax}(\mathrm{Lb}A|C)$, i.e.,

$$\begin{cases} a \preceq_C x, \ \forall x \in A \\ y \preceq_C a, \ \forall y \in \mathrm{Lb}(A|C). \end{cases}$$

The set of ideal infimal points of A is denoted by $\mathrm{IInf}(A|C)$.

iv) a is an infimal point of A with respect to C if $a \in \mathrm{Max}(\mathrm{Lb}A|C)$, i.e.,

$$\begin{cases} a \preceq_C x, \ \forall x \in A \\ \forall y \in \mathrm{Lb}(A|C), a \preceq_C y \Rightarrow y \preceq_C a. \end{cases}$$

The set of infimal points of A is denoted by $\mathrm{Inf}(A|C)$.

Like the case of Min and IMin, we should note that if $\mathrm{ISup}(A|C)$ is nonempty then $\mathrm{ISup}(A|C)$=$\mathrm{Sup}(A|C)$ and in addition, if C is pointed, then $\mathrm{ISup}(A|C)$ is a singleton. Clearly, $-\mathrm{Sup}A = \mathrm{Inf}(-A)$.

There are some different definitions of other authors on the concept of supremum. Among of them, the definition of T. Tanino ([14]) is remarkable. Definition 2.3 is an extension of the usual definition of supremum in $\mathbb{R}$ by a natural way: minimum of the set of upper bounds of A. It seems suitable for establishing several results concerning convex vector functions as shown in ([17]) and in sections 5,6 below.

From now on, when there is no afraid of confusion, we omit "with respect to C" and "$|_C$" in the definitions above.

We list here some results from ([17]) which will be needed in the sequel. A sequence $\{y_k\}_k \subset \mathbb{R}^m$ is called decreasing (with respect to C) if $y_{k+1} \preceq y_k, \forall k$. It is called bounded from below if there exists $a \in \mathbb{R}^m$ such that $a \preceq y_k, \forall k$.

Lemma 2.4 ([17], Lemma 2.8)**.** Assume that the order cone $C \subset \mathbb{R}^m$ is closed, convex and pointed. Let $\{y_k\}_k \subset \mathbb{R}^m$ be a decreasing sequence. If $\{y_k\}_k$ is bounded from below then it is convergent and

$$\lim_{k\to\infty} y_k = IInf\{y_k \mid k \in \mathbb{N}\}.$$

We say that a subset $A \subset \mathbb{R}^m$ is linearly ordered if for every $x, y \in A$, $x \preceq y$ or $y \preceq x$.

Proposition 2.5 ([17], Proposition 2.9, Remark 2.10)**.** *Assume that the order cone $C \subset \mathbb{R}^m$ is closed, convex and pointed. If a nonempty linearly ordered subset A of $\mathbb{R}^m$ is bounded from above, then ISup$A \neq \emptyset$ and there exists an increasing sequence in A converging to ISup$A$.*

We should note that an analogous result holds true for IInf.

Theorem 2.6 ([17], Theorem 2.16, Remark 2.18)**.** *Assume that the order cone $C \subset \mathbb{R}^m$ is closed, convex and pointed. Let A be a nonempty subset of $\mathbb{R}^m$. Then Sup$A \neq \emptyset$ if and only if A is bounded from above. In this case, we have*

$$Ub(A) = Sup(A) + C.$$

By this theorem, it is obvious that if Sup$A$ is a singleton then Sup$A$=ISup$A$.

Now let $D \subset \mathbb{R}^n$ be a nonempty set and let $f : D \to \mathbb{R}^m$. The epigraph of f with respect to C is defined as the set

$$\text{epi}f := \{(x, z) \in D \times \mathbb{R}^m \ : \ f(x) \preceq z\}.$$

We say that f is convex (resp., closed) with respect to C if epi$f$ is convex (resp., closed) in $\mathbb{R}^n \times \mathbb{R}^m$. It can see that f is convex if and only if D is convex and for every $x, y \in D,\ \lambda \in [0, 1]$, we have

$$f(\lambda x + (1-\lambda)y) \preceq \lambda f(x) + (1-\lambda)f(y).$$

f is called strictly convex with respect to C if

$$f(\lambda x + (1-\lambda)y) \ll \lambda f(x) + (1-\lambda)f(y), \forall x, y \in D, x \neq y, \lambda \in (0, 1).$$

The relation between scalar convex functions and vector convex functions is expressed in the following lemma which will be used in Section 4.

Lemma 2.7 ([8], Lemma 2.1)**.** Assume that the order cone $C \subset R^m$ is closed and convex. Let f be a vector function from a nonempty and convex set $D \subset R^n$ to R^m. Then

i) f is convex with respect to C if and only if ξf is convex, for every $\xi \in C' \setminus \{0\}$.

ii) Supposing that $intC \neq \emptyset$, f is strictly convex with respect to C if and only if ξf is strictly convex, for every $\xi \in C' \setminus \{0\}$.

The level set of a vector function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ at $a \in \mathbb{R}^m$ with respect to the cone C is defined as the set

$$\text{lev}_a f := \{x \in D \mid f(x) \preceq a\}.$$

It is immediately from definitions that level sets of a convex vector function are convex sets.

Lemma 2.8. Assume that $\mathbb{R}^m$ is ordered by a convex cone $C \subset \mathbb{R}^m$. Let f be a vector function from a nonempty subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. If f is closed, then $\text{lev}_a f$ is also closed for all $a \in \mathbb{R}^m$. In addition, if C is closed with $\text{int}C \neq \emptyset$, then the converse is true.

Proof. Let $a \in \mathbb{R}^m$ be arbitrary and let a sequence $\{x_k\} \subset \text{lev}_a f$ be converging to some $x \in \mathbb{R}^n$. Then the sequence $\{(x_k, a)\}_k \subset \text{epi}f$ and it converges to (x, a). Since $\text{epi}f$ is closed we have $(x, a) \in \text{epi}f$. Hence $f(x) \preceq a$, i.e., $x \in \text{lev}_a f$. Thus $\text{lev}_a f$ is closed.

Conversely, let any sequence $\{(x_k, a_k)\}_k \subset \text{epi}f$ be converging to some $(x, a) \in \mathbb{R}^n \times \mathbb{R}^m$. Let $c \in \text{int}C$ and real $t > 0$ be arbitrary. Then $a \in a + tc - \text{int}C$ which implies $a_k \in a + tc - \text{int}C$ for k sufficiently large. Hence $x_k \in \text{lev}_{a+tc} f$ for k sufficiently large. By the closedness of level sets, one get $x \in \text{lev}_{a+tc} f$, i.e.,

$$f(x) \preceq a + tc, \forall t > 0.$$

Taking t to 0, since C is closed we have $f(x) \preceq a$ which equivalent to $(x, a) \in \text{epi}f$. Thus f is closed. □

3. Continuity

From now on, we always assume that $\mathbb{R}^m$ is ordered by a convex cone C.

Let f be a vector function from a nonempty set $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$ and let $S \subset D$, $x \in S$. We say that f is lower (resp., upper) semicontinuous relative to S at x with respect to C if for every neighborhood W of $f(x)$, there exists a neighborhood V of x such that

$$y \in V \cap S \Rightarrow f(y) \in W + C \text{ (resp., } f(y) \in W - C).$$

f is called lower (resp., upper) semicontinuous relative to S if it is lower (resp., upper) semicontinuous relative to S at every $x \in S$. The concept of continuity relative to S is defined analogously. When $S = D$ we omit the phrase "relative to S" in the definitions above. In this case, if $m = 1$ and $C = \mathbb{R}_+$ then we get the usual concepts of lower and upper semicontinuity for scalar functions. But it is not like the scalar case, a vector function can be closed but not lower semicontinuous. For instant, let's consider the function $f : [0, +\infty) \to \mathbb{R}^2$ defined by

$$f(x) := \begin{cases} (0, 0), & x = 0, \\ (\frac{1}{x}, 0), & x > 0. \end{cases}$$

$\mathbb{R}^2$ is ordered by the cone $C := \text{cone}(\text{co}\{(-1, 1), (1, 1)\})$, where, $\text{co}A$ denotes the convex hull of A. Then f is closed but not lower semicontinuous at 0 with respect to C. The reason of this fact comes from the nature of the partial order generated by a cone. In ([1]) some characterizations of semicontinuity of vector functions are given. Here we present a sufficient condition for a vector function to be semicontinuous.

Proposition 3.1. *Assume that the order cone $C \subset \mathbb{R}^m$ is closed, convex and pointed with $\text{int}C \neq \emptyset$. Let f be a vector function from a nonempty set $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$ and let $S \subset D, x_0 \in S$. If f is closed and upper semicontinuous relative to S at x_0, then f is lower semicontinuous relative to S at x_0.*

To proof Proposition 3.1 we need the following lemmata.

Lemma 3.2. Let $C \subset \mathbb{R}^m$ be a closed, convex and pointed cone and let $\xi \in \text{int}C'$. Then for every number $\alpha \geq 0$, the set

$$A := \{c \in C \ : \ \xi(c) \leq \alpha\}$$

is compact.

Proof. Clearly A is closed. To complete the proof we only need to show that A is bounded. This is equivalent to showing that for every sequence in A, there exists a bounded subsequence. Let $\{c_k\}_k \subset A$ be an arbitrary sequence. Without loss of generality, we may assume that $c_k \neq 0$, for all k. Then the sequence $\{\frac{c_k}{\|c_k\|}\}_k$ has a subsequence $\{\frac{c_{k_l}}{\|c_{k_l}\|}\}_l$ which converging to some unit vector $c_0 \in C$. Since $\xi \in$intC', by Lemma 2.1 we have

$$\lim_{l \to \infty} \frac{\xi(c_{k_l})}{\|c_{k_l}\|} = \xi(c_0) > 0.$$

This fact together the boundedness of the sequence $\{\xi(c_{k_l})\}_l$ imply the boundedness of $\{c_{k_l}\}_l$. The proof is complete. □

Lemma 3.3. Assume that the order cone $C \subset \mathbb{R}^m$ is closed, convex and pointed with int$C \neq \emptyset$. Let $\xi \in \text{int}C'$, $a \in \mathbb{R}^m$ and a number $\delta > 0$ be arbitrary. Then for every number $\alpha \in \mathbb{R}$ the set

$$A := \{y \in \mathbb{R}^m \ : \ \xi(y) \geq \alpha\} \cap [\overline{B}(a, \delta) - C]$$

is compact.

Proof. The result is deduced from Lemma 3.2 and the fact that there exists $c \in \text{int}C$ such that $\bar{B}(a, \delta) - C \subset c - C$. □

Proof of Proposition 3.1. Suppose to the contrary that f is not lower semicontinuous relative to S at x_0. Then there exists a possitive number ϵ and a sequence $\{x_k\}_k \subset S$ converges to x_0 such that

$$f(x_k) \notin \text{B}(f(x_0), \epsilon) + C, \ (\forall k) \tag{3.1}$$

where $\text{B}(f(x_0), \epsilon)$ denotes the open ball with the center $f(x_0)$ and the radius ϵ. One of two following cases holds.

i) The sequence $\{f(x_k)\}_k$ is bounded. Without loss of generality we may assume that $\{f(x_k)\}_k$ converges to some $y_0 \in \mathbb{R}^m$. By (3.1),

$$y_0 \notin \text{B}(f(x_0), \epsilon) + C. \tag{3.2}$$

On other hand, since f is closed one has $(x_0, y_0) \in \text{epi}f$. Hence $y_0 \succeq f(x_0)$ which contradicts (3.2).

ii) The sequence $\{f(x_k)\}_k$ is not bounded. Since f is upper semicontinuous relative to S at x_0,

$$f(x_k) \in \overline{\text{B}}(f(x_0), \epsilon) - C$$

when $k > K$ for K sufficiently large, where $\overline{\text{B}}(f(x_0), \epsilon)$ denotes the closed ball with the center $f(x_0)$ and the radius ϵ. By Lemma 2.1, int$C' \neq \emptyset$. Pick an element $\xi \in \text{int}C'$. Put

$$\alpha := min\{\xi(y) \ : \ y \in \overline{\text{B}}(f(x_0), \epsilon)\} - 1.$$

By Lemma 3.3, the set

$$(\overline{\mathrm{B}}(f(x_0), \epsilon) - C) \cap \{y \ : \ \xi(y) \geq \alpha\}$$

is compact. Then the set

$$E := (\overline{\mathrm{B}}(f(x_0), \epsilon) - C) \cap \{y \ : \ \xi(y) = \alpha\}$$

is also compact. We note that

$$E \cap (\overline{\mathrm{B}}(f(x_0), \epsilon) + C) = \emptyset. \tag{3.3}$$

For every $k > K$, there exists $y_k \in \overline{\mathrm{B}}(f(x_0), \epsilon),\ c_k \in C$ such that

$$f(x_k) = y_k - c_k.$$

Since the sequence $\{f(x_k)\}_k$ is not bounded we may assume that

$$\|c_k\| \to +\infty.$$

Then by Lemma 3.2 we also may assume that

$$\lim \xi(c_k) = +\infty. \tag{3.4}$$

We have

$$\begin{aligned} f(x_k) &= y_k - c_k \\ &= y_k - \left(\frac{\xi(y_k) - \alpha}{\xi(c_k)}\right) c_k - \left(1 - \frac{\xi(y_k) - \alpha}{\xi(c_k)}\right) c_k \\ &= z_k - \lambda_k c_k, \end{aligned}$$

where

$$\begin{aligned} z_k &= y_k - \left(\frac{\xi(y_k) - \alpha}{\xi(c_k)}\right) c_k \in E \\ \lambda_k &= 1 - \frac{\xi(y_k) - \alpha}{\xi(c_k)} \end{aligned}$$

By (3.4), $\lambda_k > 0$ when k sufficiently large. Then we have, $f(x_k) \preceq z_k$, i.e.,

$$(x_k, z_k) \in \text{epi} f,$$

for k sufficiently large. Since $\{z_k\}_{k>K} \subset E$ and E is compact we may assume $\{z_k\}_k$ converges to some

$$z_0 \in E. \tag{3.5}$$

By the closedness of f, he have

$$z_0 \succeq f(x_0).$$

This fact together (3.3) and (3.5) give us a contradiction. The proof is complete.□

Lemma 3.4. Assume that the order cone $C \subset \mathbb{R}^m$ is closed, convex and pointed. Let f be a vector function from a nonempty set $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$ and let $S \subset D, x_0 \in S$. Then f is continuous relative to S at x_0 if and only if f is lower and upper semicontinuous relative to S at x_0.

Proof. $\Rightarrow$: It is immediate from definitions.
$\Leftarrow$: Let W be an arbitrary neibourhood of $f(x_0)$. By Lemma 2.1, there exists a neighborhood W' of $f(x_0)$ such that

$$(W' + C) \cap (W' - C) \subset W. \tag{3.6}$$

By the semicontinuity of f at x_0, there exists a neighborhood V of x_0 such that

$$x \in V \cap S \Rightarrow f(x) \in (W' + C) \cap (W' - C). \tag{3.7}$$

(3.6) and (3.7) implies the continuity relative to S of f at x_0. □

Convex vector functions have several nice properties as scalar convex functions ([5], [8]-[10], [14]-[17]). One of them is the Lipschitzian continuity property.

Proposition 3.5 ([10], Theorem 3.1)**.** *Assume that the closure clC of the order cone $C \subset \mathbb{R}^m$ is pointed. Let f be a convex vector function from a nonempty convex set $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Then f is locally Lipschitz on the relative interior riD of D.*

In general, convex vector functions are not continuous at the boundary points of the domains. However, under certain conditions this property is still valid. A such condition concerns to the concept of locally simplicial sets. We recall some background from ([13]). A subset $S \subset \mathbb{R}^n$ is called locally simplicial if for each $x \in S$, there exist a finite collection of simplices $S_1, \ldots, S_k \subset S$ such that, for some neighborhood U of x

$$U \cap S = U \cap (S_1 \cup \cdots \cup S_k).$$

The class of locally simplicial sets includes, besides line segments and other simplices, all polytopes and polyhedral convex sets. It also includes all relative open convex sets.

Let T be a simplex with vertices $x_0, x_1, \ldots, x_k$ and let $x \in T$. Then T can be triangulated into simplices having x as a vertex, i.e. each $y \in T$ belongs to a simplex whose vertices are x and m of the $m+1$ vertices of T. Base on this fact we have

Theorem 3.6. *Let f be a convex vector function from a nonempty convex set $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$ and let S be any locally simplicial subset of D. Then f is upper semicontinuous relative to S. In addition, if the order cone C is closed, pointed with $intC \neq \emptyset$ and f is closed, then f is continuous relative to S.*

Proof. Let $x \in S$ be arbitrary. Then there exist simplices $S_1, \ldots, S_k \subset S$ such that S_i contains x, for every $i = \overline{1,k}$, and $U \cap S = U \cap (S_1 \cup \cdots \cup S_k)$, for some neighborhood U of x. Each S_i can be triangulated into simplices having x as vertex. Denote these simplices by $T_1, \ldots, T_l$. Then $U \cap S = U \cap (T_1 \cup \cdots \cup T_l)$. Hence to prove the upper continuity relative to S of f at x it only needs to prove the upper continuity relative to T_i of f at x, for every $i = \overline{1,l}$. Suppose that T_i is a p-simplex with vertices $x, x_1, \ldots, x_p$. Without loss of generality we may assume that $x = 0, p = n, x_j = e_j, j = \overline{1,n}$, where $\{e_1, \ldots, e_n\}$ is the canonical base of $\mathbb{R}^n$. Let $\epsilon > 0$ be arbitrary. Put

$$r = \begin{cases} 1, & \text{if } f(e_j) - f(0) = 0, \forall j = \overline{1,n}, \\ \frac{\epsilon}{\sum_{j=1}^{n} \|f(e_j) - f(0)\|}, & \text{if } \exists j : f(e_j) - f(0) \neq 0. \end{cases}$$

Let $y = (\lambda_1, \ldots, \lambda_n) \in B(0, r) \cap T_i$. Then y can represent as

$$y = (1 - \lambda_1 - \cdots - \lambda_n)0 + \lambda_1 e_1 + \cdots + \lambda_n e_n,$$

where $\lambda_j \geq 0, j = \overline{1, n}, \lambda_1 + \cdots + \lambda_n \leqslant 1$. Hence by convexity of f, we have

$$f(y) \preceq (1 - \lambda_1 - \cdots - \lambda_n) f(0) + \lambda_1 f(e_1) + \cdots + \lambda_n f(e_n),$$

or,

$$f(y) - f(0) \preceq \sum_{i=j}^{n} \lambda_j (f(e_j) - f(0)).$$

Since

$$\begin{aligned}\|\sum_{j=1}^{n} \lambda_j (f(e_j) - f(0))\| &\leqslant \sum_{j=1}^{n} \lambda_j \|f(e_j) - f(0)\| \\ &\leqslant \|y\| \sum_{j=1}^{n} \|f(e_j) - f(0)\| \\ &< r \sum_{j=1}^{n} \|f(e_j) - f(0)\| \\ &\leqslant \epsilon\end{aligned}$$

one has $f(y) - f(0) \in B(0, \epsilon) - C$ which implies the upper semicontinuity relative to T_i of f at $x = 0$. Thus f is upper semicontinuous relative to S.

Now, assume in addition that C is closed , pointed with int$C \neq \emptyset$ and f is closed. Then by Proposition 3.1, f is lower semicontinuous relative to S. Hence by Lemma 3.4, f is continuous relative to S. The proof is complete. □

4. Characterizations

Firstly, we recall some definitions. Let $D \subset \mathbb{R}^n$ be a nonempty set and let $x \in D$. Denote by $T_0(D; x)$ the cone of feasible directions of D at x, i.e.,

$$T_0(D; x) = \{v \in \mathbb{R}^n \mid \exists t_0 > 0 \text{ such that } x + tv \in D, \forall t \in [0, t_0]\}.$$

Let $f : D \to \mathbb{R}^m, x \in D, v \in T_D(x)$. The directional derivative of f at x in the direction v is defined as the following limit

$$f'(x; v) = \lim_{t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

Suppose that int$D \neq \emptyset$ and let $x \in$ int$D$. f is said to be Gateaux differentiable at x if $f'(x; v)$ exists for every $v \in \mathbb{R}^n$ and there is a continuous linear map, say $D_G f(x)$, such that

$$f'(x; v) = D_G f(x)(v), \forall v \in \mathbb{R}^n.$$

Definition 4.1. (i) Let a map $A : D \subset \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be given. We say that A is monotone (resp., strictly monotone) with respect to C if

$$A(x)(y - x) + A(y)(x - y) \preceq 0, \forall x, y \in D.$$

$$(\textit{resp.},\ A(x)(y - x) + A(y)(x - y) \ll 0, \forall x, y \in D.)$$

(ii) Let a vector bifunction $B : D \times \mathbb{R}^n \to \mathbb{R}^m$ be given. We say that B is monotone (resp., strictly monotone) with respect to C if

$$B(x, y-x) + B(y, x-y) \preceq 0, \forall x, y \in D.$$

$$(resp.,\ B(x, y-x) + B(y, x-y) \ll 0, \forall x, y \in D.)$$

When $m = 1$ and $C = \mathbb{R}_+$ we return to the classical concept of monotonicity. A characterization of convexity of scalar functions via the monotonicity of their directional derivatives is shown out in the following result.

Lemma 4.2 ([9])**.** Let ϕ be a lower semicontinuous function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}$. Suppose that $\phi'(x; v)$ exists for every $x \in D, v \in T_0(D; x)$. Then ϕ is convex (resp.; strictly convex) if and only if $\phi'(.;.)$ is monotone (resp.; strictly monotone)in the classical sense.

Lemma 4.3. If a vector function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ is lower semicontinuous with respect to C, then ξf is also lower semicontinuous for every $\xi \in C'$.

Proof. Let $\xi \in C'$. We may assume that $\xi \neq 0$. Let $x \in D$, $\epsilon > 0$ be arbitrary. Set $\epsilon' = \frac{\epsilon}{\|\xi\|}$ (where, $\|\xi\| := \sup\limits_{\|x\| \leq 1} |\xi(x)|$). By the definition of lower semicontinuity, there exists a neighborhood V of x such that

$$x' \in V \cap D \Rightarrow f(x') \in f(x) + B(0, \epsilon') + C.$$

We have

$$\xi[f(x') - f(x)] \geq \inf \xi[B(0, \epsilon') + C] = \inf \xi[B(0, \epsilon')] = -\epsilon' \|\xi\| = -\epsilon$$

which implies the lower semicontinuity of ξf at x. Since $x \in D$ is arbitrary the proof is complete. □

Theorem 4.4. *Assume that the order cone $C \subset \mathbb{R}^m$ is convex and closed. Let f be a lower semicontinuous vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Suppose that $f'(x; v)$ exists for every $x \in D, v \in T_0(D; x)$. Then f is convex (resp.; strictly convex) if and only if $f'(.;.)$ is monotone (resp.; strictly monotone).*

Proof. For every $\xi \in C'$, ξf is lower semicontinuous by Lemma 4.3. Clearly $(\xi f)'(x; v)$ exists for every $x \in D, v \in T_D(x)$. By applying Lemma 2.7 , Lemma 4.2 and Lemma 2.1, we have

$$\begin{aligned}
f \text{ is convex} &\Leftrightarrow \xi f \text{ is convex},\ \forall \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow (\xi f)'(x; y-x) + (\xi f)'(y; x-y) \leq 0,\ \forall x, y \in D, \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow \xi\big[f'(x; y-x) + f'(y; x-y)\big] \leq 0,\ \forall x, y \in D, \xi \in C' \setminus \{0\}. \\
&\Leftrightarrow f'(x; y-x) + f'(y; x-y) \preceq 0,\ \forall x, y \in D. \\
&\Leftrightarrow f'(.;.) \text{ is monotone}.
\end{aligned}$$

Analogously, we have

$$
\begin{aligned}
&f \text{ is strictly convex} \Leftrightarrow \xi f \text{ is strictly convex}, \ \forall \xi \in C' \setminus \{0\}.\\
&\quad \Leftrightarrow (\xi f)'(x; y-x) + (\xi f)'(y; x-y) < 0, \ \forall x, y \in D, x \neq y, \xi \in C' \setminus \{0\}.\\
&\quad \Leftrightarrow \xi\big[f'(x; y-x) + f'(y; x-y)\big] < 0, \ \forall x, y \in D, x \neq y, \xi \in C' \setminus \{0\}.\\
&\quad \Leftrightarrow f'(x; y-x) + f'(y; x-y) \ll 0, \ \forall x, y \in D, \ x \neq y.\\
&\quad \Leftrightarrow f'(.;.) \text{ is strictly monotone.}
\end{aligned}
$$

The proof is complete. □

From Theorem 4.4 and definitions we have immediately the following result which generalize the corresponding famous well- known result in convex analysis.

Corollary 4.5. *Assume that the order cone $C \subset \mathbb{R}^m$ is convex and closed. Let f be a vector function from a nonempty open convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Suppose that f is lower semicontinuous and Gateaux differentiable on D. Then f is convex (resp.; strictly convex) if and only if $D_G f$ is monotone (resp.; strictly monotone).*

Example 4.6. Let $\mathbb{R}^3$ be ordered by the cone $C = con(co\{(1,0,1),(0,-1,-1),(0,0,1)\})$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $f(x_1, x_2) = ({x_1}^2 - x_1 + x_2, -{x_2}^2 + x_1 - x_2, {x_1}^2 - {x_2}^2)$. For every $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$, we have

$$
\begin{aligned}
Df(x) &= ((2x_1 - 1, 1), (1, -2x_2 - 1), (2x_1, -2x_2))\\
Df(y) &= ((2y_1 - 1, 1), (1, -2y_2 - 1), (2y_1, -2y_2)).
\end{aligned}
$$

Then

$$
\begin{aligned}
(Df(x) - Df(y))(x - y) &= (2(x_1 - y_1)^2, -2(x_2 - y_2)^2, 2(x_1 - y_1)^2 - 2(x_2 - y_2)^2)\\
&= 2(x_1 - y_1)^2(1,0,1) + 2(x_2 - y_2)^2(0,-1,-1)\\
&\in C.
\end{aligned}
$$

Hence Df is monotone with respect to C which implies the convexity of f by Corollary 4.5.

Now we investigate the second order characterization of convexity. Let X, Y be normed spaces. The space of continuous linear maps from X to Y is denoted by $\mathcal{L}(X, Y)$. The norm in $\mathcal{L}(X, Y)$ is defined as usual by

$$A \in \mathcal{L}(X, Y), \|A\| := \sup\{\|A(x)\| \ |x \in X, \|x\| \leq 1\}.$$

Let $\mathcal{A} \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$. For every $x, y \in \mathbb{R}^n$, we denote

$$\mathcal{A}(x, y) := [\mathcal{A}(x)](y).$$

Definition 4.7. We say that a map $\mathcal{A} \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ is C_definite if

$$\mathcal{A}(x, x) \in C, \ \forall x \in \mathbb{R}^n.$$

Let $D \subset \mathbb{R}^n$ be a nonempty set. An operator $\mathcal{F} : D \to \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$ is called C_ definite if $\mathcal{F}(x)$ is C_definite for every $x \in D$.

We see immediately that when m=1 and $C = \mathbb{R}_+$, $\mathcal{A}$ is C_definite if and only if the matrix representing $\mathcal{A}$ is semi positively semi-definite in the usual meaning.

Theorem 4.8. *Assume that the order cone $C \subset \mathbb{R}^m$ is closed and convex. Let $D \subset \mathbb{R}^n$ be a nonempty, convex and open set and let $F : D \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ be a continuously differentiable map. Then F is monotone with respect to C if and only if its derivative operator DF is C_definite.*

Proof. $\underline{\Rightarrow}$: Suppose to the contrary that DF is not $C-$definite. Then there exist $x_0 \in D, y_0 \in \mathbb{R}^n$ such that

$$DF(x_0)(y_0, y_0) \notin C. \tag{4.1}$$

Put $\phi(t) = DF(x_0 + ty_0)(y_0, y_0)$. Since C is closed, by (4.1) there exists $\delta > 0$ such that

$$B(\phi(0), \delta) \cap C = \emptyset. \tag{4.2}$$

Since DF is continuous on a neighborhood of x_0, ϕ is also continuous on a neighborhood of 0. Then there exists a number $\epsilon \in (0, 1)$ such that

$$\phi(t) \in B(\phi(0), \delta), \ \forall t \in [0, \epsilon]. \tag{4.3}$$

Put $\Phi(t) = F(x_0 + ty_0)(y_0)$. Then $\Phi = \varphi \circ F \circ \psi$, where,

$$\psi : t \mapsto x_0 + ty_0, \ \varphi : A \in L(\mathbb{R}^n, \mathbb{R}^m) \mapsto A(y_0).$$

Then $\Phi(t)$ is continuously differentiable on a neighborhood of $[0, \epsilon]$. By Mean value theorem for vector functions, there exist $\tau_1, \ldots, \tau_k \in [0, \epsilon], \lambda_1, \ldots, \lambda_k \geq 0, \lambda_1 + \cdots + \lambda_k = 1$ such that

$$\Phi(\epsilon) - \Phi(0) = \sum_{i=1}^{k} \lambda_i D\Phi(\tau_i)(\epsilon). \tag{4.4}$$

By chain rule,

$$D\Phi(\tau_i)(\epsilon) = \epsilon DF(x_0 + \tau_i y_0)(y_0, y_0) = \epsilon\phi(\tau_i), i = 1, \ldots, k. \tag{4.5}$$

From (4.3), (4.4) and (4.5), one has

$$\begin{aligned}
(F(x_0 + \epsilon y_0) - F(x_0))(x_0 + \epsilon y_0 - x_0) &= \epsilon(F(x_0 + \epsilon y_0) - F(x_0))(y_0) \\
&= \epsilon(\Phi(\epsilon) - \Phi(0)) \\
&= \epsilon(\sum_{i=1}^{k} \lambda_i D\Phi(\tau_i)(\epsilon)) \\
&= \epsilon^2(\sum_{i=1}^{k} \lambda_i \phi(\tau_i)) \\
&\in \epsilon^2 B(\phi(0), \delta).
\end{aligned}$$

By (4.2), $(F(x_0 + \epsilon y_0) - F(x_0))(x_0 + \epsilon y_0 - x_0) \notin C$. Hence F is not monotone on D which contradicts the assumptions.

$\underline{\Leftarrow}$: Let $x, y \in D$ be arbitrary. Consider the function

$$\Phi(t) = F(x + t(y - x))(y - x).$$

Clearly Φ is continuously differentiable on an open interval which contains $[0,1]$. By Mean value theorem, there exist $\tau_1, \dots, \tau_k \in [0,1], \lambda_1, \dots, \lambda_k \geq 0, \lambda_1 + \dots + \lambda_k = 1$ such that

$$\Phi(1) - \Phi(0) = \sum_{i=1}^{k} \lambda_i D\Phi(\tau_i)(1).$$

Hence

$$\begin{aligned}(F(y) - F(x))(y - x) &= \Phi(1) - \Phi(0) \\ &= \sum_{i=1}^{k} \lambda_i D\Phi(\tau_i)(1) \\ &= \sum_{i=1}^{k} \lambda_i DF(x + \tau_i(y - x))(y - x, y - x) \\ &\in C(\text{ since } DF \text{ is } C\_\text{definite}).\end{aligned}$$

Thus F is monotone. The proof is complete.

□

Theorem 4.9. *Assume that the order cone $C \subset \mathbb{R}^m$ is closed and convex. Let $D \subset \mathbb{R}^n$ be a nonempty convex and open set and let $f : D \to \mathbb{R}^m$ be a twice continuously differentiable vector function. Then f is convex if and only if $D^2 f$ is $C-$definite (where $D^2 f$ denotes the second order derivative map of f on D).*

Proof. By Corrolary 4.5 and Theorem 4.8, we have

$$\begin{aligned} f \text{ is convex } &\Leftrightarrow Df \text{ is monotone} \\ &\Leftrightarrow D^2 f \text{ is } C-\text{definite} . \end{aligned}$$

□

From Theorem 4.9 and from a note after Definition 4.7 we obtain immediately the following famous well-known classic result.

Corollary 4.10. *Let $D \subset \mathbb{R}^n$ be a nonempty convex and open set and let $f : D \to \mathbb{R}$ be a twice continuously differentiable function. Then f is convex if and only if the Hessian matrix $H_f(x)$ of f at every $x \in D$ is positively semi-definite.*

Example 4.11. Let $\mathbb{R}^3$ be ordered by the cone $C = con(co\{(1,0,1),(0,-1,-1),(0,0,1)\})$. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined as in Example 4.6, i.e., $f(x_1, x_2) = ({x_1}^2 - x_1 + x_2, -{x_2}^2 + x_1 - x_2, {x_1}^2 - {x_2}^2)$. By computing we have

$$D^2 f(x) = \left(\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \right), \forall x \in \mathbb{R}^2.$$

Then

$$\begin{aligned} D^2 f(x)(y,y) &= (2{y_1}^2, -2{y_2}^2, 2{y_1}^2 - 2{y_2}^2) \\ &= 2{y_1}^2(1,0,1) + 2{y_2}^2(0,-1,-1) \\ &\in C, \forall x, y \in \mathbb{R}^2. \end{aligned}$$

Hence $D^2 f$ is C_definite which implies the convexity of f by Theorem 4.9.

5. Recession maps

We recall that the recession cone of a convex subset A of a real vector space X is defined as the set

$$A_\infty := \{u \in X \mid x + tu \in A, \forall x \in A, t \geq 0\}.$$

The following basis result of recession cone in convex analysis will be needed.

Lemma 5.1 ([13])**.** (i) Let $A \subset \mathbb{R}^k$ be a nonempty, closed and convex set. Then A is compact if and only if $A_\infty = \{0\}$.

(ii) Let $A \subset \mathbb{R}^k$ be a nonempty, closed and convex set. Then A_∞ is closed and convex.

(iii) Let $A \subset \mathbb{R}^k$ be a nonempty and convex set and let $u \in \mathbb{R}^k$. Then $u \in A_\infty$ if and only if $A + u \subset A$.

(iv) Let $A \subset \mathbb{R}^k$ be a nonempty, convex and closed set and let $u \in \mathbb{R}^k$. Then $u \in A_\infty$ if and only if $\exists x \in A : x + \lambda u \in A, \forall \lambda \geq 0$.

(v) Let $(A_i)_{i\in I}$ be a family of convex subsets of $\mathbb{R}^k$ such that $\underset{i\in I}{\cap} A_i \neq \emptyset$. Then

$$\underset{i\in I}{\cap} (A_i)_\infty \subset (\underset{i\in I}{\cap} A_i)_\infty.$$

If in addition A_i is closed for every $i \in I$, then the converse holds.

Let f be a convex vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. For each $x \in \mathbb{R}^n$, set

$$S_x := \{u \in \mathbb{R}^m \mid (x, u) \in (\text{epi} f)_\infty\}.$$

The following definition is suggested from the concept of recession map of scalar convex functions.

Definition 5.2. The recession map of f is defined as follows.

$$f_\infty(x) := \begin{cases} \text{Min} S_x, & S_x \neq \emptyset, \\ \emptyset, & S_x = \emptyset. \end{cases}$$

Example 5.3. Let $\mathbb{R}^2$ be ordered by the positive orthant cone $\mathbb{R}^2{}_+$ and let $f : (0, +\infty) \to \mathbb{R}^2$ be defined by $f(x) = (x, x + \frac{1}{x})$. Then f is convex with respect to $\mathbb{R}^2{}_+$ since component functions are scalar convex (on $(0, +\infty)$).By computing we obtain

$$(\text{epi} f)_\infty = \{(x, u_1, u_2) \in \mathbb{R}^3 \mid x \geq 0, u_1 \geq x, u_2 \geq x\}.$$

From this we have

$$S_x = \{u = (u_1, u_2) \in \mathbb{R}^2 \mid (x, u) \in (\text{epi} f)_\infty\} = \begin{cases} \{(u_1, u_2) \mid u_1 \geq x, u_2 \geq x\}, \ x \geq 0 \\ \emptyset, \ x < 0. \end{cases}$$

Hence by the definition of recession map

$$f_\infty(x) = \begin{cases} \text{Min} S_x, \ x \geq 0 \\ \emptyset, \ x < 0. \end{cases} = \begin{cases} \{(x, x)\}, \ x \geq 0 \\ \emptyset, \ x < 0. \end{cases}$$

Proposition 5.4. *Assume that the order cone C is closed, convex and pointed. Let f be a convex vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Then*

$$dom f_\infty = \{x \in \mathbb{R}^n |\ S_x \neq \emptyset\}$$

and

$$f_\infty(x) = Sup\{f(y+x) - f(y)|\ y \in D\},\ \forall x \in dom f_\infty.$$

Proof. Let $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ be arbitrary. By the definition of S_x and by Lemma 5.1, one has

$$\begin{aligned} u \in S_x &\Leftrightarrow (x,u) \in (\text{epi}f)_\infty \\ &\Leftrightarrow (y,v) + (x,u) \in \text{epi}f,\ \forall (y,v) \in \text{epi}f \\ &\Leftrightarrow f(y+x) \preceq v + u,\ \forall (y,v) \in \text{epi}f \\ &\Leftrightarrow f(y+x) - f(y) \preceq u,\ \forall y \in D \\ &\Leftrightarrow u \in \text{Ub}\{f(y+x) - f(y)|\ y \in D\}. \end{aligned}$$

Thus,

$$S_x = \text{Ub}\{f(y+x) - f(y)|\ y \in D\}. \tag{5.1}$$

Hence, $S_x \neq \emptyset$ if and only if the set $\{f(y+x) - f(y)|\ y \in D\}$ is bounded above. By Theorem 2.6, this is equivalent to the fact $\text{Sup}\{f(y+x) - f(y)|\ y \in D\} \neq \emptyset$. Then by the definition of supremum and by (5.1), if $S_x \neq \emptyset$ we have

$$\begin{aligned} f_\infty(x) &= \text{Min} S_x \\ &= \text{Min}(\text{Ub}\{f(y+x) - f(y)|\ y \in D\}) \\ &= \text{Sup}\{f(y+x) - f(y)|\ y \in D\} \neq \emptyset. \end{aligned}$$

The proposition is proved. □

We note that, by the formula in Proposition 5.4, one has $f_\infty(0) = \{0\}$.

From Definition 5.2 we see that the recession maps of convex vector functions have set-valued structure. So for further investigations we need some concepts of set-valued maps. Let F be a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We define the epigraph of F with respect to C as the set

$$\text{epi}F := \{(x,u) \in \mathbb{R}^n \times \mathbb{R}^m |\ u \in F(x) + C\}.$$

F is said to be convex (respectively, closed) if epi$F$ is convex (respectively, closed). F is said to be positively homogeneous if

$$F(\lambda x) = \lambda F(x),\ \forall x \in \text{dom}F, \lambda \geq 0.$$

In the remain of this section, the order cone C is assumed convex, closed and pointed.

Proposition 5.5. *Let f be a convex vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Then*

$$epi(f_\infty) = (epi f)_\infty.$$

Proof. Let $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ be arbitrary. By Proposition 5.4, (5.1) and Theorem 2.6, one has

$$\begin{aligned}(x,u) \in \mathrm{epi}(f_\infty) &\Leftrightarrow u \in f_\infty(x) + C\\ &\Leftrightarrow u \in \mathrm{Sup}\{f(y+x) - f(y) |\ y \in D\} + C\\ &\Leftrightarrow u \in \mathrm{Ub}\{f(y+x) - f(y) |\ y \in D\}\\ &\Leftrightarrow u \in S_x\\ &\Leftrightarrow (x,u) \in (\mathrm{epi}f)_\infty.\end{aligned}$$

The proposition is proved. □

We note that, from Proposition 5.5, $\mathrm{dom}f_\infty$ is a convex cone and $\mathrm{dom}f_\infty \subset D_\infty$. The inverse inclusion is not true in general. For instant, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2$, $\forall x \in \mathbb{R}$. Then $\mathrm{dom}f_\infty = \{0\}$ while $D_\infty = \mathbb{R}$. By this proposition we also see that Definition 5.2 is an extension of the concept of recession maps for scalar convex functions to the vector case.

Lemma 5.6. Let f be a convex vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Let $x_0 \in D, x \in D_\infty$ be arbitrary. Then the set $\{\frac{f(x_0+\lambda x)-f(x_0)}{\lambda}\} \mid \lambda > 0\}$ is linearly ordered.

Proof. Let $\lambda \geq \lambda' > 0$ be arbitrary. We have

$$x_0 + \lambda' x = (1 - \frac{\lambda'}{\lambda})x_0 + \frac{\lambda'}{\lambda}(x_0 + \lambda x).$$

Then by convexity of f,

$$f(x_0 + \lambda' x) \preceq (1 - \frac{\lambda'}{\lambda})f(x_0) + \frac{\lambda'}{\lambda}f(x_0 + \lambda x)$$

which implies

$$\frac{f(x_0 + \lambda' x) - f(x_0)}{\lambda'} \preceq \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda}.$$

Hence the set $\{\frac{f(x_0+\lambda x)-f(x_0)}{\lambda}\} \mid \lambda > 0\}$ is linearly ordered. The proof is complete. □

In general, the recession maps of vector functions have set-valued structure. However, under certain conditions they reduce to single-valued maps. One such condition is the closedness as shown in the following proposition.

Proposition 5.7. *Let f be a convex vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Then the recession map f_∞ is a positively homogeneous convex set-valued map. In addition, if f is closed, then f_∞ reduces to a single-valued closed function and for any $x_0 \in D$, f_∞ is given by the following formula*

$$f_\infty(x) = ISup\left\{\frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} |\ \lambda > 0\right\},\ \forall x \in dom f_\infty.$$

Proof. The convexity of f_∞ is immediate from Proposition 5.5. By a note follows Proposition 5.4 and by following equalities

$$S_{\lambda x} = \lambda S_x,\ \forall x \in \mathbb{R}^n, \lambda > 0,$$

$$\mathrm{Min}(\lambda A) = \lambda \mathrm{Min} A, \forall A \subset \mathbb{R}^m, \lambda > 0,$$

f_∞ is positively homogeneous. Now, assume that f is closed. Then by Lemma 5.1 and by Proposition 5.5 above, epif_∞ is closed. Hence, so is f_∞. Finally, let $x_0 \in D$, $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ be arbitrary. Since epi$f$ is closed and convex, we have

$$\begin{aligned} u \in S_x &\Leftrightarrow (x, u) \in (\mathrm{epi} f)_\infty \\ &\Leftrightarrow (x_0, f(x_0)) + \lambda(x, u) \in \mathrm{epi} f, \ \forall \lambda > 0 \\ &\Leftrightarrow \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} \preceq u, \ \forall \lambda > 0 \\ &\Leftrightarrow u \in \mathrm{Ub} \left\{ \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} \mid \lambda > 0 \right\}. \end{aligned}$$

Hence,

$$S_x = \mathrm{Ub} \left\{ \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} \mid \lambda > 0 \right\}. \tag{5.2}$$

If $x \in \mathrm{dom} f_\infty$, then by Proposition 5.4, $S_x \neq \emptyset$. From (5.2), the set $\{\frac{f(x_0+\lambda x)-f(x_0)}{\lambda} \mid \lambda > 0\}$ is bounded from above. On other hand, by Lemma 5.6, the set $\{\frac{f(x_0+\lambda x)-f(x_0)}{\lambda} \mid \lambda > 0\}$ is linearly ordered. Then by Proposition 2.5, there exists ISup$\{\frac{f(x_0+\lambda x)-f(x_0)}{\lambda} \mid \lambda > 0\}$. From definitions of supremum and recession map together (5.2) and the note which follows Definition 2.3 we have

$$\begin{aligned} f_\infty(x) &= \mathrm{Min} S_x \\ &= \mathrm{Min} \left(\mathrm{Ub} \left\{ \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} \mid \lambda > 0 \right\} \right) \\ &= \mathrm{Sup} \left\{ \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} \mid \lambda > 0 \right\} \\ &= \mathrm{ISup} \left\{ \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda} \mid \lambda > 0 \right\}. \end{aligned}$$

Since C is pointed, ISup$\{\frac{f(x_0+\lambda x)-f(x_0)}{\lambda} \mid \lambda > 0\}$ is a singleton. Hence f_∞ is a single-valued map. The proposition is proved. □

Example 5.8. Let $\mathbb{R}^2$ be ordered by the positive orthant cone $\mathbb{R}^2{}_+$ and let $f : (0, +\infty) \to \mathbb{R}^2$ be defined as in Example 5.3, i.e., $f(x) = (x, x + \frac{1}{x})$. For every nonempty level set lev$_a f$ one has

$$\begin{aligned} \mathrm{lev}_a f &= \{x \in (0, +\infty) \mid f(x) \preceq a\} \\ &= \{x \in (0, +\infty) \mid x \leq a_1, x + \frac{1}{x} \leq a_2\} \\ &= (0, a_1] \cap [\alpha_1, \alpha_2] \end{aligned}$$

(where, $\alpha_i > 0, i = 1, 2$, are the solutions of the equation $x + \frac{1}{x} = a_2$). Hence lev$_a f$ is closed. Then by Lemma 2.8, f is closed too. Applying Proposition 5.7, for every

$x \in \mathrm{dom} f_\infty = \mathbb{R}_+$, we have

$$f_\infty(x) = \mathrm{IIsup}\{\frac{f(1+\lambda x) - f(1)}{\lambda} \mid \lambda > 0\}$$
$$= \mathrm{IIsup}\{\left(x, x - \frac{1}{\lambda} + \frac{1}{\lambda(1+\lambda x)}\right) \mid \lambda > 0\}.$$

By the definition of IIsup, we can verify that

$$\mathrm{IIsup}\{\left(x, x - \frac{1}{\lambda} + \frac{1}{\lambda(1+\lambda x)}\right) \mid \lambda > 0\} = \{(x,x)\}.$$

Hence

$$f_\infty(x) = \begin{cases} \{(x,x)\}, \ x \geq 0 \\ \emptyset, \ x < 0, \end{cases}$$

which coincides to the results in Example 5.3.

A function $\phi : A \subset \mathbb{R} \to \mathbb{R}^m$ is called decreasing (with respect to the cone C) if

$$\forall r, s \in A, r > s \Rightarrow \phi(r) \preceq \phi(s).$$

Proposition 5.9. *Let f be a convex vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$ and let $x_0 \in D_\infty$. Then $f(y + \lambda x_0)$ is a decreasing function of $\lambda(\lambda \geq 0)$ for every $y \in D$ if and only if*

$$f_\infty(x_0) \cap (-C) \neq \emptyset.$$

Proof. $\underline{\Rightarrow}$: Since $f(y + x_0) - f(y) \preceq 0$ for every $y \in D$, we have $0 \in Ub\{f(y+x_0) - f(y) | y \in D\}$. Then by Theorem 2.6, one has

$$\mathrm{Sup}\{f(y+x_0) - f(y) | y \in D\} \cap (-C) \neq \emptyset.$$

By Proposition 5.4, this implies

$$f_\infty(x_0) \cap (-C) \neq \emptyset.$$

$\underline{\Leftarrow}$: Let $y \in D$ be arbitrary and let $\lambda, \lambda' \in \mathbb{R}$ such that $\lambda > \lambda' \geq 0$. Since f_∞ is positively homogeneous we have

$$f_\infty\left((\lambda - \lambda')x_0\right) \cap (-C) \neq \emptyset. \tag{5.3}$$

By Proposition 5.4,

$$f_\infty\left((\lambda - \lambda')x_0\right) = \mathrm{Sup}\{f(z + (\lambda - \lambda')x_0) - f(z) | \ z \in D\}$$

which together (5.3) and the definition of supremum imply

$$f(z + (\lambda - \lambda')x_0) - f(z) \preceq 0, \forall z \in D.$$

Observe that

$$f(y + \lambda x_0) - f(y + \lambda' x_0) = f(y + \lambda' x_0 + (\lambda - \lambda')x_0) - f(y + \lambda' x_0)$$

then one has

$$f(y + \lambda x_0) - f(y + \lambda' x_0) \preceq 0.$$

Hence $f(y + \lambda x_0)$ is a decreasing function of $\lambda \in \mathbb{R}_+$. The proof is complete. □

Corollary 5.10. *Let f be a closed convex vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$ and let $x_0 \in D_\infty$. If there is a point $\bar{y} \in D$ having the property that $f(\bar{y} + \lambda x_0)$ is a decreasing function of $\lambda(\lambda \geq 0)$, then this property holds for every $y \in D$.*

Proof. Since f is closed convex and the set $\left\{\frac{f(\bar{y}+\lambda x_0)-f(\bar{y})}{\lambda} \mid \lambda > 0\right\}$ is bounded above by 0, we have $S_{x_0} \neq \emptyset$ by (5.2), hence $x_0 \in \mathrm{dom} f_\infty$. From Proposition 5.7 and the definition of ISup, one has

$$f_\infty(x_0) = \mathrm{ISup}\left\{\frac{f(\bar{y} + \lambda x_0) - f(\bar{y})}{\lambda} \mid \lambda > 0\right\} \preceq 0.$$

Apply Proposition 5.9, we complete the proof. □

The set of all vectors $x \in \mathbb{R}^n$ such that $f_\infty(x) \cap (-C) \neq \emptyset$ is said to be the recession cone of f and denoted by $\mathrm{Rec}(f)$. It is easy to see that it is a convex cone. The directions of vectors in $\mathrm{Rec}(f)$ is called the recession directions of f.

Proposition 5.11. *Let f be a closed convex vector function from a nonempty convex subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$. Then all the nonempty level sets of f have the same recession cone, namely the recession cone of f, i.e., for every $a \in \mathbb{R}^m$ such that $lev_a f \neq \emptyset$ one has*

$$(lev_a f)_\infty = Rec(f).$$

Proof. Let $a \in \mathbb{R}^m$ such that $\mathrm{lev}_a f \neq \emptyset$ and let $y \in \mathrm{lev}_a f$. Then $(y, a) \in \mathrm{epi} f$. Since f is closed, by Lemma 2.8, $\mathrm{lev}_a f$ is closed also. Then for every vector $x \in \mathbb{R}^n$, one has

$$\begin{aligned}
x \in (\mathrm{lev}_a f)_\infty &\Leftrightarrow y + \lambda x \in \mathrm{lev}_a f,\ \forall \lambda > 0 \\
&\Leftrightarrow f(y + \lambda x) \preceq a,\ \forall \lambda > 0 \\
&\Leftrightarrow (y, a) + \lambda(x, 0) \in \mathrm{epi} f,\ \forall \lambda > 0 \\
&\Leftrightarrow (x, 0) \in (\mathrm{epi} f)_\infty \\
&\Leftrightarrow (x, 0) \in \mathrm{epi}(f_\infty),\ (\text{ by Proposition 5.5}) \\
&\Leftrightarrow 0 \in f_\infty(x) + C \\
&\Leftrightarrow f_\infty(x) \cap (-C) \neq \emptyset \\
&\Leftrightarrow x \in \mathrm{Rec}(f).
\end{aligned}$$

The proposition is proved. □

6. Applications

In this section we investigate the sufficient conditions for the existence of optimal solutions of vector optimization problems based on recession directions of objective functions. Let f be a vector function from a nonempty subset $D \subset \mathbb{R}^n$ to $\mathbb{R}^m$, where $\mathbb{R}^m$ is ordered by a convex cone C. Firstly, we consider the following unconstraint vector optimization problem

$$\begin{cases} \mathrm{Min} f(x) \\ \text{s.t. } x \in D. \end{cases} \tag{VP}$$

We need to find a point $x^* \in D$, called an optimal (or minimal, or efficient) solution of (VP), such that

$$f(x^*) \in \text{Min}(f(D)|C).$$

In the remain of this section we assume that the order cone $C \subset \mathbb{R}^m$ is convex, closed and pointed with $\text{int}C \neq \emptyset$.

Lemma 6.1. Let $c \in \text{int}C$ and $x \in \mathbb{R}^m$ be arbitrary. Then there exists $k \in \mathbb{N}$ such that

$$-kc \preceq x.$$

Proof. Since $0 \in -c + \text{int}C$ there exists $r > 0$ such that the open ball $B(0, r) \subset -c + \text{int}C$. Then one can find a number $k \in \mathbb{N}$ such that $\frac{x}{k} \in B(0, r)$. This implies

$$x \in -kc + k\text{int}C \subset -kx + C$$

which complete the proof. □

Theorem 6.2. *Assume that f is convex and closed. If f has no nonzero direction of recession, i.e., Rec$(f) = \{0\}$, then the vector optimization problem (VP) has an optimal solution.*

Proof. Let $y \in f(D)$ be arbitrary. Set $B := (y - C) \cap f(D)$. Clearly $\text{Min}B \subset \text{Min}f(D)$. Then to complete the proof of the theorem, it is sufficient to show that $\text{Min}B \neq \emptyset$. For this, let S be any nonempty linearly ordered subset of B. We shall show that S is bounded below. Indeed, suppose to the contrary that S is not bounded below. Let $c \in \text{int}C$. Then by induction, we can construct a sequence $\{y_k\}_k \subset S$ such that

$$-kc \npreceq y_k, \tag{6.1}$$

$$y_{k+1} \preceq y_k, \tag{6.2}$$

for every k. Since $\text{Rec}(f) = \{0\}$, by Proposition 5.11, Lemma 2.7 and Lemma 5.1, $\text{lev}_{y_k} f$ is a nonempty compact subset, for every k. From (6.2), one has

$$\text{lev}_{y_{k+1}} f \subset \text{lev}_{y_k} f, \ \forall k.$$

Hence,

$$\bigcap_{k=1}^{\infty} \text{lev}_{y_k} f \neq \emptyset.$$

Let $x \in \bigcap_{k=1}^{\infty} \text{lev}_{y_k} f$. Then $f(x) \preceq y_k$, for every k. Hence by (6.1),

$$-kc \npreceq f(x), \ \forall k. \tag{6.3}$$

Since $c \in \text{int}C$, by Lemma 6.1 there exists k_0 such that $-k_0 c \preceq f(x)$ which contradicts to (6.3). Thus, S is bounded below. By Proposition 2.5 and the note which follows it, $\text{IInf}S$ exists and there is a decreasing sequence $\{f(x_k)\}_k \subset S$ converging to $\text{IInf}S$. Since $\text{lev}_y f$ is compact and $\{x_k\}_k \subset \text{lev}_y f$, without loss of generality, we may assume that $\{x_k\}_k$ converges to some $x_0 \in \text{lev}_y f$. By the closedness of f, one has $(x_0, \text{IInf}S) \in \text{epi}f$. Hence S is bounded below from $f(x_0) \in B$. Applying Zorn's Lemma we obtain $\text{Min}B \neq \emptyset$. The theorem is proved. □

Example 6.3. Let $\mathbb{R}^2$ be ordered by the positive orthant cone $\mathbb{R}^2{}_+$. Consider the vector problem

$$\begin{cases} \text{Min} f(x) \\ \text{s.t. } x \in (0, +\infty), \end{cases}$$

where, $f : (0, +\infty) \to \mathbb{R}^2$ is defined as in Example 5.3. Since

$$f_\infty(x) = \begin{cases} \{(x, x)\}, \ x \geq 0 \\ \emptyset, \ x < 0 \end{cases}$$

we have $\text{Rec}(f) = \{0\}$. Then by Theorem 6.2 the problem above has an optimal solution. Here we can see that $x = 1$ is an optimal solution of the problem.

Now, we consider the following vector optimization problem with a general set constraint.

$$\begin{cases} \text{Min} f(x) \\ \text{s.t. } x \in D \\ x \in E. \end{cases} \qquad (SVP)$$

where, E is a subset of $\mathbb{R}^n$. Denote by S the feasible set of (SVP), i.e., $S = \{x \in D \mid x \in E\}$. We should note that if f and E are closed then $f_{D\cap E}$ is also closed, where, $f_{D\cap E}$ denotes the restriction of f on $D \cap E$.

Corollary 6.4. *Assume that f and E are convex, closed and the feasible set S is nonempty. If Rec$(f) \cap E_\infty = \{0\}$ then the vector optimization problem (SVP) has an optimal solution.*

Proof. We only need to show that

$$\text{Rec}(f_{D\cap E}) = \text{Rec}(f) \cap E_\infty$$

then applying Theorem 6.2 we obtain the result.

Let $x \in \text{Rec}(f_{D\cap E})$ and $y \in D \cap E$ be arbitrary. Then $x \in (D \cap E)_\infty$ and by Proposition 5.9, $f(y + \lambda x)$ is a decreasing function of λ ($\lambda \geq 0$). By Corollary 5.10, $x \in \text{Rec}(f)$. On other hand from the closedness of E, applying Lemma 5.1 we have $x \in E_\infty$.

Conversely, let $x \in \text{Rec}(f) \cap E_\infty$ and let $y \in D \cap E$ be arbitrary. Then $x \in (D \cap E)_\infty$ (by Lemma 6.1) and $f(y + \lambda x)$ is a decreasing function of λ ($\lambda \geq 0$). By Proposition 5.9, $x \in \text{Rec}(f_{D\cap E})$. The proof is complete.

□

Finally, we consider a vector optimization problem with inequality constraints as follows

$$\begin{cases} \text{Min} f(x) \\ \text{s.t. } x \in D \\ x \in D_i, f_i(x) \preceq_{C_i} 0, \ i \in I. \end{cases} \qquad (IVP)$$

where, $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$, $f_i : D_i \subset \mathbb{R}^n \to \mathbb{R}^{m_i}, i \in I$, be vector functions with I is an arbitrary index set and for every $i \in I$, $\mathbb{R}^{m_i}$ is ordered by a closed, pointed and convex cone C_i with $\text{int}C_i \neq \emptyset$. Set $E_i := \{x \in D_i | f_i(x) \preceq_{C_i} 0\}, \forall i \in I$, $E := \bigcap_{i\in I} E_i$. Denote by T the feasible set of (IVP),i.e.,

$$T = \{x \in D \mid x \in D_i, f_i(x) \preceq_{C_i} 0, \forall i \in I\}.$$

Corollary 6.5. *Assume that the feasible set T is nonempty and the functions $f, f_i, i \in I$, are convex and closed. If $f, f_i, i \in I$, have no nonzero direction of recession in common, i.e.,*

$$Rec(f) \cap \left(\bigcap_{i\in I} Rec(f_i) \right) = \{0\},$$

then the problem (IVP) has an optimal solution.

Proof. By Lemma 2.8, E_i is closed for every $i \in I$. Then by Lemma 5.1, one has $E_\infty = \bigcap_{i\in I} (E_i)_\infty$. Applying Proposition 5.11, we get $E_\infty = \bigcap_{i\in I} \text{Rec}(f_i)$. Then using Corollary 6.4 we complete the proof. □

Example 6.6. Let $X = \mathbb{R}^2$ be ordered by the positive orthant cone $C = {\mathbb{R}^2}_+$ and let $X_1 = \mathbb{R}^2$ be ordered by the cone $C_1 = \{\alpha(1, 1) + \beta(-1, 1) \mid \alpha, \beta \geq 0\}$. Consider the following inequality constraint vector problem

$$\begin{cases} \text{Min} f(x) \\ \text{s.t., } x \in \mathbb{R}^2 \\ f_1(x) \preceq_{C_1} 0 \\ f_2(x) \preceq_C 0, \end{cases}$$

where, $f, f_2 : \mathbb{R}^2 \to X$, $f_1 : \mathbb{R}^2 \to X_1$ are defined as follows.

$$f(x, y) = (x + y, e^x - y), f_1(x, y) = (-x, -2y), f_2(x, y) = (y - x, e^{-x} - 1 - y).$$

By computing we obtain

$$\begin{aligned} \text{Rec}(f) &= \{\alpha(-1, 0) + \beta(-1, 1) \mid \alpha, \beta \geq 0\} \\ \text{Rec}(f_1) &= \{\alpha(-1, \frac{1}{2}) + \beta(1, \frac{1}{2}) \mid \alpha, \beta \geq 0\} \\ \text{Rec}(f_2) &= \{\alpha(1, 0) + \beta(1, 1) \mid \alpha, \beta \geq 0\}. \end{aligned}$$

Hence $\text{Rec}(f) \cap \text{Rec}(f_1) \cap \text{Rec}(f_2) = \{0\}$. Then by Corollary 6.5 we know that the problem above has an optimal solution.

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