Journal of Nonlinear and Convex Analysis
Volume 14, Number 1, 2013, 115–122

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

SKEWNESS AND JAMES CONSTANT OF BANACH SPACES

KEN-ICHI MITANI, KICHI-SUKE SAITO*, AND YASUJI TAKAHASHI

Abstract. Let X be a real Banach space. The notion of skewness of X was introduced by Fitzpatrick and Reznick. Also, the notion of James constant of X was introduced by Gao and Lau. In this paper we give some relations between these two constants of X.

1. Introduction and preliminaries

Throughout this paper let X be a real Banach space with $\dim X \geq 2$ and $S_X = \{x \in X : \|x\| = 1\}$. Fitzpatrick and Reznick [2] introduced the skewness $s(X)$ of X, which describes the asymmetry of norm:

$$s(X) = \sup\left\{ \lim_{t\to 0^+} \frac{\|x+ty\| - \|y+tx\|}{t} : \ x, y \in S_X \right\}$$
$$= \sup\left\{ \langle x, y\rangle - \langle y, x\rangle : \ x, y \in S_X \right\},$$

where

$$\langle x, y\rangle = \|x\| \lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

for $x, y \in X$. As stated in [2], if X is smooth, then $\langle \cdot, \cdot \rangle$ is a generalized inner product by Ritt [10]. It is clear that $0 \leq s(X) \leq 2$ for any Banach space X. They showed that $s(X) = 0$ if and only if X is a Hilbert space, and calculated the skewness for L_p spaces where $1 \leq p \leq \infty$. Moreover, the uniform non-squareness of X can be described in terms of the skewness $s(X)$. Indeed, they showed that $s(X) < 2$ if and only if X is uniformly non-square, that is, there exists a $\delta > 0$ such that for any $x, y \in S_X$, either $\|x + y\| \leq 2(1-\delta)$ or $\|x - y\| \leq 2(1-\delta)$. A modified version of skewness was introduced and studied by Baronti and Papini [1].

The James constant $J(X)$ of X is defined by

$$J(X) = \sup\left\{ \min\{\|x+y\|, \|x-y\|\} : x, y \in S_X \right\},$$

(Gao and Lau [3]). It is well-known that $\sqrt{2} \leq J(X) \leq 2$ for any Banach space X. If X is a Hilbert space, then $J(X) = \sqrt{2}$, and the converse is not true. The James constant for L_p spaces where $1 \leq p \leq \infty$ was calculated. Namely, if $1 \leq p \leq \infty$ and dim $L_p \geq 2$, then $J(L_p) = \max\{2^{1/p}, 2^{1/q}\}$, where $1/p + 1/q = 1$. Also, X is uniformly non-square if and only if $J(X) < 2$ (see [3]). Moreover, some relations between the James constant and other constants are discussed by several authors (see [4, 8, 9]).

2010 *Mathematics Subject Classification.* 46B20.
Key words and phrases. Skewness, James constant, modulus of smoothness.
*The second author was supported in part by Grants-in-Aid for Scientific Research (No. 23540189), Japan Society for the Promotion of Science.

Our aim in this paper is to give some relations between the skewness $s(X)$ and James constant $J(X)$ of X. Namely, we show that for any Banach space X,

$$2+4(2-J(X))-4\sqrt{(2-J(X))(4-J(X))}\le s(X)\le 4\Big\{1-\frac{1}{J(X)}\Big\} \tag{1.1}$$

(see Theorems 2.6 and 2.9). From the inequalities (1.1) we can directly obtain the result that $s(X) < 2$ if and only if X is uniformly non-square, given by [2]. To show the second inequality of (1.1) we use some results in Baronti and Papini [1], Takahashi and Kato [9].

We recall some definitions (cf. [6]). A Banach space X is called uniformly convex if, for every $\varepsilon > 0$ there is a $\delta > 0$ such that for any $x, y \in S_X$, $\|x+y\| > 2-\delta$ implies $\|x-y\| < \varepsilon$. It is well-known that X is uniformly convex if and only if, for any sequences $\{x_n\}, \{y_n\}$ in S_X with $\|x_n+y_n\| \to 2$, we have $\|x_n-y_n\| \to 0$.

The modulus of smoothness of X is defined by

$$\rho_X(\tau)=\sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2}-1 : x,y\in S_X\right\}.$$

It is known that X is uniformly non-square if and only if $\rho_X(1) < 1$([4]).

The following lemmas will be used later.

Lemma 1.1 ([5], Lemma 2)**.** *Let $\{x_n\}, \{y_n\}$ be sequences in a Banach space X such that $\{\|x_n\|\}_{n=1}^{\infty}$ and $\{\|y_n\|\}_{n=1}^{\infty}$ are convergent to non-zero limits, respectively. Then the following are equivalent.*
(i) $\lim_{n\to\infty}\|x_n+y_n\| = \lim_{n\to\infty}(\|x_n\|+\|y_n\|)$.
(ii) $\lim_{n\to\infty}\left\|\frac{x_n}{\|x_n\|}+\frac{y_n}{\|y_n\|}\right\| = 2$.

Lemma 1.2 ([7], Lemma 5.4.14)**.** *Let X be a Banach space and $x \in X$ with $x \neq 0$. Then for each y in X, the function*

$$t \mapsto \frac{\|x+ty\|-\|x\|}{t}$$

from $\mathbb{R}\backslash\{0\}$ into $\mathbb{R}$ is non-decreasing.

2. Skewness and James constant

We first consider a relation between skewness and the modulus of smoothness. Baronti and Papini [1] estimated $s(X)$ from above by $\rho_X(1)$, as follows:

Proposition 2.1 ([1], Proposition 6.3)**.** *Let X be a Banach space. Then*

$$s(X) \le 2\rho_X(1). \tag{2.1}$$

Remark 2.2. Let X be a Hilbert space. It is known that $\rho_X(1) = \sqrt{2}-1$ (see [6]). We also have $s(X) = 0$. Thus we obtain that the inequality (2.1) is strict in this case.

Proposition 2.3. *If X is uniformly convex, then the inequality* (2.1) *in Proposition 2.1 is strict, that is, $s(X) < 2\rho_X(1)$.*

Proof. Suppose that X is uniformly convex but $s(X) = 2\rho_X(1)$. By Remark 2.2 we note that X is not a Hilbert space, that is, $s(X) \neq 0$. From the definition of skewness we can find sequences $\{x_n\}$, $\{y_n\}$ in S_X such that $s(X) - 1/n < \langle x_n, y_n\rangle - \langle y_n, x_n\rangle$. Fix t_0 with $0 < t_0 < 1$. Since $\{\|x_n + y_n\|\}$ is bounded, without loss of generality, we may assume that $\|x_n + y_n\| \to a$ for some a. Similarly, we may assume that $\|x_n - y_n\| \to b$ and $\|x_n + t_0 y_n\| \to c$ for some b, c. By Lemma 1.2,

$$\begin{aligned}\langle x_n, y_n\rangle - \langle y_n, x_n\rangle &\leq \frac{\|x_n + t_0 y_n\| - \|x_n\|}{t_0} - \frac{\|y_n - x_n\| - \|y_n\|}{-1} \\ &\leq \frac{\|x_n + y_n\| - \|x_n\|}{1} + \|y_n - x_n\| - \|y_n\| \\ &\leq 2\rho_X(1).\end{aligned}$$

As $n \to \infty$, we get

$$c - 1 = t_0(a - 1) \tag{2.2}$$

and

$$s(X) = a + b - 2 = 2\rho_X(1). \tag{2.3}$$

From

$$\|x_n + t_0 y_n\| = \|t_0(x_n + y_n) + (1 - t_0)x_n\| \leq t_0\|x_n + y_n\| + (1 - t_0)\|x_n\|$$

and the equality (2.2), we obtain

$$\lim_{n\to\infty} \|z_n + w_n\| = \lim_{n\to\infty} \|z_n\| + \lim_{n\to\infty} \|w_n\|,$$

where $z_n = t_0(x_n + y_n)$ and $w_n = (1 - t_0)x_n$.

Case 1. $\|z_n\| \nrightarrow 0$.

By Lemma 1.1,

$$\lim_{n\to\infty} \left\| \frac{z_n}{\|z_n\|} + \frac{w_n}{\|w_n\|} \right\| = 2.$$

Since X is uniformly convex, we have

$$\lim_{n\to\infty} \left\| \frac{z_n}{\|z_n\|} - \frac{w_n}{\|w_n\|} \right\| = 0,$$

that is,

$$\lim_{n\to\infty} \left\| \frac{x_n + y_n}{\|x_n + y_n\|} - x_n \right\| = 0.$$

By

$$\begin{aligned}\left\| \frac{x_n + y_n}{\|x_n + y_n\|} - x_n \right\| &= \left\| \Big(\frac{1}{\|x_n + y_n\|} - 1\Big)x_n + \frac{1}{\|x_n + y_n\|} y_n \right\| \\ &\geq \left| \left|\frac{1}{\|x_n + y_n\|} - 1\right| - \frac{1}{\|x_n + y_n\|} \right|,\end{aligned}$$

we have

$$\left| \left|\frac{1}{a} - 1\right| - \frac{1}{a} \right| = 0$$

and hence $a = 2$. Thus $\|x_n + y_n\| \to 2$. Since X is uniformly convex, we obtain $\|x_n - y_n\| \to 0$, that is, $b = 0$.
Case 2. $\|z_n\| \to 0$.
Then we clearly have $a = 0$. Moreover, it follows from the inequalities

$$2 \geq \|x_n - y_n\| = \|x_n + y_n - 2y_n\| \geq |\|x_n + y_n\| - 2\|y_n\||$$

that we get $\|x_n - y_n\| \to 2$, that is, $b = 2$.

Thus $a + b = 2$ is valid for any cases. By (2.3) we have $s(X) = 0$, which is a contradiction. □

Remark 2.4. (i) If X is not uniformly non-square, then by $s(X) = 2$ and $\rho_X(1) = 1$, we have $s(X) = 2\rho_X(1)$.
(ii) There is a uniformly non-square (not uniformly convex) Banach space X such that $s(X) = 2\rho_X(1)$. In fact, let $X_0 = \mathbb{R}^2$ with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x_1x_2 \geq 0 \\ \|x\|_1 & \text{if } x_1x_2 \leq 0 \end{cases}$$

for $x = (x_1, x_2)$. It is clear that X_0 is uniformly non-square and is not uniformly convex. By Example 4 in [4] we have $\rho_{X_0}(1) = 1/2$. From Proposition 2.1 we obtain $s(X_0) \leq 2\rho_{X_0}(1) = 1$. We next show $s(X_0) \geq 1$. For $0 < \varepsilon < 1$ we put $x = (1, 1)$ and $y = (-\varepsilon, 1 - \varepsilon)$. Let $t > 0$ be sufficiently small. It is clear that $x, y \in S_{X_0}$. Moreover $\|x + ty\| = 1 + t - t\varepsilon$ and $\|y + tx\| = 1$. Hence

$$s(X_0) \geq \langle x, y\rangle - \langle y, x\rangle = 1 - \varepsilon.$$

As $\varepsilon \to 0$ we have $s(X_0) \geq 1$. Thus $s(X_0) = 2\rho_{X_0}(1) = 1$.

Recently, Takahashi and Kato in [9] estimated $\rho_X(1)$ from above by $J(X)$.

Proposition 2.5 ([9], Theorem 1)**.** *Let X be a Banach space. Then*

$$\rho_X(1) \leq 2\Big\{1 - \frac{1}{J(X)}\Big\}.$$

From Propositions 2.1 and 2.5, we obtain the following.

Theorem 2.6. *Let X be a Banach space. Then*

$$s(X) \leq 4\Big\{1 - \frac{1}{J(X)}\Big\}. \tag{2.4}$$

Remark 2.7. (i) Immediately from Proposition 2.3 we obtain that the inequality (2.4) in Theorem 2.6 is strict for all uniformly convex spaces.
(ii) If X is not uniformly non-square, then by $s(X) = 2$ and $J(X) = 2$ we have equality in (2.4).
(iii) We consider the space X_0 in Remark 2.4 (ii). By Example 4 in [4] it follows that $J(X_0) = 3/2$. We also have $s(X_0) = 1$. Thus the inequality (2.4) is strict for the space X_0.

(iv) We do not know whether there is a uniformly non-square (not uniformly convex) Banach space X such that the inequality (2.4) becomes equality.

In the following, we shall present an estimate $s(X)$ from below by $J(X)$. To do this we need the following lemma.

Lemma 2.8. *Let X be a Banach space. Then*

$$s(X) > \frac{2(J(X) - 2 + t - t^2)}{t(1+t)}$$

for all t with $0 < t \leq 1$.

Proof. Let $0 < t_0 \leq 1$. We first show that

$$s(X) \geq \frac{2(J(X) - 2 + t_0 - t_0^2)}{t_0(1+t_0)}. \tag{2.5}$$

By $s(X) \geq 0$, we may assume that $J(X) - 2 + t_0 - t_0^2 > 0$. Let $\varepsilon > 0$ with $\varepsilon < \min(J(X)-2+t_0-t_0^2, \sqrt{2}-1)$. Then there exist $u, v \in S_X$ such that $J(X)-\varepsilon < \min(\|u+v\|, \|u-v\|)$. Let $w = u+t_0 v$ and $z = v-t_0 u$. Using the triangle inequality we have

$$\|w\| \leq \|u\| + t_0\|v\| = 1 + t_0. \tag{2.6}$$

Also,

$$\|w\| = \|u + v - (1 - t_0)v\| \geq \|u + v\| - (1 - t_0) > J(X) - \varepsilon - 1 + t_0. \tag{2.7}$$

Similarly,

$$\|z\| \leq 1 + t_0 \tag{2.8}$$

and

$$\|z\| \geq \|v - u\| - 1 + t_0 > J(X) - \varepsilon - 1 + t_0. \tag{2.9}$$

Note that w and z are non-zero elements, because

$$J(X) - \varepsilon - 1 + t_0 \geq \sqrt{2} - 1 - \varepsilon + t_0 > t_0 > 0$$

holds. Moreover,

$$\|w - t_0 z\| = \|u + t_0^2 u\| = 1 + t_0^2 \tag{2.10}$$

and

$$\|z + t_0 w\| = \|v + t_0^2 v\| = 1 + t_0^2. \tag{2.11}$$

It follows by Lemma 1.2 and from the inequalities (2.7) and (2.10) that for t with $0 < t < t_0$,

$$\begin{aligned}\frac{\|w + tz\| - \|w\|}{t} &\geq \frac{\|w - t_0 z\| - \|w\|}{-t_0} = \frac{\|w\| - \|w - t_0 z\|}{t_0} \\ &> \frac{J(X) - \varepsilon - 1 + t_0 - (1 + t_0^2)}{t_0}\end{aligned}$$

and so

$$\frac{\|w + tz\| - \|w\|}{t} > \frac{J(X) - 2 - \varepsilon + t_0 - t_0^2}{t_0}. \tag{2.12}$$

We define

$$A_\varepsilon = J(X) - 2 - \varepsilon + t_0 - t_0^2$$

for $\varepsilon > 0$. Note that $A_\varepsilon > 0$. Put $x = \frac{w}{\|w\|}$ and $y = \frac{z}{\|z\|}$. From the inequalities (2.8) and (2.12),

$$\langle x, y\rangle = \frac{\langle w, z\rangle}{\|w\|\|z\|} = \frac{1}{\|z\|}\lim_{t\to 0^+}\frac{\|w + tz\| - \|w\|}{t} \geq \frac{A_\varepsilon}{t_0\|z\|} \geq \frac{A_\varepsilon}{t_0(1+t_0)}. \tag{2.13}$$

By Lemma 1.2, (2.9) and (2.11) we have for t with $0 < t < t_0$,

$$\begin{aligned}\frac{\|z + tw\| - \|z\|}{t} &\leq \frac{\|z + t_0 w\| - \|z\|}{t_0} \\ &< \frac{1 + t_0^2 - (J(X) - \varepsilon - 1 + t_0)}{t_0} \\ &= \frac{-A_\varepsilon}{t_0}\end{aligned}$$

and hence

$$-\langle y, x\rangle = -\frac{\langle z, w\rangle}{\|z\|\|w\|} \geq \frac{A_\varepsilon}{t_0\|w\|} \geq \frac{A_\varepsilon}{t_0(1+t_0)}. \tag{2.14}$$

Consequently we have

$$s(X) \geq \langle x, y\rangle - \langle y, x\rangle \geq \frac{2A_\varepsilon}{t_0(1+t_0)}. \tag{2.15}$$

As $\varepsilon \to 0$, we obtain (2.5).

We next prove that the inequality (2.5) is strict. Suppose that

$$s(X) = \frac{2(J(X) - 2 + t_0 - t_0^2)}{t_0(1+t_0)} \tag{2.16}$$

for some t_0 with $0 < t_0 \leq 1$. By $s(X) \geq 0$ it follows that $J(X) - 2 + t_0 - t_0^2 \geq 0$. From $J(X) - 7/4 \geq (t_0 - 1/2)^2 \geq 0$, we obtain $J(X) \geq 7/4$. This inequality implies that X is not a Hilbert space. Hence $s(X) > 0$. By (2.16), $J(X) - 2 + t_0 - t_0^2 > 0$ holds. Take a number n_0 such that $n \geq n_0$ imples

$$\frac{1}{n} < \min(J(X) - 2 + t_0 - t_0^2, \sqrt{2} - 1).$$

For each n with $n \geq n_0$ we take u_n, v_n in S_X with

$$J(X) - 1/n < \min(\|u_n + v_n\|, \|u_n - v_n\|).$$

Put

$$w_n = u_n + t_0 v_n,\ z_n = v_n - t_0 u_n,\ x_n = \frac{w_n}{\|w_n\|},\ y_n = \frac{z_n}{\|z_n\|}.$$

As in the inequalities (2.13), (2.14) and (2.15), we have for each $n \geq n_0$,

$$\langle x_n, y_n\rangle \geq \frac{B_n}{t_0\|z_n\|} \geq \frac{B_n}{t_0(1+t_0)},\ -\langle y_n, x_n\rangle \geq \frac{B_n}{t_0\|w_n\|} \geq \frac{B_n}{t_0(1+t_0)} \tag{2.17}$$

and

$$s(X) \geq \langle x_n, y_n\rangle - \langle y_n, x_n\rangle \geq \frac{2B_n}{t_0(1+t_0)}, \tag{2.18}$$

where $B_n = A_{1/n} = J(X) - 2 - 1/n + t_0 - t_0^2$. Since $\{\|z_n\|\}_{n=1}^{\infty}$ and $\{\|w_n\|\}_{n=1}^{\infty}$ are bounded, we may assume that these sequences have limits. As $n \to \infty$, the inequalities (2.17) and (2.18) imply

$$\lim_{n\to\infty} \|w_n\| = \lim_{n\to\infty} \|u_n + t_0 v_n\| = 1 + t_0$$

and

$$\lim_{n\to\infty} \|z_n\| = \lim_{n\to\infty} \|v_n - t_0 u_n\| = 1 + t_0.$$

By Lemma 1.1,

$$\lim_{n\to\infty} \|u_n + v_n\| = \lim_{n\to\infty} \|u_n - v_n\| = 2.$$

Hence X is not uniformly non-square and so $s(X) = J(X) = 2$. Thus it follows from (2.16) that

$$2 = 2\frac{t_0 - t_0^2}{t_0(1 + t_0)}.$$

By $t_0 > 0$, this is a contradiction. Thus the inequality (2.5) is strict. □

Theorem 2.9. *Let X be a Banach space. Then*

$$s(X) \geq 2 + 4(2 - J(X)) - 4\sqrt{(2 - J(X))(4 - J(X))}, \tag{2.19}$$

where equality holds only when X is not uniformly non-square.

Proof. We define a function f on $(0, 1]$ as

$$f(t) = \frac{2(J(X) - 2 + t - t^2)}{t(1 + t)}.$$

To prove the inequality (2.19) we calculate the supremum of $f(t)$ on $(0, 1]$. If X is not uniformly non-square, that is, $J(X) = 2$, then

$$f(t) = \frac{2(1 - t)}{1 + t}$$

for all t with $0 < t \leq 1$ and so the function f is decreasing on $(0, 1]$. Hence, by Lemma 2.8 we have

$$s(X) \geq \lim_{t\to 0^+} f(t) = 2.$$

By $s(X) \leq 2$ we obtain $s(X) = 2$. Thus (2.19) is valid for this case and then (2.19) becomes equality. Let X be uniformly non-square, that is, $J(X) < 2$. By

$$f(t) = -2 + \frac{2(2t + J(X) - 2)}{t^2 + t} \tag{2.20}$$

the derivative of f is

$$f'(t) = 2 \cdot \frac{2(t^2 + t) - (2t + J(X) - 2)(2t + 1)}{(t^2 + t)^2}.$$

Put

$$t_0 = \frac{2 - J(X) + \sqrt{J(X)^2 - 6J(X) + 8}}{2}.$$

Note that $0 < t_0 < 1$ by $\sqrt{2} \leq J(X) < 2$. Then $f'(t_0) = 0$ and f has the maximum at $t = t_0$. From (2.20) and the equality $2(t_0^2 + t_0) - (2t_0 + J(X) - 2)(2t_0 + 1) = 0$, we have

$$\begin{aligned} f(t_0) &= -2 + \frac{4}{2t_0 + 1} \\ &= -2 + \frac{4}{3 - J(X) + \sqrt{J(X)^2 - 6J(X) + 8}} \\ &= 2 + 4(2 - J(X)) - 4\sqrt{(2 - J(X))(4 - J(X))}. \end{aligned}$$

By Lemma 2.8 it holds that $s(X) > f(t_0)$ and thus this completes the proof. □

From Theorems 2.6 and 2.9 we direct have the following.

Corollary 2.10 ([2], Theorem 3.1). *Let X be a Banach space. Then X is uniformly non-square if and only if $s(X) < 2$.*

References

[1] M. Baronti and P. L. Papini, *Projections, skewness and related constants in real normed spaces*, Math. Pannonica **3** (1992), 31–47.

[2] S. Fitzpatrick and B. Reznick, *Skewness in Banach spaces*, Trans. Amer. Math. Soc. **275** (1983), 587–597.

[3] J. Gao and K. S. Lau, *On the geometry of spheres in normed linear spaces*, J. Aust. Math. Soc. A **48** (1990), 101–112.

[4] M. Kato, L. Maligranda and Y. Takahashi, *On James and Jordan–von Neumann constants and the normal structure coefficient of Banach spaces*, Stud. Math. **144** (2001), 275–295.

[5] M. Kato, K.-S. Saito and T. Tamura, *Uniform non-squareness of ψ-direct sums of Banach spaces $X \oplus_\psi Y$*, Math. Inequal. Appl. **7** (2004), 429–437.

[6] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II* : Function spaces, Springer, Berlin. 1979.

[7] R. E. Megginson, *An introduction to Banach space theory*, Graduate Texts in Mathematics, 183, New York, Springer, 1998.

[8] Y. Takahashi, *Some geometric constants of Banach spaces -a unified approach*, Banach and function spaces II, Yokohama Publ., Yokohama, 2007, pp. 191–220.

[9] Y. Takahashi and M. Kato, *A simple inequality for the von Neumann–Jordan and James constants of a Banach space*, J. Math. Anal. Appl. **359** (2009), 602–609.

[10] R. K. Ritt, *A generalization of inner product*, Michigan Math. J. **3** (1955), 23–26.

Manuscript received October 11, 2011
revised July 27, 2012

Ken-Ichi Mitani
Department of Systems Engineering, Okayama Prefectural University, Soja 719-1197, Japan
E-mail address: `mitani@cse.oka-pu.ac.jp`

Kichi-Suke Saito
Department of Mathematics, Faculty of Science, Niigata University, Niigata 950-2181, Japan
E-mail address: `saito@math.sc.niigata-u.ac.jp`

Yasuji Takahashi
Department of Systems Engineering, Okayama Prefectural University, Soja 719-1197, Japan
E-mail address: `ym-takahashi@clear.ocn.ne.jp`