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A NEW ITERATIVE PROCESS FOR A FINITE FAMILY OF GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. In this paper, we introduce a new iterative process for approximating a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in a convex metric space. A necessary and sufficient condition for strong convergence of the propose iterative process is established. We give characterization of a uniformly convex metric space with continuous convex structure and by using this characterization, we also prove convergence results of the propose iterative process in a uniformly convex metric space with continuous convex structure. Moreover, we apply our main results to obtain strong convergence theorems in hyperbolic spaces and CAT(0) spaces. Our results generalize and refine many known results in the current literature.

1. Introduction

In 1970, Takahashi [20] introduced the concept of convexity in a metric space and the properties of the space. The convex metric space is a more general space and each normed linear space is a special example of a convex metric space. Then since many authors discussed the existence of the fixed point and the convergence of the iterative processes for nonexpansive mappings, quasi-nonexpansive mappings and their generalized mappings in convex metric spaces, (see [1, 4, 6, 7, 9, 10, 11, 15, 18, 19, 20, 21, 22]).

Definition 1.1 ([20])**.** Let (X, d) be a metric space. A mapping $W : X\times X\times[0, 1] \to X$ is said to be a *convex structure* on X if for each $x, y, z \in X$ and $\lambda \in [0, 1]$,

$$d\left(z, W\left(x, y, \lambda\right)\right) \leq \lambda d\left(z, x\right) + \left(1 - \lambda\right) d\left(z, y\right).$$

A metric space (X, d) together with a convex structure W is called a *convex metric space* which will be denoted by (X, d, W). A nonempty subset C of X is said to be *convex* if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. Clearly, a normed space and each of its convex subsets are convex metric space, but the converse does not hold. In 1996, Shimizu and Takahashi [19] introduced the concept of uniform convexity in convex metric spaces and studies the properties of these spaces.

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Definition 1.2 ([19]). A convex metric space (X, d, W) is said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta_\varepsilon \in (0, 1]$ such that for all $r > 0$ and $x, y, z \in X$ with $d(z, x) \leq r$, $d(z, y) \leq r$ and $d(x, y) \geq r\varepsilon$ imply that

$$d\left(z, W\left(x, y, \frac{1}{2}\right)\right) \leq (1 - \delta_\varepsilon)\, r.$$

Obviously, uniformly convex Banach spaces are uniformly convex metric spaces.

In 2010, Kettapun et al. [8] introduced the iterative process to approximate a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces. Yatakoat and Suantai [24] extended the results of Kettapun et al. to a finite family of generalized asymptotically quasi-nonexpansive mappings. Later on, Khan and Ahmed [10] introduced the iterative process in convex metric spaces for finding a common fixed point of a finite family of asymptotically quasi-nonexpansive mappings. Recently, Khan et al. [11] given some sufficient and necessary conditions for a general iteration scheme of a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces and CAT(0) spaces. Many authors has studied the specific space of a convex metric space, that is, the class of hyperbolic spaces, which contain the class of CAT(0) spaces, see [2, 5, 11, 12, 13, 14, 16].

Motivated by above results, we introduce a new iterative process for finding a common fixed point in a convex metric space as follows: Let C be a convex subset of a convex metric space (X, d, W). Suppose that $\alpha_n^{(i)} \in [0, 1]$ for all $n \in \mathbb{N}$ and all $i = 1, 2, \ldots, N$. Let $\{T_i\}_{i=1}^N$ be a finite family of self-mappings of C. For $x_1 \in C$, let $\{x_n\}$ be the sequence defined by

$$(1.1)\qquad \begin{cases} y_n^{(0)} = x_n, \\ y_n^{(1)} = W\left(T_1^n y_n^{(0)}, y_n^{(0)}, \alpha_n^{(1)}\right), \\ y_n^{(2)} = W\left(T_2^n y_n^{(1)}, y_n^{(1)}, \alpha_n^{(2)}\right), \\ y_n^{(3)} = W\left(T_3^n y_n^{(2)}, y_n^{(2)}, \alpha_n^{(3)}\right), \\ \quad\vdots \\ y_n^{(N-1)} = W\left(T_{N-1}^n y_n^{(N-2)}, y_n^{(N-2)}, \alpha_n^{(N-1)}\right), \\ x_{n+1} = W\left(T_N^n y_n^{(N-1)}, y_n^{(N-1)}, \alpha_n^{(N)}\right), \end{cases}$$

for all $n \in \mathbb{N}$.

The purpose of this paper is to establish strong convergence of the iterative process (1.1) for a finite family of generalized asymptotically quasi-nonexpansive mapping in a convex metric space. Moreover, we give characterization of a uniformly convex metric space with continuous convex structure and prove a convergence theorem of the iterative process (1.1) under some suitable control conditions. Finally, we apply our results to obtain strong convergence theorems in hyperbolic spaces and CAT(0) spaces.

2. Preliminaries

Let C be a nonempty subset of a metric space (X, d). $T : C \to C$ be a mapping. The fixed point set of T is denote by $F(T) = \{x \in C : x = Tx\}$. A mapping T is called:

(i) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$;
(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$ for all $x \in C$ and $p \in F(T)$;
(iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $d(T^n x, T^n y) \leq k_n d(x, y)$ for all $x, y \in C$ and $n \in \mathbb{N}$;
(iv) *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ such that $\lim_{n\to\infty} k_n = 1$ and $d(T^n x, p) \leq k_n d(x, p)$ for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$;
(v) *generalized asymptotically nonexpansive* if there exists two sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ and $d(T^n x, T^n y) \leq k_n d(x, y) + s_n$ for all $x, y \in C$ and $n \in \mathbb{N}$;
(vi) *generalized asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists two sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ and $d(T^n x, p) \leq k_n d(x, p) + s_n$ for all $x \in C, p \in F(T)$ and $n \in \mathbb{N}$;
(vii) *uniformly L-Lipschitzian* if there exists constant $L > 0$ such that $d(T^n x, T^n y) \leq L d(x, y)$ for all $x, y \in C$ and $n \in \mathbb{N}$.

From the above definitions, it is clear that:
(i) a nonexpansive mapping is asymptotically nonexpansive;
(ii) a quasi-nonexpansive mapping is asymptotically quasi-nonexpansive;
(iii) an asymptotically quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive;
(iv) if $F(T) \neq \emptyset$, then a nonexpansive mapping is quasi-nonexpansive, an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive and a generalized asymptotically nonexpansive mapping is generalized asymptotically quasi-nonexpansive.

Remark 2.1. If T is a generalized asymptotically quasi-nonexpansive mapping, it is know that $F(T)$ is not necessarily closed, see [17].

We state the following conditions in metric spaces:

Condition (A): Let C be a subset of a metric space (X, d). A finite family of self-mappings $\{T_i\}_{i=1}^N$ of C is said to have *Condition (A)* if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$ such that $d(x, T_i x) \geq f(d(x, F))$ for some i, $1 \leq i \leq N$ and for all $x \in C$, where $d(x, F) = \inf\{d(x, p) : p \in F = \bigcap_{i=1}^N F(T_i)\}$.

Semi-compact: Let C be a subset of a metric space (X, d). A mapping T is *semi-compact* if for a sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to p \in C$.

The following results are needed for proving our results.

Lemma 2.2 ([23]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences of nonnegative real numbers satisfy:*

$$a_{n+1} = (1 + b_n)\, a_n + c_n, \ \forall n \in \mathbb{N}$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then

(i) *$\lim_{n\to\infty} a_n$ exists;*
(ii) *If $\liminf_{n\to\infty} a_n = 0$, then $\lim_{n\to\infty} a_n = 0$.*

Definition 2.3 ([17]). Let $\{x_n\}$ be a sequence in a metric space (X, d) and F be a subset of X. We say that $\{x_n\}$ is of

(i) *monotone type (I) with respect to F* if there exist sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative real numbers suth that $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $d(x_{n+1}, p) \le (1 + r_n)d(x_n, p) + s_n$ for all $n \in \mathbb{N}$ and $p \in F$;
(ii) *monotone type (II) with respect to F* if for each $p \in F$ there exist sequences $\{r_n\}$ and $\{s_n\}$ of nonnegative real numbers suth that $\sum_{n=1}^{\infty} r_n < \infty$, $\sum_{n=1}^{\infty} s_n < \infty$ and $d(x_{n+1}, p) \le (1 + r_n)d(x_n, p) + s_n$ for all $n \in \mathbb{N}$.

Theorem 2.4 ([17, Theorem 2.4]). Let (X, d) be a complete metric space, F be a subset of X and $\{x_n\}$ be a sequence in X. Then one has the following assertions.

(i) If $\{x_n\}$ is of monotone type (I) with respect to F, then $\lim_{n\to\infty} d(x_n, F)$ exists.
(ii) If $\{x_n\}$ is of monotone type (I) with respect to F and $\liminf_{n\to\infty} d(x_n, F) = 0$, then $x_n \to p$ for some $p \in X$ satisfying $d(p, F) = 0$. In particular, if F is closed, then $p \in F$.
(iii) If $\{x_n\}$ is of monotone type (II) with respect to F, then $\lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F$.

Lemma 2.5 ([1, 20]). *Let (X, d, W) be a convex metric space. For each $x, y \in X$ and $\lambda \in [0, 1]$,*

$$d\,(x, W(x, y, \lambda)) = (1 - \lambda)d(x, y) \ \text{and} \ d\,(y, W(x, y, \lambda)) = \lambda d(x, y).$$

Lemma 2.6 ([9, 18]). *Let (X, d, W) be a uniformly convex metric space with a continuous convex structure $W : X \times X \times [0, 1] \to X$. Then for arbitrary positive number ε and r, there exists $\eta_\varepsilon \in (0, 1]$ such that*

$$d\,(z, W\,(x, y, \lambda)) \le (1 - 2\min\{\lambda, 1 - \lambda\}\eta_\varepsilon)\, r$$

for all $x, y, z \in X$, $d(z, x) \le r$, $d(z, y) \le r$, $d(x, y) \ge r\varepsilon$ and $\lambda \in [0, 1]$.

By using Lemma 2.6, we get the characterization of a uniformly convex metric space as follows.

Lemma 2.7. *Let (X, d, W) be a convex metric space with continuous convex structure. Then (X, d, W) is uniformly convex if and only if for each $x \in X$ and $r > 0$, if $\{t_n\}$ is a sequence in $[a, b]$ with $0 < a < b < 1$ and $\{x_n\}$, $\{y_n\}$ are sequences in X with $\limsup_{n\to\infty} d\,(x_n, x) \le r$, $\limsup_{n\to\infty} d\,(y_n, x) \le r$ and $\lim_{n\to\infty} d\,(W\,(x_n, y_n, t_n)\,, x) = r$ imply $\lim_{n\to\infty} d\,(x_n, y_n) = 0$.*

Proof. $(\Rightarrow)$: Let $r > 0$. Suppose that $\lim_{n\to\infty} d(x_n, y_n) \neq 0$. Then there are subsequences, denoted by $\{x_n\}$ and $\{y_n\}$ such that $\inf_n d(x_n, y_n) > 0$. Then there is an $\varepsilon \in (0,1]$ such that

$$d(x_n, y_n) \geq \varepsilon(r+1) > 0, \ \forall n \in \mathbb{N}.$$

Since there exists $\eta_\varepsilon \in (0,1]$ and $0 < 2a(1-b) < 1$, we have $0 < 1 - 2a(1-b)\eta_\varepsilon < 1$. We can also choose $R \in (r, r+1)$ such that $(1 - 2a(1-b)\eta_\varepsilon) R < r$. Since $\limsup_{n\to\infty} d(x_n, x) \leq r$, $\limsup_{n\to\infty} d(y_n, x) \leq r$ and $r < R$, there are futher subsequences again denoted by $\{x_n\}$ and $\{y_n\}$ such that $d(x_n, x) \leq R$, $d(y_n, x) \leq R$, $d(x_n, y_n) \geq \varepsilon R$, $\forall n \in \mathbb{N}$. Then by Lemma 2.6, we have

$$\begin{aligned} d(W(x_n, y_n, t_n), x) &\leq (1 - 2\min\{t_n, 1-t_n\}\eta_\varepsilon) R \\ &\leq (1 - 2t_n(1-t_n)\eta_\varepsilon) R \\ &\leq (1 - 2a(1-b)\eta_\varepsilon) R < r \end{aligned}$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we obtain $\lim_{n\to\infty} d(W(x_n, y_n, t_n), x) < r$, which contradiction to the hypothesis.

$(\Leftarrow)$: Suppose that the condition holds. We will show that X is uniformly convex. To show this, suppose not. Then, there exist $\varepsilon > 0$, $r > 0$ and $z \in X$ such that each $n \in \mathbb{N}$, there are $x_n, y_n \in X$ such that $d(x_n, z) \leq r$, $d(y_n, z) \leq r$, $d(x_n, y_n) \geq r\varepsilon$ and $d\left(z, W\left(x_n, y_n, \frac{1}{2}\right)\right) > \left(1 - \frac{1}{n}\right) r$. It follows that $\limsup_{n\to\infty} d(x_n, z) \leq r$, $\limsup_{n\to\infty} d(y_n, z) \leq r$ and $\lim_{n\to\infty} d\left(z, W\left(x_n, y_n, \frac{1}{2}\right)\right) = r$. By the assumption, we have $\lim_{n\to\infty} d(x_n, y_n) = 0$, which is a contradiction with $d(x_n, y_n) \geq r\varepsilon \neq 0$. Hence, we have X is uniformly convex. □

3. Main Results

In this section, we prove strong convergence theorems of the proposed iteration method in convex metric spaces. We first note that if $\{T_i\}_{i=1}^N$ is a finite family of generalized asymptotically quasi-nonexpansive self-mappings of C with $F = \bigcap_{i=1}^N F(T_i)$ is nonempty, where C is a nonempty convex subset of a convex metric space (X, d, W). Then, for $p \in F$, we have $d(T_i^n x, p) \leq k_n^{(i)} d(x, p) + s_n^{(i)}$ for all $x \in C$ and all $i = 1, 2, \ldots, N$, where $\{k_n^{(i)}\} \subset [1, \infty)$, $\{s_n^{(i)}\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n^{(i)} = 1$ and $\lim_{n\to\infty} s_n^{(i)} = 0$. Put $k_n = \max_{1\leq i\leq N}\{k_n^{(i)}\}$ and $s_n = \max_{1\leq i\leq N}\{s_n^{(i)}\}$. It is clear that $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ and

$$d(T_i^n x, p) \leq k_n d(x, p) + s_n$$

for all $x \in C, p \in F$, $i = 1, 2, \ldots, N$ and all $n \in \mathbb{N}$.

In order to prove our main results, the following lemmas are needed.

Lemma 3.1. *Let (X, d, W) be a convex metric space and C be a nonempty convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of generalized asymptotically quasi-nonexpansive self-mappings of C with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$. Suppose $F = \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1). Then, we heve the following:*

(i) $d(y_n^{(i)}, p) \leq k_n d(y_n^{(i-1)}, p) + s_n$, $\forall i = 1, 2, \ldots, N-1$, $\forall n \in \mathbb{N}$ *and* $\forall p \in F$*;*

(ii) $d(y_n^{(i)}, p) \le k_n^i d(x_n, p) + s_n \sum_{j=1}^{i} k_n^{j-1}$, $\forall i = 1, 2, \ldots, N-1$, $\forall n \in \mathbb{N}$ *and* $\forall p \in F$;

(iii) $d(x_{n+1}, p) \le k_n^i d(y_n^{(N-i)}, p) + s_n \sum_{j=1}^{i} k_n^{j-1}$, $\forall i = 1, 2, \ldots, N-1$, $\forall n \in \mathbb{N}$ *and* $\forall p \in F$.

Proof. (i) : For $i = 1, 2, \ldots, N-1$, we have

$$\begin{aligned} d\left(y_n^{(i)}, p\right) &= d\left(W\left(T_i^n y_n^{(i-1)}, y_n^{(i-1)}, \alpha_n^{(i)}\right), p\right) \\ &\le \alpha_n^{(i)} d\left(T_i^n y_n^{(i-1)}, p\right) + \left(1 - \alpha_n^{(i)}\right) d\left(y_n^{(i-1)}, p\right) \\ &\le \alpha_n^{(i)} \left(k_n d\left(y_n^{(i-1)}, p\right) + s_n\right) + \left(1 - \alpha_n^{(i)}\right) d\left(y_n^{(i-1)}, p\right) \\ &\le \left(\frac{1 - \alpha_n^{(i)}}{k_n} + \alpha_n^{(i)}\right) k_n d\left(y_n^{(i-1)}, p\right) + s_n. \end{aligned}$$

Since $0 \le \frac{1-\alpha_n^{(i)}}{k_n} + \alpha_n^{(i)} \le 1$ for all $i = 1, 2, \ldots, N-1$, we obtain $d(y_n^{(i)}, p) \le k_n d(y_n^{(i-1)}, p) + s_n$.

(ii) : By (i), we have

$$\begin{aligned} d\left(y_n^{(1)}, p\right) &\le k_n d\left(y_n^{(0)}, p\right) + s_n \\ &= k_n d\left(x_n, p\right) + s_n \sum_{j=1}^{1} k_n^{j-1}, \end{aligned}$$

and so

$$\begin{aligned} d\left(y_n^{(2)}, p\right) &\le k_n d\left(y_n^{(1)}, p\right) + s_n \\ &\le k_n \left(k_n d\left(x_n, p\right) + s_n \sum_{j=1}^{1} k_n^{j-1}\right) + s_n \\ &= k_n^2 d\left(x_n, p\right) + s_n \left(k_n \sum_{j=1}^{1} k_n^{j-1} + 1\right) \\ &= k_n^2 d\left(x_n, p\right) + s_n \sum_{j=1}^{2} k_n^{j-1}. \end{aligned}$$

Assume that $d(y_n^{(m)}, p) \le k_n^m d(x_n, p) + s_n \sum_{j=1}^{m} k_n^{j-1}$ for some m, $1 \le m \le N-2$. By (i), we have

$$\begin{aligned} d\left(y_n^{(m+1)}, p\right) &\le k_n d\left(y_n^{(m)}, p\right) + s_n \\ &\le k_n \left(k_n^m d\left(x_n, p\right) + s_n \sum_{j=1}^{m} k_n^{j-1}\right) + s_n \end{aligned}$$

$$= k_n^{m+1} d(x_n, p) + s_n \left(k_n \sum_{j=1}^{m} k_n^{j-1} + 1 \right)$$
$$= k_n^{m+1} d(x_n, p) + s_n \sum_{j=1}^{m+1} k_n^{j-1}.$$

By induction, we obtain $d(y_n^{(i)}, p) \le k_n^i d(x_n, p) + s_n \sum_{j=1}^{i} k_n^{j-1}$ for all $i = 1, 2, \dots, N-1$.

(iii) : By (1.1) and (i), we have

$$\begin{aligned}
d(x_{n+1}, p) &= d\left(W\left(T_N^n y_n^{(N-1)}, y_n^{(N-1)}, \alpha_n^{(N)}\right), p\right) \\
&\le \alpha_n^{(N)} d\left(T_N^n y_n^{(N-1)}, p\right) + \left(1 - \alpha_n^{(N)}\right) d\left(y_n^{(N-1)}, p\right) \\
&\le \alpha_n^{(N)} \left(k_n d\left(y_n^{(N-1)}, p\right) + s_n\right) + \left(1 - \alpha_n^{(N)}\right) d\left(y_n^{(N-1)}, p\right) \\
&\le \left(\frac{1 - \alpha_n^{(N)}}{k_n} + \alpha_n^{(N)}\right) k_n d\left(y_n^{(N-1)}, p\right) + s_n \\
&\le k_n d\left(y_n^{(N-1)}, p\right) + s_n \sum_{j=1}^{1} k_n^{j-1} \\
&\le k_n \left(k_n d\left(y_n^{(N-2)}, p\right) + s_n\right) + s_n \sum_{j=1}^{1} k_n^{j-1} \\
&= k_n^2 d\left(y_n^{(N-2)}, p\right) + s_n \sum_{j=1}^{2} k_n^{j-1} \\
&\vdots \\
&\le k_n^i d\left(y_n^{(N-i)}, p\right) + s_n \sum_{j=1}^{i} k_n^{j-1}
\end{aligned}$$

for all $i = 1, 2, \dots, N-1$. □

Lemma 3.2. *Let (X, d, W) be a convex metric space and C be a nonempty convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of generalized asymptotically quasi-nonexpansive self-mappings of C with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose $F = \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1). Then, we have the following:*

(i) *There exist two sequences $\{\delta_n\}$ and $\{\varepsilon_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and $d(x_{n+1}, p) \le (1 + \delta_n) d(x_n, p) + \varepsilon_n$ for all $p \in F$ and $n \in \mathbb{N}$;*
(ii) $\lim_{n \to \infty} d(x_n, p)$ *exists, for all $p \in F$.*

Proof. (i) : By Lemma 3.1(ii)-(iii), we have

$$\begin{aligned} d(x_{n+1}, p) &\le k_n d\left(y_n^{(N-1)}, p\right) + s_n \\ &\le k_n \left(k_n^{N-1} d(x_n, p) + s_n \sum_{j=1}^{N-1} k_n^{j-1}\right) + s_n \\ &= k_n^N d(x_n, p) + s_n \left(k_n \sum_{j=1}^{N-1} k_n^{j-1} + 1\right) \\ &= k_n^N d(x_n, p) + s_n \sum_{j=1}^{N} k_n^{j-1} \\ &= (1 + (k_n - 1))^N d(x_n, p) + s_n \sum_{j=1}^{N} k_n^{j-1} \\ &= (1 + \delta_n) d(x_n, p) + \varepsilon_n, \end{aligned}$$

where $\delta_n = \sum_{j=1}^{N} \binom{N}{j} (k_n - 1)^j$ and $\varepsilon_n = s_n \sum_{j=1}^{N} k_n^{j-1}$. Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, it follows that $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Hence, we obtain the desired result.

(ii) : By (i) and Lemma 2.2(i), we obtain that $\lim_{n\to\infty} d(x_n, p)$ exists. □

Theorem 3.3. *Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of generalized asymptotically quasi-nonexpansive self-mappings of C with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose $F = \bigcap_{i=1}^N F(T_i)$ is nonempty and closed. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1). Then $\{x_n\}$ converges to a common fixed point of the family $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.*

Proof. The necessity is obvious and then we prove only the sufficiency. Suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. By Lemma 3.2(i), we obtain that the sequence $\{x_n\}$ is of monotone type (I) with respect to F. It follows by Theorem 2.4(ii), that $\{x_n\}$ converges to a point $p \in F$. □

The following result is obtained direclty from Theorem 3.3. Clearly, the closedness of $\bigcap_{i=1}^N F(T_i)$ can be dropped if T_i is asymptotically quasi-nonexpansive mappings for all $i = 1, 2, \ldots, N$.

Corollary 3.4. *Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of asymptotically quasi-nonexpansive self-mappings of C with a sequence $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$. Suppose $F = \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1). Then $\{x_n\}$ converges to a common fixed point of the family $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Since any quasi-nonexpansive mapping is asymptotically quasi-nonexpansive, the next corollary is obtained immediately from above corollary.

Corollary 3.5. *Let (X, d, W) be a complete convex metric space and C be a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^{N}$ be a finite family of quasi-nonexpansive self-mappings of C with $F = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1). Then $\{x_n\}$ converges to a common fixed point of the family $\{T_i\}_{i=1}^{N}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Next, we prove a strong convergence theorem in a uniformly convex metric space. The following lemma is very useful for our main result.

Lemma 3.6. *Let (X, d, W) be a complete uniformly convex metric space with continuous convex structure and C be a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^{N}$ be a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive self-mappings of C with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose $F = \bigcap_{i=1}^{N} F(T_i)$ is nonempty. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1) with $\{\alpha_n^{(i)}\} \subset [a, b]$ for all $i = 1, 2, \ldots, N$, where $0 < a < b < 1$. Then $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, N$.*

Proof. Let $p \in F$. By Lemma 3.2(ii), we get $\lim_{n\to\infty} d(x_n, p)$ exists. Then there is $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, p) = c. \tag{3.1}$$

By Lemma 3.1(ii), we get

$$\limsup_{n\to\infty} d\left(y_n^{(i)}, p\right) \leq c \text{ for } i = 1, 2, \ldots, N-1. \tag{3.2}$$

Since $d(T_i^n y_n^{(i-1)}, p) \leq k_n d(y_n^{(i-2)}, p) + s_n$ and (3.2), we obtain

$$\limsup_{n\to\infty} d\left(T_i^n y_n^{(i-1)}, p\right) \leq c \text{ for } i = 1, 2, \ldots, N. \tag{3.3}$$

Since $\lim_{n\to\infty} d(x_{n+1}, p) = c$, we have

$$\lim_{n\to\infty} d\left(W\left(T_N^n y_n^{(N-1)}, y_n^{(N-1)}, \alpha_n^{(N)}\right), p\right) = c. \tag{3.4}$$

By (3.2), (3.3), (3.4) and Lemma 2.7, we can conclude that

$$\lim_{n\to\infty} d\left(T_N^n y_n^{(N-1)}, y_n^{(N-1)}\right) = 0.$$

Assume that

$$\lim_{n\to\infty} d\left(T_j^n y_n^{(j-1)}, y_n^{(j-1)}\right) = 0, \text{ for some } j,\ 2 \leq j \leq N.$$

By Lemma 3.1(iii) and $\lim_{n\to\infty} d(x_{n+1}, p) = c$, we have

$$c \leq \liminf_{n\to\infty} d\left(y_n^{(j-1)}, p\right) \text{ for } 2 \leq j \leq N.$$

By (3.2), we get $\lim_{n\to\infty} d(y_n^{(j-1)}, p) = c$ for $2 \leq j \leq N$. It follows that

$$\lim_{n\to\infty} d\left(W\left(T_{j-1}^n y_n^{(j-2)}, y_n^{(j-2)}, \alpha_n^{(j-1)}\right), p\right) = \lim_{n\to\infty} d\left(y_n^{(j-1)}, p\right) = c. \tag{3.5}$$

Using (3.2), (3.3), (3.5) and Lemma 2.7, we can conclude that

$$\lim_{n\to\infty} d\left(T_{j-1}^n y_n^{(j-2)}, y_n^{(j-2)}\right) = 0.$$

Therefore, we obtain by induction that

$$\lim_{n\to\infty} d\left(T_i^n y_n^{(i-1)}, y_n^{(i-1)}\right) = 0 \text{ for } i = 1, 2, \ldots, N. \tag{3.6}$$

By (1.1) and Lemma 2.5, we get

$$\begin{aligned} d\left(y_n^{(i)}, y_n^{(i-1)}\right) &= d\left(W\left(T_i^n y_n^{(i-1)}, y_n^{(i-1)}, \alpha_n^{(i)}\right), y_n^{(i-1)}\right) \\ &= \alpha_n^{(i)} d\left(T_i^n y_n^{(i-1)}, y_n^{(i-1)}\right) \end{aligned}$$

for $i = 1, 2, \ldots, N-1$. It follows by (3.6) that

$$\lim_{n\to\infty} d\left(y_n^{(i)}, y_n^{(i-1)}\right) = 0 \text{ for } i = 1, 2, \ldots, N-1. \tag{3.7}$$

From

$$d\left(x_n, y_n^{(i)}\right) \le d\left(x_n, y_n^{(1)}\right) + d\left(y_n^{(1)}, y_n^{(2)}\right) + \cdots + d\left(y_n^{(i-1)}, y_n^{(i)}\right)$$

for $i = 1, 2, \ldots, N-1$. This implies by (3.7) that

$$\lim_{n\to\infty} d\left(x_n, y_n^{(i)}\right) = 0 \text{ for } i = 1, 2, \ldots, N-1. \tag{3.8}$$

For $1 \le i \le N$, we have

$$\begin{aligned} d\left(x_n, T_i^n x_n\right) &\le d\left(x_n, y_n^{(i-1)}\right) + d\left(y_n^{(i-1)}, T_i^n y_n^{(i-1)}\right) + d\left(T_i^n y_n^{(i-1)}, T_i^n x_n\right) \\ &\le d\left(x_n, y_n^{(i-1)}\right) + d\left(y_n^{(i-1)}, T_i^n y_n^{(i-1)}\right) + Ld\left(y_n^{(i-1)}, x_n\right). \end{aligned}$$

By (3.6) and (3.8), we get

$$\lim_{n\to\infty} d\left(x_n, T_i^n x_n\right) = 0 \text{ for } i = 1, 2, \ldots, N. \tag{3.9}$$

Using (1.1), we have

$$\begin{aligned} d\left(x_{n+1}, x_n\right) &= d\left(W\left(T_N^n y_n^{(N-1)}, y_n^{(N-1)}, \alpha_n^{(N)}\right), x_n\right) \\ &\le \alpha_n^{(N)} d\left(T_N^n y_n^{(N-1)}, x_n\right) + \left(1 - \alpha_n^{(N)}\right) d\left(y_n^{(N-1)}, x_n\right) \\ &\le \alpha_n^{(N)} \left(d\left(T_N^n y_n^{(N-1)}, y_n^{(N-1)}\right) + d\left(y_n^{(N-1)}, x_n\right)\right) \\ &\quad + \left(1 - \alpha_n^{(N)}\right) d\left(y_n^{(N-1)}, x_n\right) \\ &= \alpha_n^{(N)} d\left(T_N^n y_n^{(N-1)}, y_n^{(N-1)}\right) + d\left(y_n^{(N-1)}, x_n\right). \end{aligned}$$

By (3.6) and (3.8), we have

$$\lim_{n\to\infty} d\left(x_{n+1}, x_n\right) = 0. \tag{3.10}$$

For $1 \le i \le N$, we have

$$\begin{aligned} d\left(x_n, T_i x_n\right) &\le d(x_n, x_{n+1}) + d\left(x_{n+1}, T_i^{n+1} x_{n+1}\right) \\ &\quad + d\left(T_i^{n+1} x_{n+1}, T_i^{n+1} x_n\right) + d\left(T_i^{n+1} x_n, T_i x_n\right) \end{aligned}$$

$$\leq d(x_n, x_{n+1}) + d\left(x_{n+1}, T_i^{n+1}x_{n+1}\right) + Ld\left(x_{n+1}, x_n\right) + Ld\left(T_i^n x_n, x_n\right)$$
$$= (1+L)d(x_n, x_{n+1}) + d\left(T_i^{n+1}x_{n+1}, x_{n+1}\right) + Ld\left(T_i^n x_n, x_n\right).$$

By (3.9) and (3.10), we obtain $\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0$ for all $i = 1, 2, \ldots, N$. □

Theorem 3.7. *Let (X, d, W) be a complete uniformly convex metric space with continuous convex structure and C be a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive self-mappings of C with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F = \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1) with $\{\alpha_n^{(i)}\} \subset [a,b]$ for all $i = 1, 2, \ldots, N$, where $0 < a < b < 1$. If one of the following is satisfied:*

(i) *$\{T_i\}_{i=1}^N$ satisfies Condition (A),*
(ii) *one member of the family $\{T_i\}_{i=1}^N$ is semi-compact,*

then $\{x_n\}$ converges to a common fixed point of the family $\{T_i\}_{i=1}^N$.

Proof. By Lemma 3.6, $\lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0$ for all $i = 1, 2, \ldots, N$.

(i) : By the Condition (A), there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that

$$\lim_{n\to\infty} f\left(d\left(x_n, F\right)\right) \leq \lim_{n\to\infty} d\left(x_n, T_i x_n\right) = 0,$$

for some i, $1 \leq i \leq N$. It follows that $\lim_{n\to\infty} d\left(x_n, F\right) = 0$. By Theorem 3.3, we can conclude that $\{x_n\}$ converges to a point $p \in F$.

(ii) : Without loss of generality, we assume that T_1 is semi-compact. Then there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in C$. Hence, for each $i = 1, 2, \ldots, N$, we have

$$d(p, T_i p) \leq d(p, x_{n_j}) + d(x_{n_j}, T_i x_{n_j}) + d(T_i x_{n_j}, T_i p)$$
$$\leq (1+L)d(p, x_{n_j}) + d(x_{n_j}, T_i x_{n_j}) \to 0.$$

Thus $p \in F$. By continuity of $x \mapsto d(x,F)$, we obtain $\lim_{j\to\infty} d\left(x_{n_j}, F\right) = d(p,F) = 0$. It follows by Lemma 3.2(ii) that $\lim_{n\to\infty} d\left(x_n, F\right) = 0$. By Theorem 3.3, we can conclude that $\{x_n\}$ converges to a point $p \in F$. □

4. Applications

In this section, we apply our main results to obtain strong convergence theorems in both hyperbolic spaces and CAT(0) spaces.

Definition 4.1 ([12, 14, 16])**.** A hyperbolic space (X, d, W) is a metric space (X, d) together with a convexity mapping $W : X \times X \times [0,1] \to X$ satisfying

(i) $d\left(z, W\left(x, y, \lambda\right)\right) \leq \lambda d\left(z, x\right) + (1-\lambda)\, d\left(z, y\right)$;
(ii) $d\left(W\left(x, y, \lambda_1\right), W\left(x, y, \lambda_2\right)\right) = |\lambda_1 - \lambda_2| d\left(x, y\right)$;
(iii) $W\left(x, y, \lambda\right) = W\left(y, x, 1-\lambda\right)$;
(iv) $d\left(W\left(x, z, \lambda\right), W\left(y, w, \lambda\right)\right) \leq \lambda d\left(x, y\right) + (1-\lambda)\, d\left(z, w\right)$.

If a triple (X, d, W) satisfies (i)-(iii), then we get the notion of space of hyperbolic type in the sense of Goebel and Kirk [6]. It is easy to see that hyperbolic spaces are covex metric spaces. Then, our results in the previous section are also true for hyperbolic spaces as follows.

Theorem 4.2. *Let (X, d, W) be a complete hyperbolic space and C be a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of generalized asymptotically quasi-nonexpansive self-mappings of C with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose $F = \bigcap_{i=1}^N F(T_i)$ is nonempty and closed. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1). Then $\{x_n\}$ converges to a common fixed point of the family $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{d(x, p) : p \in F\}$.*

The following result is a useful property in uniformly convex hyperbolic spaces, see [13]. It can be applied to a CAT(0) space as well.

Lemma 4.3 ([13, Lemma 2.9])**.** *Let (X, d, W) be a uniformly convex hyperbolic space with modulus of uniform convexity η such that η increases with r (for a fixed ε). Let $x \in X$ and suppose that $\{t_n\}$ is a sequence in $[a, b]$ for some $a, b \in (0, 1)$ and $\{x_n\}, \{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$ and $\lim_{n\to\infty} d(W(x_n, y_n, t_n), x) = r$, where $r \ge 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Using Lemma 4.3 and the same arguments as in the proof of Theorem 3.7, the following result is obtained.

Theorem 4.4. *Let (X, d, W) be a complete uniformly convex hyperbolic space with modulus of uniform convexity η such that η increases with r (for a fixed ε) and C be a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive self-mappings of C with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Suppose that $F = \bigcap_{i=1}^N F(T_i)$ is nonempty. Let $x_1 \in C$ and the sequence $\{x_n\}$ be defined by (1.1) with $\{\alpha_n^{(i)}\} \subset [a, b]$ for all $i = 1, 2, \ldots, N$, where $0 < a < b < 1$. If one of the following is satisfied:*

(i) *$\{T_i\}_{i=1}^N$ satisfies Condition (A),*

(ii) *one member of the family $\{T_i\}_{i=1}^N$ is semi-compact,*

then $\{x_n\}$ converges to a common fixed point of the family $\{T_i\}_{i=1}^N$.

Next, we apply our main results to CAT(0) spaces. We first recall CAT(0) spaces, see more details in [2]. Let (X, d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t_1), c(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a *geodesic segment* joining x and y. When it is unique this geodesic is denoted by $[x, y]$. The space (X, d) is said to be a *geodesic metric space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset Y of X is said to be *convex* if Y includes every geodesic segment joining any two of its points.

A *geodesic triangle* $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X and a geodesic segment between each pair of vertices. A *comparison triangle* for geodesic triangle $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\bar{x}_i, \bar{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space is said to be a CAT(0) space if all geodesic triangles satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$, $d(x, y) \le d_{\mathbb{E}^2}(\bar{x}, \bar{y})$.

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \le \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

This is the (CN) inequality of Bruhat and Tits [3]. By using the (CN) inequality, it is easy to see the CAT(0) spaces are uniformly convex. In fact [2], a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality. Moreover, if X is CAT(0) space and $x, y \in X$, then for any $\lambda \in [0, 1]$, there exists a unique point $\lambda x \oplus (1 - \lambda)y \in [x, y]$ such that

$$d(z, \lambda x \oplus (1 - \lambda)y) \le \lambda d(z, x) + (1 - \lambda)d(z, y),$$

for any $z \in X$.

Remark 4.5. In view of the above inequality, CAT(0) spaces have convex structure $W(x, y, \lambda) = \lambda x \oplus (1 - \lambda)y$. Then, the iterative process (1.1) can be translated to CAT(0) spaces as follows:

$$\tag{4.1} \begin{cases} y_n^{(0)} = x_n, \\ y_n^{(1)} = \alpha_n^{(1)} T_1^n y_n^{(0)} \oplus \left(1 - \alpha_n^{(1)}\right) y_n^{(0)}, \\ y_n^{(2)} = \alpha_n^{(2)} T_2^n y_n^{(1)} \oplus \left(1 - \alpha_n^{(2)}\right) y_n^{(1)}, \\ y_n^{(3)} = \alpha_n^{(3)} T_3^n y_n^{(2)} \oplus \left(1 - \alpha_n^{(3)}\right) y_n^{(2)}, \\ \quad \vdots \\ y_n^{(N-1)} = \alpha_n^{(N-1)} T_{N-1}^n y_n^{(N-2)} \oplus \left(1 - \alpha_n^{(N-1)}\right) y_n^{(N-2)}, \\ x_{n+1} = \alpha_n^{(N)} T_N^n y_n^{(N-1)} \oplus \left(1 - \alpha_n^{(N)}\right) y_n^{(N-1)}, \end{cases}$$

for all $n \in \mathbb{N}$.

In 2007, Leuştean [14] proved that CAT(0) spaces are uniformly convex hyperbolic spaces with modulus of uniform convexity $\eta := \frac{\varepsilon^2}{8}$. Thus, Theorem 4.2 and 4.4 can be applied to CAT(0) spaces as follows.

Theorem 4.6. *Let X be a complete CAT(0) space and C be a nonempty closed convex subset of X. Let $\{T_i\}_{i=1}^N$ be a finite family of generalized asymptotically quasi-nonexpansive self-mappings of C with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset$*

$[0,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n-1)<\infty$ *and* $\sum_{n=1}^{\infty}s_n<\infty$. *Suppose* $F=\bigcap_{i=1}^{N}F(T_i)$ *is nonempty and closed. Let* $x_1\in C$ *and the sequence* $\{x_n\}$ *be defined by* (4.1). *Then* $\{x_n\}$ *converges to a common fixed point of the family* $\{T_i\}_{i=1}^{N}$ *if and only if* $\liminf_{n\to\infty} d(x_n,F)=0$, *where* $d(x,F)=\inf\{d(x,p):p\in F\}$.

Theorem 4.7. *Let* X *be a complete CAT(0) space and* C *be a nonempty closed convex subset of* X. *Let* $\{T_i\}_{i=1}^{N}$ *be a finite family of uniformly L-Lipschitzian and generalized asymptotically quasi-nonexpansive self-mappings of* C *with sequences* $\{k_n\}\subset[1,\infty)$ *and* $\{s_n\}\subset[0,\infty)$ *such that* $\sum_{n=1}^{\infty}(k_n-1)<\infty$ *and* $\sum_{n=1}^{\infty}s_n<\infty$. *Suppose that* $F=\bigcap_{i=1}^{N}F(T_i)$ *is nonempty. Let* $x_1\in C$ *and the sequence* $\{x_n\}$ *be defined by* (4.1) *with* $\{\alpha_n^{(i)}\}\subset[a,b]$ *for all* $i=1,2,\ldots,N$, *where* $0<a<b<1$. *If one of the following is satisfied:*

(i) $\{T_i\}_{i=1}^{N}$ *satisfies Condition (A),*
(ii) *one member of the family* $\{T_i\}_{i=1}^{N}$ *is semi-compact,*

then $\{x_n\}$ *converges to a common fixed point of the family* $\{T_i\}_{i=1}^{N}$.

Remark 4.8. The results in Section 3 and Section 4 hold true in a Banach space, if we set $W(x,y,\lambda)=\lambda x+(1-\lambda)y$. Theorem 3.3 extends and generalizes Theorem 3.2 of Kettapun et al. [8] to a finite family of generalized asymptotically quasi-nonexpansive mappings and to a convex metric space setting. Theorem 3.3 extends and generalizes Theorem 3.2 of Yatakoat and Suantai [24] from a Banach space to a convex metric space setting.

References

[1] K. Aoyama, K. Eshita and W. Takahashi, *Iteration processes for nonexpansive mappings in convex metric spaces*, in Proceedings of the International Conference on Nonlinear and Convex Analysis Okinawa, Yokohama Pub. Yokohama, 2005, pp. 31–39.

[2] M. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, Heidelberg, 1999.

[3] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251.

[4] S.-S. Chang, L. Yang and X. R. Wang, *Stronger convergence theorems for an infinite family of uniformly quasi-Lipschitzian mappings in convex metric spaces*, Applied Mathematics and Computation **217** (2010), 277–282.

[5] Y. J. Cho, Lj. Ćirić and S.-H. Wang, *Convergence theorems for nonexpansive semigroups in CAT(0) spaces*, Nonlinear Anal. **74** (2011), 6050–6059.

[6] K. Goebel and W.A. Kirk, *Iteration processes for nonexpansive mappings*, in Topological methods in nonlinear functional analysis (Toronto, 1982), S. P. Singh, S. Thomeier, B. Watson (Eds.), Contemp. Math. vol. 21, Amer. Math. Soc., 1983, pp. 115-123,.

[7] A. Kaewcharoen and B. Panyanak, *Fixed points for multivalued mappings in uniformly convex metric spaces*, International Journal of Mathematics and Mathematical Sciences, vol. 2008, Article ID 163580, 9 pages.

[8] A. Kettapun, A. Kananthai and S. Suantai, *A new approximation method for common fixed points of a finite family of asymptotically quasi-nonexpansive mappings in Banach spaces*, Computers and Mathematics with Applications **60** (2010), 1430–1439.

[9] A. R. Khan, H. Fukhar-ud-din and A. A. Domlo, *Approximating fixed points of some maps in uniformly convex metric spaces*, Fixed Point Theory and Applications Volume 2010, Article ID 385986, 11 pages.

[10] A. R. Khan and M. A. Ahmed, *Convergence of a general iterative scheme for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces and applications*, Computers and Mathematics with Applications **59** (2010), 2990–2995.

[11] A. R. Khan, M. A. Khamsi and H. Fukhar-ud-din, *Strong convergence of a general iteration scheme in CAT(0) spaces*, Nonlinear Anal. **74** (2011), 783–791.

[12] W. A. Kirk, *Krasnoselskii's iteration process in hyperbolic spaces*, Numer. Funct. Anal. and Optimiz. **4** (1982), 371–381.

[13] W. Laowang and B. Panyanak, *Approximating fixed points of nonexpansive nonself mappings in CAT(0) spaces*, Fixed Point Theory and Applications Volume 2010, Article ID 367274, 11 pages.

[14] L. Leuştean, *A quadratic rate of asymptotic regularity for CAT(0)-spaces*, J. Math. Anal. Appl. **325** (2007), 386–399.

[15] Q.-Y. Liu, Z.-B. Liu and N.-J. Huang, *Approximating the common fixed points of two sequences of uniformly quasi-Lipschitzian mappings in convex metric spaces*, Applied Mathematics and Computation **216** (2010), 883–889.

[16] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. **15** (1990), 537–558.

[17] S. Saejung, S. Suantai and P. Yotkaew, A Note on "Common Fixed Point of Multistep Noor Iteration with Errors for a Finite Family of Generalized Asymptotically Quasi-Nonexpansive Mapping", Abstract and Applied Analysis, vol. 2009, Article ID 283461, 9 pages.

[18] T. Shimizu, *A convergence theorem to common fixed points of families of nonexpansive mappings in convex metric spaces*, in Nonlinear and Convex Analysis, Yokohama Pub. Yokohama, 2007, pp. 575–585,

[19] T. Shimizu and W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces*, Topological Methods in Nonlinear Analysis **8** (1996), 197–203.

[20] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, Kodai Math. Sem. Rep. **22** (1970), 142–149.

[21] C. Wang and L.-W. Liu, *Convergence theorems for fixed points of uniformly quasi-Lipschitzian mappings in convex metric spaces*, Nonlinear Anal. **70** (2009), 2067–2071.

[22] C. Wang, J. Zhu, B. Damjanović and L.-G. Hu, *Approximating fixed points of a pair of contractive type mappings in generalized convex metric spaces*, Applied Mathematics and Computation **215** (2009), 1522–1525.

[23] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109–113.

[24] P. Yatakoat and S. Suantai, *A new approximation method for common fixed points of a finite family of generalized asymptotically quasi-nonexpansive mappings in Banach spaces*, Abstract and Applied Analysis Volume 2010, Article ID 706587, 15 pages.

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